# Spectral Approach to $H_\infty$-Optimal SISO Synthesis Problem

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*Abstract:* - This paper is devoted to a particular case of $H_\infty$-optimization problem for LTI systems with scalar control, external disturbance and measurement noise. This problem can be numerically solved with the help of the well-known universal approaches based on Riccati equations, LMI or maximum entropy technique. Nevertheless, in our opinion there exists a possibility to implement a special form of spectral approach, using polynomial factorization for the mentioned particular situation. The correspondent technique is proposed with the aim to increase a computational efficiency of the synthesis and to present an optimal solution in a specific form, which is convenient for investigation. Some theoretical details are discussed and numerical algorithms are developed for practical implementation. Theirs applicability and effectiveness are illustrated by the examples of $H_\infty$-optimal synthesis.

*Key-Words:* - control system, control law, synthesis, stability, optimization, H infinity control

## 1 Introduction

One of the most important problems in a practice of controlled systems analytical design is an LTI synthesis problem of the optimal rejection of external disturbances and measurement noises using feedback connection. This problem has determined the vast area of investigations in control theory and signal processing from the beginning of 40-th years of the previous century. Starting directions of this area are known as optimal filtering and mean square synthesis and nowadays theirs multiple descendants are joined in the framework of the modern *H*-optimization theory.

The founders of the optimal filtering and synthesis theory are A.N. Kolmogorov and N. Wiener. This theory has received its development in numerous works of the given direction and in modern treatment optimal filtering and mean square synthesis are the partial cases of $H_2$-optimization ideology. Besides that, many questions of the mentioned direction can be considered with the help of $H_\infty$-optimization methods.

Within the framework of the *H*-optimization theory two computational approaches are widely used: first of them is based on a solution of the algebraic matrix Riccati equations ("2-Riccati" approach) [1], [2], [3], and second – on a solution of linear matrix inequalities ("LMI" technique) [4]. Correspondent methods have successful implementation in MATLAB package.

As for the stationary laboratory conditions, computational effectiveness of these methods is quite enough to provide control laws design and investigations. However, we cannot say the same with respect to theirs implementation in real-time regime of operating for control systems with adoptive changeover of control laws. Here the computational running time of synthesis is highly crucial issue, and commonly used universal algorithms can be not fully satisfactory for practical applications. We usually deal with the similar difficulties for various kinds of embedded systems or for onboard control systems of autonomous moving robots.

Nevertheless, in our opinion there exists some possibility to improve computational effectiveness of $H_\infty$-synthesis by means of using the certain alternative variant of optimization problem. Similar to the standard situation, this variant reflects our desire to suppress external disturbances with respect to output variables. However, it simplifies a considering of some analytical and computational issues for a practical implementation.

Such a possibility can be realised for SISO $H_\infty$-optimization problem, where controlled plant has scalar controlling and disturbing inputs. In this connection, the paper is devoted to the spectral approach to the synthesis on the base of polynomial methods ([5], [6]) presented in original spectral form. In particular, here the mentioned alternative variant of synthesis is used as an auxiliary instrument to obtain the upper estimates for the standard situation to reduce the computational running time. In addition, the spectral approach allows to overcome the degenerate essence of a standard situation with no noise in measurements.

The paper is organized as follows. In the next section, equations of a controlled plant are presented and the standard problem of $H_\infty$-optimal synthesis is posed. Section 3 is devoted to the statement of an alternative problem, which is used as the basis for investigations provided below. Here we focus special attention on the relationship between standard and alternative problems. In section 4, we develop special spectral approach to the synthesis of the $H_\infty$-optimal controller. As a particular result, easy calculated upper estimates for the standard problem are proposed. Section 5 is devoted to the issue of the optimal transfer function design. In Section 6, a degenerate variant of a standard situation with no noise in measurements is discussed. In section 7, the numerical examples of synthesis are presented on the base of the obtained results. Finally, Section 8 concludes this paper by discussing the overall results of the investigation.

## 2 The Problem of $H_\infty$-Synthesis

Let us consider LTI controlled plant with a mathematical model of the form

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{p}d(t), \\ &y = \mathbf{c}\mathbf{x} + \psi(t),\ \xi = \mathbf{c}\mathbf{x}, \\ &e_1 = \xi, \\ &e_2 = ku, \end{aligned} \tag{1}$$

where $\mathbf{x} \in E^n$ is a state space vector, $y$, $u$, $\xi$, $d$ and $\psi$ are scalar values: $y$ and $\xi$ are measured and controlled variables respectively, $u$ is a control, $d(t)$ represents an external disturbance, $\psi(t)$ is a measurement noise. All the components of the matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{p}$, $\mathbf{c}$ and parameter $k$ are given constants. Let suppose that the pairs $\{\mathbf{A},\mathbf{b}\}$ and $\{\mathbf{A},\mathbf{c}\}$ are controllable and observable respectively.

External inputs $d$ and $\psi$ we shall treated bellow as the outputs of additional systems

$$d = S_{d1}(s)i_1,\quad \psi = S_{\psi 1}(s)i_2 \tag{2}$$

correspondently, where $i_1, i_2$ are the components of the vector $\mathbf{i} = (i_1 \quad i_2)'$ of a new disturbances,

$$S_{d1}(s) = N_d(s)/T_d(s),\quad S_{\psi 1}(s) = N_\psi(s)/T_\psi(s).$$

Here polynomials $N_d, T_d, N_\psi, T_\psi$ are Hurwitz. One can easy see that, if the inputs $i_1(t)$ and $i_2(t)$ are Gaussian white noises, disturbances $d$ and $\psi$ can be treated as random stationary processes with rational spectral power densities

$$S_d(\omega) = S_{d1}(s)S_{d1}(-s)\big|_{s=j\omega},\ S_\psi(\omega) = S_{\psi 1}(s)S_{\psi 1}(-s)\big|_{s=j\omega}.$$

Let accept that controller to be designed has a form

$$u = W(s)y, \tag{3}$$

where $W = W_1/W_2$, $W_1, W_2$ are polynomials. The transfer function $W$ of the controller (3) should be found as a solution of the analytical synthesis problem. If any, we obtain a closed-loop connection (1) – (3) presented in Fig. 1 by its block-scheme with the input $\mathbf{i} = (i_1 \quad i_2)'$ and the output $\mathbf{e} = (e_1 \quad e_2)'$, having mathematical model of the form $\mathbf{e} = \mathbf{H}(s,W)\mathbf{i}$, where $\mathbf{H}$ is a transfer matrix of the system.

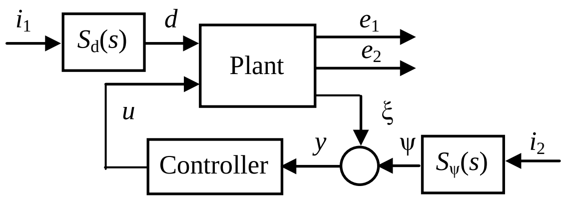


Fig. 1. The closed-loop connection scheme.

Let us introduce auxiliary transfer functions of the system (1) – (3) with respect to inner input $\{d,\psi\}$ and output $\{\xi,u\}$:

$$F_{d\xi}(s) = P/(A - BW),\ F_{\psi\xi}(s) = BW/(A - BW),$$

$$F_{du}(s) = PW/(A - BW),\ F_{\psi u}(s) = AW/(A - BW),$$

where $A(s) = \det(Es - \mathbf{A})$,

$$B(s) = A(s)\mathbf{c}(Es - \mathbf{A})^{-1}\mathbf{b},\ P(s) = A(s)\mathbf{c}(Es - \mathbf{A})^{-1}\mathbf{p}.$$

Using the introduced functions, we can represent the transfer matrix $\mathbf{H}(s,W)$ as follows:

$$\mathbf{H}(s,W) \equiv \begin{pmatrix} F_{d\xi}(s) & F_{\psi\xi}(s) \\ kF_{du}(s) & kF_{\psi u}(s) \end{pmatrix} \cdot \begin{pmatrix} S_{d1}(s) & 0 \\ 0 & S_{\psi 1}(s) \end{pmatrix}. \tag{4}$$

The matter of the standard problem of $H_\infty$-optimal synthesis is to find any solution of the following optimization problem

$$\begin{aligned} &J_\infty(W) = \|\mathbf{H}(s,W)\|_\infty^2 \to \min_{W\in\Omega_\infty}, \|\mathbf{H}\|_\infty = \sup_{\omega\in[0,\infty)} \sigma_m(\omega), \\ &\Omega_\infty = \{\, W : \mathbf{H}(s,W) \in \mathbf{RH}_\infty \}, \end{aligned} \tag{5}$$

where Hardy space $\mathbf{RH}_\infty$ [1] consists of $2\times 2$

matrices with proper fractionally rational components with Hurwitz denominators, $\sigma_m(\omega)=\sigma_m(\omega,W)$ is the maximum singular value of the matrix $\mathbf{H}(j\omega,W)$.

We shall suppose bellow that the plant (1) satisfies all the requirements for the existence of the optimal controller. This guaranties that using any known approach ("2-Riccati", LMI or maximum entropy [5], [7]) we can find the transfer function $W=W_{\infty 0}$ of the optimal controller.

Numerical solution of the problem with all mentioned methods uses a standard iteration technique to determine the optimal value of the functional $J_\infty$. Starting with high and low estimates of the optimum, a bisection algorithm allows to find this one as a minimum value $J_m$, for which the relationship $J_\infty(W)\le J_m$ can be satisfied by any $W\in\Omega_\infty$. It is evident that the overall running time of calculations essentially depends on the choice of the mentioned estimates: the less is the relative difference between them, the less running time is.

A particular purpose of this paper is to obtain upper $J_{\infty u}$ and lower $J_{\infty w}$ estimates for the minimal value $J_{\infty 0}=J_\infty(W_{\infty 0})$ of the functional $J_\infty$ for considered partial situation. These estimates should reduce the number of iterations in a bisection algorithm for the solution of a standard $H_\infty$-optimization problem, using mentioned techniques.

To begin with, let us consider another $H_\infty$-optimization problem with respect to the same plant (1), inputs (2), and controller (3), aimed to the maximum suppression of the disturbance **i** action to the output **e**.

## 3 Alternative H∞-Optimization

First, let us introduce the generalized transfer function $H_w(s,W)$, satisfying the identity

$$|H_w(j\omega,W)|^2\equiv\left[|F_{d\xi}(j\omega)|^2+k^2|F_{du}(j\omega)|^2\right]S_d(\omega)+ \\ +\left[|F_{\psi\xi}(j\omega)|^2+k^2|F_{\psi u}(j\omega)|^2\right]S_\psi(\omega). \tag{6}$$

Second, let us pose the following optimization problem:

$$J(W)=\|H_w(s,W)\|_\infty^2\to\min_{W\in\Omega}, \\ \|H_w\|_\infty=\sup_{\omega\in[0,\infty)}|H_w(j\omega)|, \\ \Omega=\{\,W:H_w(s,W)\in RH_\infty\}, \tag{7}$$

where the set $RH_\infty$ consists of proper rational fractions with Hurwitz denominators.

One can see that the functional $J(W)$ similar to the functional $J_\infty(W)$ represents a measure of a disturbances suppression by the closed-loop connection. Therefore, the problems (5) and (7) have the same practical essence, but different mathematical formalization. Additionally note that for the both problems parameter $k$ plays the role of a weight multiplier, governing the relationship between the intensity of control action and the achieved accuracy of suppression.

The following statement assigns a connection between the both optimization problems:

**Lemma 1:** For any controller (3) with the transfer function $W\in\Omega$ and for any frequency $\omega\in[0,\infty)$ the following relationship holds:

$$\sigma_m^2(\omega,W)\le|H_w(j\omega,W)|^2. \tag{8}$$

**Proof:** In accordance with the definition of the singular value, we have $\sigma_m^2(\omega,W)=\lambda_m(\omega,W)$; here real number $\lambda_m>0$ is the maximum eigenvalue of the Hermitian matrix $\mathbf{H}(j\omega,W)\mathbf{H}'(-j\omega,W)$. Next, omitting an explicit dependency from variables $j\omega$, $W$, and using notation $\overline{\rho}=\overline{\rho}(s)=\rho(-s)$ for any rational fraction $\rho(s)$, on the base of (4) obtain

$$\mathbf{H}\overline{\mathbf{H}}'=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix},\alpha_{11}=F_{d\xi}\overline{F}_{d\xi}S_d+F_{\psi\xi}\overline{F}_{\psi\xi}S_\psi, \tag{9}$$

$$\alpha_{12}=kF_{d\xi}\overline{F}_{du}S_d+kF_{\psi\xi}\overline{F}_{\psi u}S_\psi,\alpha_{21}= \\ =k\overline{F}_{d\xi}F_{du}S_d+k\overline{F}_{\psi\xi}F_{\psi u}S_\psi,$$

$$\alpha_{22}=k^2F_{du}\overline{F}_{du}S_d+k^2F_{\psi u}\overline{F}_{\psi u}S_\psi.$$

Characteristic polynomial of this matrix is quadratic trinomial $\Delta_H(s)=s^2-d_1s+d_0$, where

$$d_1=\alpha_{11}+\alpha_{22},\ d_0=\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}$$

with the discriminant $D=d_1^2/4-d_0\ge 0$. It follows from this relationship that $\sqrt{D}=d_1/2-\delta$, where $\delta$ is real number such that $d_1/2\ge\delta>0$.

As a result, we obtain the following expression for the maximum root of the polynomial $\Delta_H(s)$:

$$\sigma_m^2(\omega,W)=\lambda_m(\omega,W)=d_1/2+\sqrt{D}=d_1-\delta,$$

i.e. in accordance with (6) and (9) we have

$$\sigma_m^2(\omega,W)\le d_1=\alpha_{11}+\alpha_{22}=|H_w(j\omega,W)|^2,$$

and the lemma is proven. ■

**Theorem 1:** The solution $J_0 = J(W_0)$ of the optimization problem (7) can be used as an upper bound estimate for the solution $J_{\infty 0} = J_\infty(W_{\infty 0})$ of the initial optimization problem (5), i.e. we have

$$J_{\infty 0} = J_\infty(W_{\infty 0}) \le J_0 = J(W_0)\,. \qquad (10)$$

**Proof:** In accordance with Lemma 1 on the base of (8) we can claim that $\forall W \in \Omega$ the following relationships hold:

$$\begin{aligned} J_\infty(W) &= \|\mathbf{H}(s,W)\|_\infty^2 = \sup_{\omega\in[0,\infty)} \sigma_m^2(\omega,W) \le \\ &\le \sup_{\omega\in[0,\infty)} |H_w(j\omega,W)|^2 = \|H_w(s,W)\|_\infty^2 = J(W), \end{aligned} \qquad (11)$$

Because of $J_{\infty 0} = J_\infty(W_{\infty 0}) \le J_\infty(W)$, $\forall W \in \Omega$, it follows from (11) that $J_{0\infty} \le J(W),\ \forall W \in \Omega$, and in particular the same is valid for $W = W_0$, i.e. $J_{0\infty} \le J(W_0) = J_0$. ■

Thus, as a preliminary result it is possible to claim that in the sense of the relationship (10) we can consider the problem (7) instead of the problem (5) to obtain desirable estimates. This transition allows us to apply polynomial technique that essentially reduces a running time of calculations for the low order systems $(n \le 5)$.

However, we have some difficulty in the accepted way, because of a direct solution of the problem (7) is appreciably obstructed by the nonlinear dependency of the functional $J$ from the adjustable function $W$. To avoid this difficulty, it seems suitable to employ any parameterization technique for the stabilizing controllers set. The most popular approach is based on the results discussed in Youla [8]. However, here we shall use other method, firstly described in 1971, with modern interpretation presented in Aliev and Larin [6].

In accordance with this method, let introduce the adjustable function-parameter $\Phi$ as

$$\Phi = L_\Phi^{-1}(W) = \frac{\alpha + \beta W}{A - BW} \ \Rightarrow\ W = L_\Phi(\Phi) = \frac{A\Phi - \alpha}{B\Phi + \beta}, \qquad (12)$$

where $\alpha$ and $\beta$ are any polynomials such that the polynomial

$$Q(s) = A(s)\beta(s) + B(s)\alpha(s) \qquad (13)$$

is Hurwitz. Formulae (12) allow us to express transfer functions of the closed system as

$$F_{d\xi} = P(B\Phi + \beta)/Q\,,\ F_{\psi\xi} = B(A\Phi - \alpha)/Q\,, \qquad (14)$$

$$F_{du} = P(A\Phi - \alpha)/Q\,,\ F_{\psi u} = A(A\Phi - \alpha)/Q\,.$$

It is easy to see that optimization problem (7) is equivalent to the following problem:

$$I(\Phi) = \|H(s,\Phi)\|_\infty^2 \to \min_{\Phi\in\Omega_\Phi}, \qquad (15)$$

where admissible set $\Omega_\Phi = L_\Phi^{-1}(\Omega)$ includes rational fractions $\Phi$ with Hurwitz denominators. The function $H(s,\Phi) = H_w(s, L_\Phi(\Phi))$ in accordance with (5) satisfies the identity

$$\begin{aligned} |H(j\omega,\Phi)|^2 &\equiv \left(F_{d\xi}\overline{F}_{d\xi} + k^2 F_{du}\overline{F}_{du}\right)S_d + \\ &+ \left(F_{\psi\xi}\overline{F}_{\psi\xi} + k^2 F_{\psi u}\overline{F}_{\psi u}\right)S_\psi. \end{aligned} \qquad (16)$$

**Lemma 2:** The identity (16) can be converted to the following form:

$$H(\Phi)\overline{H}(\Phi) \equiv (T_1 - T_2\Phi)(\overline{T}_1 - \overline{T}_2\overline{\Phi}) + T_3\,, \qquad (17)$$

where the rational fraction $T_2(s)$ with Hurwitz denominator and functions $T_1(s) \in RL$, $T_3(s) \in RL$ are determined by the formulae

$$\begin{aligned} T_1 &= \left(k^2\alpha\overline{A} - \beta\overline{B}\right)D/(GQ) + S_\psi \overline{A}\overline{B}/\left(G\overline{D}\right), \\ T_2 &= \overline{G}D/Q, \end{aligned} \qquad (18)$$

$$\begin{aligned} T_3 &= k^2 D\overline{D}/\left(G\overline{G}\right) + S_\psi (B\overline{B} - k^2 A\overline{A})/\left(G\overline{G}\right) - \\ &- S_\psi^2 A\overline{A}B\overline{B}/\left(G\overline{G}D\overline{D}\right). \end{aligned} \qquad (19)$$

Here the Hurwitz polynomial $G(s)$ and the fraction $D(s) \equiv N(s)/T(s)$ with Hurwitz polynomials $N$, $T$ are the results of the following factorizations:

$$k^2 A\overline{A} + B\overline{B} \equiv G\overline{G}\,,\ \ S_\psi A\overline{A} + S_d P\overline{P} \equiv D\overline{D}\,. \qquad (20)$$

**Proof:** A possibility of the mentioned representation directly follows from the substitution of the formulae (16), (18) – (20) to the right part of (17). ■

# 4 Spectral Approach to the Synthesis

Let us now consider the problems (7) and (15), which also can be transformed to an equivalent form. Really, in accordance with Lemma 2 we have

$$\begin{aligned} I(\Phi) &= \|H(s,\Phi)\|_\infty^2 = \sup_{\omega\in[0,\infty)} |H(j\omega)|^2 = \\ &= \sup_{\omega\in[0,\infty)} \left[\left|T_1 - T_2\Phi\right|_{s=j\omega}^2 + T_3(\omega)\right] \to \min_{\Phi\in\Omega_\Phi}, \end{aligned} \qquad (21)$$

where the rational fraction $T_2(s)$ with Hurwitz denominator and functions $T_1(s) \in RL$, $T_3(s) \in RL$ are determined by the formulas (18) and (19).

Using a technique proposed in [2], let instead the problem (21) consider the choice of the function

$\Phi \in \Omega_\infty^\Phi$ such that the following relation holds:

$$I(\Phi) \le \rho^2,\ \rho^2 = J_a + \varepsilon, \quad (22)$$

where $\varepsilon$ is nonnegative real value,

$$J_a = \max_{\omega\in[0,\infty)} T_3(\omega)\,. \quad (23)$$

It is evident that the minimum of the functional $I(\Phi)$ in (21) is equal to the smallest value $\rho_0^2$ of a number $\rho^2$ for which a solution of the problem (22) exists:

$$I_0(\Phi) = \min_{\Phi\in\Omega_\Phi} I(\Phi) = \min\{\rho^2 : \exists\Phi\in\Omega_\Phi : \\ : \forall\omega\in[0,\infty)\ |T_1 - T_2\Phi|^2 + T_3 \le \rho^2\}. \quad (24)$$

To provide further discussion let address to the well known Nevanlinna-Pick interpolation problem (NP-problem) [2], [3]. The matter of this problem is to search a function $H(s)\in RH_\infty$ such that the following conditions hold:

$$\|H\|_\infty \le 1,\ H(\xi_i) = \zeta_i,\ \mathrm{Re}\,\xi_i > 0,\ i = \overline{1,\mu},$$

where $\xi_i$ and $\zeta_i$ are given complex numbers. Naturally, it is necessary that $|\zeta_i| \le 1, i = \overline{1,\mu}$, however this is not sufficient for the existence of a solution. The famous Pick's theorem [2] presents the necessary and sufficient conditions of NP-problem solvability.

On the base of the mentioned theorem, we arrive at the following statement.

**Theorem 2:** The problem (24) is solvable if and only if the value $\rho$ is such that Hermitian matrix $L_h(\rho^2) = \{l_{ij}(\rho^2)\}$ is non-negative, where

$$l_{ij} = (1-\zeta_i\bar\zeta_j)/(\xi_i + \bar\xi_j),$$
$$\xi_i = \begin{cases} g_i, if\ i\le n; \\ \nu_{i-n}, if\ n < i \le n+p; \end{cases}$$
$$\zeta_i = \begin{cases} d_i, if\ i\le n; \\ c_{i-n}, if\ n < i \le n+p. \end{cases}$$

Here $g_i$ $(i=\overline{1,n})$ and $\nu_i$ $(i=\overline{1,p})$ $(p = \deg N(-s))$ are the roots of polynomials $G(-s)$ and $N(-s)$ correspondently (recall that for simplicity we assume that the complex points $g_1, g_2, \ldots, g_n$, $\nu_1, \nu_2, \ldots, \nu_p$ are distinct),

$$d_i = -\left.\frac{\bar B S_d P\bar P T\bar T}{A R_\rho T_\psi}\right|_{s=g_i},\ i=\overline{1,n}; \quad (25)$$

$$c_i = \left.\frac{N_\psi \bar N_\psi \bar A \bar B T_d \bar T_d}{R_\rho T_\psi}\right|_{s=\nu_i},\ i=\overline{1,p}. \quad (26)$$

**Proof:** In accordance with (22) it is necessary and sufficient that there exists a function $\Phi\in\Omega_\Phi$ such that for any $\omega\in[0,\infty)$ the following relationship holds:

$$|T_1(j\omega) - T_2(j\omega)\Phi(j\omega)|^2 \le \rho^2 - T_3(\omega)\,. \quad (27)$$

Because of $\rho^2 \ge J_a$, it follows from (22) and (23) that $\rho^2 - T_3(\omega) > 0\ \ \forall\omega\in[0,\infty)$, moreover there exists a rational fraction $\tilde L(s)$ with Hurwitz numerator and denominator satisfying the identity

$$\tilde L(s)\tilde L(-s) \equiv \rho^2 - T_3(s)\,. \quad (28)$$

Really, it is easy to transform the expression (19) taking into account formulae (2) and (20) that give us the function $T_3(s)$ of the form

$$T_3 = \frac{N_d\bar N_d P\bar P\left(k^2 N\bar N + N_\psi \bar N_\psi B\bar B T_d \bar T_d\right)}{G\bar G N\bar N T_d \bar T_d}\,. \quad (29)$$

This transformation allows presenting a function $\tilde L(s)$ satisfying the identity (28) as follows

$$\tilde L(s) = R_\rho(s)/[G(s)N(s)T_d(s)]\,, \quad (30)$$

where the polynomial $R_\rho(s)$ is a Hurwitz result of the factorization

$$R_\rho\bar R_\rho \equiv \rho^2 G\bar G N\bar N T_d\bar T_d - \\ - N_d\bar N_d P\bar P\left(k^2 N\bar N + N_\psi\bar N_\psi B\bar B T_d\bar T_d\right). \quad (31)$$

Next, let introduce an inverse function

$$L(s) = \tilde L^{-1}(s) = G(s)N(s)T_d(s)/R_\rho(s)\,, \quad (32)$$

and on the base of (28) convert the problem (27) to the form

$$|[T_1(j\omega) - T_2(j\omega)\Phi(j\omega)]L(j\omega)|^2 \le 1\ \ \forall\omega\in[0,\infty)\,,$$

that is equivalent to the problem

$$\left|[T_1(j\omega) - T_2(j\omega)\Phi(j\omega)]L(j\omega)\frac{N(-j\omega)}{N(j\omega)}\right|^2 \le 1, \\ \forall\omega\in[0,\infty), \quad (33)$$

because of $|N(-j\omega)/N(j\omega)|^2 \equiv 1$.

Now let us introduce a rational fraction

$$Z(s) \equiv [T_1(s) - T_2(s)\Phi(s)]L(s)\frac{N(-s)}{N(s)} \equiv$$
$$\equiv \left[\frac{(k^2\alpha\bar{A} - \beta\bar{B})\bar{N}}{Q} + \frac{N_\psi \bar{N}_\psi \bar{A}\bar{B}\bar{T}_\psi T_d \bar{T}_d}{N} - \right. \quad (34)$$
$$\left. - \frac{G\bar{G}\bar{N}}{Q}\Phi\right]\frac{N}{R_\rho T_\psi}.$$

It is a matter of simple calculation to verify that the following equalities hold:

$$Z(g_i) = d_i,\ i = \overline{1,n};\ Z(\nu_i) = c_i,\ i = \overline{1,p}, \quad (35)$$

where the complex values $d_i$ and $c_i$ are determined by the formulas (25) and (26).

As a result, the initial problem (27) can be treated as the following NP-problem

$$\|Z(s)\|_\infty \le 1,\ Z(g_i) = d_i,\ i = \overline{1,n}\ ; \quad (36)$$
$$Z(\nu_i) = c_i,\ i = \overline{1,p}$$

with respect to a function $Z(s)$. However, in accordance with Pick's theorem, the NP-problem is solvable if and only if the Pick matrix $L_h(\rho) = \{l_{ij}(\rho)\}$ is non-negative that proves this theorem. ■

**Theorem 3:** The mentioned smallest value $\rho_0^2$, for which a solution of the problem (24) exists, belongs to the segment $[J_a,\ J_a + \varepsilon^*]$, where

$$\varepsilon^* = \min_{\Phi\in\Omega_\Phi} \|T_1 - T_2\Phi\|_\infty^2. \quad (37)$$

**Proof:** Let us address to the problem (21) and consider the evident relationships

$$I(\Phi) = \sup_{\omega\in[0,\infty)} \left[ \left| T_1 - T_2\Phi \right|^2_{s=j\omega} + T_3(\omega) \right] \le$$
$$\le \sup_{\omega\in[0,\infty)} \left| T_1 - T_2\Phi \right|^2_{s=j\omega} + \max_{\omega\in[0,\infty)} T_3(\omega), \quad (38)$$

which is valid for any function-parameter $\Phi \in \Omega_\Phi$. In particular, we can accept $\Phi = \Phi^*$, where

$$\Phi^* = \arg\min_{\Phi\in\Omega_\Phi} \|T_1 - T_2\Phi\|_\infty^2, \quad (39)$$

and from (38) we obtain

$$I(\Phi^*) = \sup_{\omega\in[0,\infty)} \left[ \left| T_1 - T_2\Phi^* \right|^2_{s=j\omega} + T_3(\omega) \right] \le \quad (40)$$
$$\le \sup_{\omega\in[0,\infty)} \left| T_1 - T_2\Phi^* \right|^2_{s=j\omega} + J_a,$$

taking into account (23). Because of the function $\Phi^*$ is not a solution of the problem (21), we have $\rho_0^2 = I(\Phi_0) \le I(\Phi^*)$, i.e.

$$\rho_0^2 \le J_a + \varepsilon^* \quad (41)$$

in accordance with (40) and (37). ■

**Corollary 1:** The value $\rho_m^2 = J_a + \varepsilon^*$ is an upper estimate of the minimal value $J_{\infty 0}$ of the functional $J_\infty(W)$ for initial optimization problem (5).

**Proof:** This claim directly follows from Theorem 1 and Theorem 3. ■

**Remark 1:** One can easy see that the optimization issue (39), on which the proof of Theorem 3 is based on, can be treated as well known Nehari problem [1].

This problem has effective solution presented by numerical algorithm in [2]. To use this algorithm directly, it is convenient to make the following transformation

$$\|T_1 - T_2\Phi\|_\infty^2 \equiv \left\| (T_1 - T_2\Phi)\frac{\bar{N}}{N} \right\|_\infty^2$$

similar to the proof of Theorem 2 that yields to the Nehari problem

$$I^*(\Phi) = \|T_1' - T_2'\Phi\|_\infty^2 \to \min_{\Phi\in\Omega_\Phi}, \quad (42)$$

where $T_1' = T_1\bar{N}/N$, $T_2' = T_2\bar{N}/N$. The initial data for the mentioned algorithm of its solution consist of all the zeros of the function $T_2'$ which are $g_i$, $(i = \overline{1,n})$ and $\nu_i$, $(i = \overline{1,p})$. Besides, these data also include the following values:

$$d_i^* = T_1'(g_i) = -\left.\frac{\bar{B}N_d\bar{N}_d P\bar{P}\bar{T}_\psi}{GAT_d N}\right|_{s=g_i},\ i = \overline{1,n}; \quad (43)$$

$$c_i^* = T_1'(\nu_i) = \left.\frac{N_\psi \bar{N}_\psi \bar{A}\bar{B}\bar{T}_d}{T_\psi NG}\right|_{s=\nu_i},\ i = \overline{1,p}. \quad (44)$$

Note that all the values (43) and (44) do not depend on the auxiliary polynomials $\alpha(s)$, $\beta(s)$, and $Q(s)$.

**Remark 2:** If the estimate $\rho_m^2$ seems to be too pessimistic, one can find the more realistic estimate $\rho_0^2$, using bisectional algorithm for the segment $[J_a,\ \rho_m^2]$ with checkout of the matrix $L_h(\rho^2) = \{l_{ij}(\rho^2)\}$ non-negative definiteness in accordance with Theorem 2.

## 5 Optimal Transfer Function

Note that the Theorem 2 gives not only the conditions of solvability, but also specifies direct way to find an optimal transfer function $W_0(s)$.

Remark that if there exists a solution of the NP-problem for given value $\rho$ such that $\|Z(s)\|_\infty = 1$, then this solution has the form

$$Z(s) = Z(s,\rho) \equiv m(-s,\rho)/m(s,\rho), \qquad (45)$$

where $m(s,\rho)$ is Hurwitz polynomial [2].

**Theorem 4:** Let we know the minimal value $\rho = \rho_0$ for which the problem (36) has a unique solution

$$Z_0(s) = Z(s,\rho_0) = m(-s)/m(s), m(s) \equiv m(s,\rho_0) \quad (46)$$

with the polynomial $R_\rho = R_{\rho 0}(s)$ (31) such that

$$R_{\rho 0}\overline{R}_{\rho 0} \equiv \rho_0^2 G\overline{G}N\overline{N}T_d\overline{T}_d - \\ - N_d\overline{N}_d P\overline{P}\left(k^2 N\overline{N} + N_\psi \overline{N}_\psi B\overline{B}T_d\overline{T}_d\right).$$

Then there exists a unique optimal controller (3) with respect to the $H_\infty$-problem (7), having the transfer function

$$W_0(s) = \frac{[A(s)M(s) - N(s)B(-s)m_0(s)]/\overline{G}}{[B(s)M(s) + k^2 N(s)A(-s)m_0(s)]/\overline{G}}, \quad (47)$$

where an auxiliary polynomial $M(s)$ is determined as follows

$$M(s) = [m_0(s)N_\psi(s)N_\psi(-s)A(-s)B(-s) \times \\ \times T_d(s)T_d(-s) - m_0(-s)R_{\rho 0}(s)T_\psi(s)]/N(-s). \quad (48)$$

Here divisions to the polynomials $G(-s)$ and $N(-s)$ are realized totally (without a reminder).

**Proof:** First, let substitute the known value $\rho_0$, polynomial $R_{\rho 0}(s)$, and function $Z_0(s)$ to the identity (34) that yields the following equation with respect to the variable $\Phi$:

$$Z_0(s) \equiv \left[\frac{(k^2\alpha\overline{A} - \beta\overline{B})\overline{N}}{Q} + \frac{N_\psi\overline{N}_\psi\overline{A}\overline{B}T_d\overline{T}_d}{N} - \frac{G\overline{G}\overline{N}}{Q}\Phi\right]\frac{N}{R_{\rho 0}T_\psi}. \quad (49)$$

A solution of this equation gives us an expression for the optimal parameter $\Phi = \Phi_0(s)$:

$$\Phi_0 = \frac{1}{G\overline{G}}\left(k^2\alpha\overline{A} - \beta\overline{B} + \frac{N_\psi\overline{N}_\psi\overline{A}\overline{B}T_d\overline{T}_d - Z_0R_{\rho 0}T_\psi}{N\overline{N}}Q\right). \quad (50)$$

Observe that using formulae (26) it is easy to verify that the expression $N_\psi\overline{N}_\psi\overline{A}\overline{B}T_d\overline{T}_d - Z_0R_{\rho 0}T_\psi$ is equal to zero in the complex points $s = \nu_i$ $(i = \overline{1,p})$. This implies that

$$\frac{N_\psi\overline{N}_\psi\overline{A}\overline{B}T_d\overline{T}_d - Z_0R_{\rho 0}T_\psi}{N\overline{N}} \equiv \frac{M}{Nm},$$

where $M$ is polynomial given by the formula (48) because of a division to $N(-s)$ here is realized totally, and we obtain

$$\Phi_0 = \frac{1}{G\overline{G}}\left(k^2\alpha\overline{A} - \beta\overline{B} + \frac{M}{Nm}Q\right). \quad (51)$$

Besides, one can check that the whole expression in the brackets for the parameter $\Phi_0$ in (50) and (51) is equal to zero in the complex points $s = g_i$ $(i = \overline{1,n})$: this follows from the formulae (25). Therefore, we can conclude that the rational fraction $\Phi_0(s)$ has Hurwitz denominator $N(s)G(s)m(s)$. Because of we have $I_\infty(\Phi_0) = \rho_0^2 < \infty$, it means that $\Phi_0 \in \Omega_\Phi$, and we can find a correspondent optimal transfer function with the help of the formulae (12), (51):

$$W_0(s) = \frac{A\Phi_0 - \alpha}{B\Phi_0 + \beta} = \frac{QGNm\left(AM - \overline{B}Nm\right)/\overline{G}}{QGNm\left(BM + k^2\overline{A}Nm\right)/\overline{G}}$$

i.e. the equality (47) holds. Observe that the function $W_0$ does not depend on the auxiliary polynomials $\alpha$ and $\beta$, i.e. the result is determined by only the initial data. ■

**Remark 3:** Despite the fact that the formula (47) seems to be not very complicated, detailed analysis shows certain disadvantage of the obtained optimal solution, because an order of the controller is greater than for the solution of standard $H_\infty$-problem (5).

Nevertheless, proposed controller with the optimal transfer function (47) could be also useful for practical applications if there is any reason for direct using of the optimization problem (7) instead of the standard one (5). To overcome its drawback, it is not difficult to compute a reduced order approximation of the $H_\infty$-optimal controller, excluding close roots of numerator and denominator for the transfer function (47).

Specially note that the presentation (47) is very convenient for various investigations of the optimal solution features such as a structure of the transfer function, its limit behaviour with respect to $k \to 0$ and $k \to \infty$, robust peculiarities of the controller, situations with non-fractional representation of the disturbances spectrums, transport delays, etc.

Let us summarise the above discussion introducing the following computational algorithm for the solution of the $H_\infty$-optimal synthesis problem (7).

**Algorithm 1:**

1. Execute the factorizations (20) and construct the polynomial $T(s) = T_d(s)T_\psi(s)$.

2. Construct the function $T_3(\omega)$ (29) and find its maximum value $J_a = \max\limits_{\omega\in[0,\infty)} T_3(\omega)$.

3. Decide Nehari problem $\varepsilon^* = \min\limits_{\Phi\in\Omega_\Phi} \|T_1' - T_2'\Phi\|_\infty^2$ with the help of algorithm (based on two Lyapunov equations), presented in [1], [2], using the initial data (43), (44). Determine the upper estimate $\rho_m^2 = J_a + \varepsilon^*$ for the minimal value $\rho_0^2$ of the functional $J(W)$.

4. Consider the segment $\rho^2 \in [J_a,\ \rho_m^2]$ and using bisectional algorithm determine the minimal value $\rho^2 = \rho_0^2$ guarantying non-negative definiteness of the matrix $L_h(\rho^2) = \{l_{ij}(\rho^2)\}$ with initial data (25), (26). Here $R_\rho(s)$ is a Hurwitz result of the factorization (31).

5. If the problem (7) is used only as an auxiliary instrument for the solution of the standard problem (5), accept the values $J_a$ and $J_0 = \rho_0^2$ as the lower and upper estimates correspondently for the minimum value $J_{\infty 0}$ of the functional $J_\infty(W)$. Then use any standard algorithm to solve the problem (5) with accepted estimates. Recall that it is possible to employ more pessimistic upper estimate $\rho_m^2$: this allows omitting step 4.

6. Execute factorization (31) accepting $\rho^2 = \rho_0^2$, find the values (25), (26) and solve NP-problem

$$\|H\|_\infty \le 1,\ H(\xi_i) = \zeta_i\,,\ i = \overline{1, n+p}\,,$$

with respect to the Hurwitz polynomial $m(s)$: $H(s) \equiv m_0(-s)/m_0(s)$. This action can be executed with the help of the same algorithm as for step 3.

7. Construct the auxiliary polynomial $M(s)$ (48) and obtain the numerator

$$W_{10}(s) = [A(s)M(s) - N(s)B(-s)m_0(s)]/G(-s)$$

and the denominator

$$W_{20}(s) = [B(s)M(s) + k^2 N(s)A(-s)m_0(s)]/G(-s)$$

of the optimal transfer function.

# 6 Irregular Situation

Remark that spectral form (47) plays a distinct role for the particular case of $H_\infty$-optimization problem with no measurement noise in the plant model (1). For this non-standard case [9] we have $\psi(t) \equiv 0$, i.e. $N_\psi(s) \equiv 0$, $T_\psi(s) \equiv 1$.

From the practical point of view, this irregular situation is not senseless. However, the standard problem (5) is directly unsolvable by «2-Riccati» approach for this case due to the degeneration essence of the statement. Of course, there exist many ways to overcome this difficulty (one of them is realized in MATLAB package), but these ways actually give us only the minimizing sequences of regular controllers with no accurate low bound.

As for the alternative variant (7), one can easy see that the mentioned situation is not irregular here, and can be treated as the usual particular case with the initial data $N_\psi \equiv 0$, $T_\psi \equiv 1$.

Let us address to this situation separately. One can easy make correspondent modification in the statement of the problem (7), changing (6) by the following identity:

$$|H_w(j\omega, W)|^2 \equiv \left[\,|F_{d\xi}(j\omega)|^2 + k^2 |F_{du}(j\omega)|^2\,\right] S_d(\omega)\,.$$

It is a matter of simple calculation to make correspondent transformation of all the formulas presented above. As a result, we arrive at the following algorithm of synthesis.

**Algorithm 2:**

1. Execute the following factorizations:

$$k^2 A(s)A(-s) + B(s)B(-s) \equiv G(s)G(-s),$$
$$P(s)P(-s) \equiv P_1(s)P_1(-s),$$

where $G$ and $P_1$ are Hurwitz polynomials.

2. Construct the function

$$T_3(\omega) = k^2 |N_d(j\omega)P(j\omega)/T_d(j\omega)G(j\omega)|^2$$

and find its maximum value $J_a = \max\limits_{\omega\in[0,\infty)} T_3(\omega)$.

3. Decide Nehari problem $\varepsilon^* = \min\limits_{\Phi\in\Omega_\Phi} \|T_1 - T_2\Phi\|_\infty^2$, where $T_2 = \overline{G} N_d P_1 / Q T_d$, using initial data

$$d_i^* = T_1(g_i) = -\overline{B} N_d \overline{N}_d P_1 / GAT_d \big|_{s=g_i}, \ i = \overline{1,n}.$$

Determine the upper estimate $\rho_m^2 = J_a + \varepsilon^*$ for the minimal value $\rho_0^2$ of the functional $J(W)$.

4. Consider the segment $\rho^2 \in [J_a, \ \rho_m^2]$ and using bisectional algorithm determine the minimal value $\rho^2 = \rho_0^2$ providing non-negative definiteness of the matrix $L_h(\rho^2) = \{l_{ij}(\rho^2)\}$, where

$$l_{ij} = (1 - d_i \overline{d}_j)/(g_i + \overline{g}_j), \ i, j = \overline{1,n};$$
$$d_i = -\overline{B} N_d \overline{N}_d P_1 / AR_\rho \big|_{s=g_i}, \ i = \overline{1,n}. \quad (52)$$

Here $R_\rho(s)$ is a Hurwitz result of the factorization

$$R_\rho \overline{R}_\rho \equiv \rho^2 G \overline{G} T_d \overline{T}_d - k^2 N_d \overline{N}_d. \quad (53)$$

5. Execute factorization (53) accepting $\rho^2 = \rho_0^2$, find complex values (52) and solve NP-problem

$$\|H\|_\infty \le 1, \ H(g_i) = d_i, \ i = \overline{1,n},$$

with respect to the Hurwitz polynomial $m(s)$: $H(s) \equiv m_0(-s)/m_0(s)$.

6. Construct the polynomials

$$W_{10}(s) = \left(A\overline{m}_0 R_{\rho 0} + \overline{B} m_0 N_d P_1\right)/\overline{G},$$
$$W_{20}(s) = \left(B\overline{m}_0 R_{\rho 0} - k^2 \overline{A} m_0 N_d P_1\right)/\overline{G}$$

and form the transfer function $W_0 = W_{10}/W_{20}$ of the optimal controller (3).

The alternative numerical algorithm of irregular synthesis is also presented in [10].

# 7 Synthesis Examples

**Example 1:** Let us consider a control plant (1) with given matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0.200 \\ 0.250 & 0 & 0.290 \\ 0 & 0.500 & -1.15 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2.00 \\ 0 \\ 0.250 \end{pmatrix}, \mathbf{p} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},$$
$$\mathbf{c} = \begin{pmatrix} 0 & 0 & 4 \end{pmatrix},$$

and the value $k = 1$ of a weight parameter. As for external disturbances, let suppose that $S_d \equiv S_\psi \equiv 1$.

Here we have the initial polynomials

$$A(s) = s^3 + 1.15s^2 - 0.145s - 0.0250; \ B(s) = s^2 - 1;$$
$$P(s) = 1; \ T_d(s) = T_\psi(s) = 1; \ \ N_d(s) = N_\psi(s) = 1.$$

Consequently obtain the following polynomials in accordance with Algorithm 1:

$$G(-s) = -s^3 + 2.86s^2 - 2.79s + 1$$

with the roots $g_{1,2} = 0.770 \pm 0.403j$, $g_3 = 1.32$;

$$N(-s) = -s^3 + 2.47s^2 - 2.24s + 1;$$

with the roots $\nu_{1,2} = 0.552 \pm 0.655j$, $\nu_3 = 1.36$. Then we determine a lower estimate for the value $\rho_0^2$ $J_a = \max_{\omega \in [0,\infty)} T_3(\omega) = 2.00$, and upper bound $\rho_m^2 = 26.78$. Executing step 4 of the algorithm 1, obtain $\rho_0^2 = 25.778$. Correspondently we have the polynomial

$$R_{\rho 0}(s) = 5.07s^6 + 27.0s^5 + 61.0s^4 + 77.0s^3 +$$
$$+ 57.6s^2 + 24.8s + 4.88$$

and the following initial data for NP-problem (36):

$g_1 = 1.32$, $g_{2,3} = 0.770 \pm 0.403j$; $\nu_1 = 1.36$,
$\nu_{2,3} = 0.552 \pm 0.654j$; $d_1 = -0.285 \cdot 10^{-3}$,
$d_{2,3} = -(4.15 \pm 1.22j) \cdot 10^{-3}$;
$c_1 = -0.272 \cdot 10^{-3}$, $c_{2,3} = (5.40 \pm 10.4j) \cdot 10^{-3}$.

As a solution of this problem obtain polynomial

$$m_0(s) = -0.458s^5 - 2.26s^4 - 4.61s^3 -$$
$$- 5s^2 - 2.94s - 0.739.$$

Then we can calculate an auxiliary polynomial

$$M(s) = 2.33s^8 + 6.16s^7 - 1.41s^6 - 20.1s^5 -$$
$$- 23.8s^4 - 4.24s^3 + 13.4s^2 + 12.1s + 3.58$$

and, at last, to form a transfer function $W_{0\infty} = W_{1\infty}/W_{2\infty}$ of $H_\infty$-optimal controller, where

$$W_{1\infty}(s) = 2.33s^8 + 16.0s^7 + 48.0s^6 + 83.1s^5 + 91.4s^4 +$$
$$+ 65.6s^3 + 29.8s^2 + 7.73s + 0.828,$$

$$W_{2\infty}(s) = 0.458s^8 + 6.50s^7 + 30.8s^6 + 75.2s^5 + 110s^4 +$$
$$+ 104s^3 + 62.2s^2 + 22.0s + 3.57.$$

Controller $u = W_{0\infty}(p)y$ provides the value $J_0 = J(W_{0\infty}) = \rho_0^2 = 25.778$ of the functional $J(W)$. Remark that we can reduce an order of the optimal controller using MATLAB function ***balred***:

$$W_{*\infty}(s) = \frac{5.08s^3 + 11.7s^2 + 7.75s + 1.38}{s^3 + 9.63s^2 + 14.6s + 5.96}.$$

Controller (3) with obtained transfer function gives practically the same value of performance index $J(W_{*\infty}) \approx J(W_{0\infty}) = 25.779$.

As for the solution of the standard problem (5), with the help of MATLAB function ***hinfsyn*** obtain the optimal transfer function

$$W_{h\infty}(s) = \frac{2380s^2 + 3630s + 846}{s^3 + 472s^2 + 416s + 3650},$$

providing minimal value $J_{\infty 0} = 25.716$ of the functional $J_\infty(W)$. Remark that the average running time of synthesis for the default initial segment $J_{\infty 0} \in [0, 88.9]$ is equal to 0.08 s. By using the estimates obtained above, we can shrink initial segment, for example $J_{\infty 0} \in [0.95 J_0, J_0]$, that reduces the running time more then twice.

**Example 2:** Consider the same control plant (1) as for the proceeding example and address to the irregular problem (7) with no noise. Using Algorithm 2, we consequently obtain $J_a = 0.9994$, $\rho_m^2 = 2.288$, $\rho_0^2 = 1.669$,

$$W_{0\infty}(s) = \frac{s^5 + 4.61s^4 + 8.55s^3 + 8.06s^2 + 3.88s + 0.764}{s^4 + 3.46s^3 + 4.49s^2 + 2.66s + 0.593}.$$

These data can be used for standard synthesis (5) to reduce a running time of calculations.

## 8 Conclusion

The main goal of the paper is to propose and to discuss a special spectral approach in frequency domain to partial case of $H_\infty$-optimization problem for LTI controlled plants. Proposed approach is based on a polynomial representation of initial and temporary data and on a special parameterization of stabilizing controllers set. Instead of the Riccati equation (or linear matrix inequalities) solutions, here polynomial factorizations are used, that substantially simplifies algorithms of synthesis.

Proposed spectral approach is implemented to the $H_\infty$-problem statement, which is some differed from the commonly used standard variant. The mentioned difference allows using the alternative problem and the algorithm of its solving as an auxiliary instrument with respect to the standard situation. First, this instrument gives the initial estimates for the standard $H_\infty$-norm minimum that can essentially reduce the running time of synthesis.

Second, the spectral approach is quite suitable to solve the irregular problem with no measurement noise. This situation is directly unsolvable by «2-Riccati» method, however the spectral approach allows to overcome this difficulty additionally reducing the running time of synthesis.

Third, the spectral representation is convenient for various investigations of the system features.

Some disadvantage of proposed approach is an evident structure imperfection that can be overcome by reducing an order of the controller.

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