Containment control of second-order multi-agent systems in a sampled-data setting

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Abstract: In this paper, we study the containment control issue of second-order multi-agent systems in a sampled-data setting under the assumption that only the sampled position information can be measured. The communication topology graph between the leaders and followers is assumed to be directed, while the topology among the followers is assumed to be undirected. Necessary and sufficient containment control criteria for multi-agent systems with periodic sampling are first established, where we present the specific range size of parameters and the relationship among the eigenvalues of Laplacian matrix, sampling interval, and scaling parameter. Then we extend our results to multi-agent systems with non-periodic sampling interval. Finally, experimental results are provided to illustrate the effectiveness of the theoretical results.

Key–Words: Multi-agent systems, position information, containment control, time-varying sampling interval.

1 Introduction

The past decades have witnessed rapidly growing papers concerning distributed control and cooperative control of multi-agent systems [1-5]. This is due to its broad applications in many areas such as computer science, sensor networks, robotic teams, and so on. Some basic topics such as consensus, distributed formation control, distributed estimation and control and so on have attracted extensive attention.

Recently, containment control issue becomes a hot topic in distributed cooperative control of multi-agent systems, where a group of followers is driven by a group of leaders to be in the convex hull spanned by the leaders. The study of containment control is motivated by numerous natural phenomena and potential applications. For example, a group of heterogenous agents moves from one target to another when only a portion of the agents is equipped with necessary sensors to detect the hazardous obstacles such that the agents who are not equipped will be driven into a safety area spanned by the equipped agents[6-14].

At present, there have been many results concerning the containment control of second-order multi-agent systems. Cao et al. (2011) studied the containment control issue of second-order multi-agent systems in the presence of both dynamic and stationary leaders. Based on a linear control protocol, Liu et al. (2012) studied the containment control issue of continuous-time second-order multi-agent systems and extended the results to discrete-time multi-agent systems in a sampling setting. However, both papers are based on a strict assumption: **both the position and velocity information can be measured in the control protocol.** In practice, it is more difficult to obtain velocity and acceleration measurements than position measurements. Hence, Li et al. (2012) studied the distributed containment control issue of multiple dynamic leaders for double-integrator dynamics using only position measurements, whereas Zhao et al. (2013) further studied the finite-time containment control issue. However, Li et al. (2012) and Zhao et al. (2013) didn't consider the corresponding containment control issue in a sampling setting. In practical engineering, information may not be transmitted continuously due to the unreliability of communication channels, the limitations of sensing ability of agents, and the constraints of total cost. Thus, it is more practical to consider the case of intermittent information transmission. Sampled control has been widely developed and applied in many areas, such as radar tracking systems and temperature control. This motivates us to write this paper.

Motivated by the above analysis, this paper will mainly study the containment control issue of second-order multi-agent systems with only position information available in a periodic sampling setting and non-periodic sampling setting, respectively. Some necessary and sufficient conditions will be established to guarantee the achievement of containment control for both sampling settings. Our main results not only give the convex hull spanned by the leaders, but also show that the convergence depends on the topology structure, the eigenvalues of Laplacian matrix, sampling interval, and gain parameters.

The remaining of this paper is organized as follows. In Section 2, we review some concepts in graph theory, give some useful lemmas and formulate our problems. Main results are stated in Sections 3. In Section 4, some examples are provided to illustrate the effectiveness of the theoretical results. Conclusions are given in Section 5.

Notations: Let I_n and O_n be $n-$order identity matrix and zero matrix, respectively. Given a complex number $\lambda \in \mathcal{C}, Re(\lambda), Im(\lambda)$ and $|\lambda|$ are the real part, the imaginary part, and the modulus of λ, respectively. $1_n \in R^n$ is the column vector with all entries being 1. $diag\{a_1, \cdots, a_n\}$, represents the diagonal matrix $\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$. For any square matrix H, $\Lambda(H)$ denotes the set of all eigenvalues of H. $det(\cdot)$ represents the determinant of a matrix. $\otimes$ denotes the Kronecker product.

2 Preliminaries

In this section, we first review some basic knowledge on graph theory, definitions, lemmas as the preliminaries of this paper.

2.1 Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote a graph, where $\mathcal{V} = (v_1, v_2, \cdots, v_n)$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set and $\mathcal{A} = [a_{ij}]$ is a adjacency matrix with nonnegative elements. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (j, i)$. The adjacency elements associated with the edges are positive, i.e., $e_{ij} \in \mathcal{E} \Longleftrightarrow a_{ij} > 0$. Here we assume $a_{ii} = 0$ for all $i \in \mathcal{V}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. An agent is called a leader if the agent has no neighbor. Otherwise, an agent is called a follower if the agent has at least one neighbor. In this paper, we suppose that there are $n - m$ followers, labelled as agents 1 to $n - m$, and m leaders labelled as agents $n - m + 1$ to n. Denote the set of leaders as $\mathcal{R}$ and the set of followers as $\mathcal{F}$. A path in $\mathcal{G}$ is a sequence $i_0, i_1, \cdots, i_m$ of distinct nodes such that $(i_{j-1}, i_j) \in \mathcal{E}$ for $j = 1, \cdots, m$. A digraph $\mathcal{G}$ contains a spanning tree if there exists at least one node having a directed path to all other nodes.

The Laplacian matrix of the graph is defined as $L = [l_{ij}] \in R^{n \times n}$, where

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum\limits_{k=1, k\neq i}^{n} a_{ik}, & i = j \end{cases}$$

It is easy to verify that L has at least one zero eigenvalue with a corresponding eigenvector 1_n, where 1_n is an all-one column vector with a compatible size. In this paper, L can be rewritten as: $L = \begin{bmatrix} L_{\mathcal{FF}} & L_{\mathcal{FR}} \\ 0_{m\times(n-m)} & 0_{m\times m} \end{bmatrix}$.

2.2 Definitions and lemmas

Definition 1. *A set $\mathcal{C} \subseteq R^p$ is said to be convex if for any x, y in $\mathcal{C}$, the point $(1-t)x + ty \in \mathcal{C}$ for any $t \in [0, 1]$. The convex hull of a finite set of points $x_1, \cdots, x_m \in R^p$ is the minimal convex set containing all points x_i, $i = 1, \cdots, n$, denoted by $co\{x_1, \cdots, x_n\} = \{\sum_{i=1}^n a_i x_i | a_i \in R, a_i \geq 0, \sum_{i=1}^n a_i = 1\}$.*

Definition 2. *(Alefeld G and Schneider N, 1982) Let $\alpha \in R$ and $C \in R^{n\times n}$. A matrix $B = [b_{ij}] \in R^{n\times n}$ is called an M-matrix if it can be written as*

$$B = \alpha I_n - C,$$

where $\alpha > 0, C \geq 0$, and $\rho(C) \leq \alpha$. The matrix B is called a nonsingular M-matrix if $\rho(C) < \alpha$.

Lemma 1. *(Mei et al., 2011) $L_{\mathcal{FF}}$ is a nonsingular M-matrix if and only if for each of the $n - m$ followers, there exists at least one leader that has a directed path to the follower. In addition, if $L_{\mathcal{FF}}$ is a nonsingular matrix, then each entry of $-L_{\mathcal{FF}}^{-1} L_{\mathcal{FR}}$ is nonnegative and all row sums of $-L_{\mathcal{FF}}^{-1} L_{\mathcal{FR}}$ equal to one.*

Lemma 2. *(Horn RA and Johnson CR, 1985) For third-order real coefficient polynomial $f(s) = b_0 s^3 + b_1 s^2 + b_2 s + b_3$, where $b_0 > 0$, $f(s)$ is Hurwitz stable if and only if $b_1, b_2, b_3 > 0$ and $b_1 b_2 - b_0 b_3 > 0$.*

2.3 System modeling and problem formulations

Consider a second-order multi-agent system as follows:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = u_i, \quad i \in \{1, \cdots, n\}, \end{cases} \tag{1}$$

where $x_i \in R, v_i \in R, u_i \in R$ denote the position, velocity and control input of agent i, respectively.

Suppose that the communication topology graph satisfies the following assumption.

Assumption 1. *The communication topology between the leader and follower is assumed to be directed, while the communication topology among followers is assumed to be undirected. For every follower, there exists at least one leader having a directed path to it.*

In some practical situations, the relative velocity information is impossible or quite difficult to measure because of technological limitations and the environmental disturbances. Therefore, this paper needs to satisfy the following assumption.

Assumption 2. *Only the position information can be obtained at the sampling instants, while the corresponding velocity information is unavailable.*

2.4 System model with periodic sampling

Inspired by the control protocol Gao et al. (2009) and Ma et al. (2013), for $t \in [t_k, t_{k+1})$, we consider the following containment control protocol:

$$u_i(t) = \begin{cases} -\alpha \sum\limits_{j=1}^{n} l_{ij} x_j(t_{k-1}) - \beta \times \\ \sum\limits_{j=1}^{n} l_{ij} \dfrac{x_j(t_{k-1}) - x_j(t_{k-2})}{h}, \\ t \in [kh, kh+\tau), i \in \mathcal{F} \\ -\alpha \sum\limits_{j=1}^{n} l_{ij} x_j(t_k) - \beta \times \\ \sum\limits_{j=1}^{n} l_{ij} \dfrac{x_j(t_k) - x_j(t_{k-1})}{h}, \\ t \in [kh+\tau, kh+h), i \in \mathcal{F}, \\ 0, \ i \in \mathcal{R}, \end{cases} \tag{2}$$

where α, $\beta > 0$ are gain parameters, $h = t_{k+1} - t_k$ is the sampling interval, t_k is the sampling instant satisfying $0 = t_0 < t_1 < \cdots < t_k < \cdots$.

Then, using the control protocol (2), system (1) becomes

$$\begin{cases} x_i(k+1) = x_i(k) + h v_i(k) - \dfrac{(h-\tau)^2}{2} \sum\limits_{j=1}^{n} l_{ij} \\ \quad (\alpha x_j(t_k) + \beta \dfrac{x_j(t_k) - x_j(t_{k-1})}{h}) \\ \quad - \dfrac{(2h-\tau)\tau}{2} \sum\limits_{j=1}^{n} l_{ij} (\alpha x_j(t_{k-1}) \\ \quad + \beta \dfrac{x_j(t_{k-1}) - x_j(t_{k-2})}{h}), \\ v_i(k+1) = v_i(k) - (h-\tau) \sum\limits_{j=1}^{n} l_{ij} (\alpha x_j(t_k) \\ \quad + \beta \dfrac{x_j(t_k) - x_j(t_{k-1})}{h}) - \tau \sum\limits_{j=1}^{n} l_{ij} \\ \quad (\alpha x_j(t_{k-1}) + \beta \dfrac{x_j(t_{k-1}) - x_j(t_{k-2})}{h}), \end{cases} \tag{3}$$

Denote $x(k) =: [x_1^T(k), x_2^T(k), \cdots, x_n^T(k)]^T$, $v(k) =: [v_1^T(k), v_2^T(k), \cdots, v_n^T(k)]^T$, $z(k+1) =: [x^T(k+1), v^T(k+1), x^T(k), v^T(k), x^T(k-1), v^T(k-1)]^T$.

Then, system (1) can be written as the following matrix form

$$z(k+1) = (I_n \otimes H_1 - L \otimes G_1) z(k), \tag{4}$$

where $H_1 = \begin{bmatrix} 1 & h & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$,

$$G_1 = \begin{bmatrix} \dfrac{(\alpha h+\beta)(h-\tau)^2}{2h} & 0 & \dfrac{\tau(2h-\tau)(\alpha h+\beta) - (h-\tau)^2\beta}{2h} & 0 & -\dfrac{\beta\tau(2h-\tau)}{2h} & 0 \\ \dfrac{(h-\tau)(\alpha h+\beta)}{h} & 0 & -\dfrac{\beta h - 2\tau\beta - \tau\alpha h}{h} & 0 & -\dfrac{\beta\tau}{h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, system (4) can be rewritten as

$$\begin{cases} z_{\mathcal{F}}(k+1) = (I_{n-m} \otimes H_1 - L_{\mathcal{FF}} \otimes G_1) z_{\mathcal{F}}(k) \\ -(L_{\mathcal{FR}} \otimes G_1) z_{\mathcal{R}}(k), \\ z_{\mathcal{R}}(k+1) = (I_m \otimes H_1) z_{\mathcal{R}}(k). \end{cases} \tag{5}$$

2.5 System model with non-periodic sampling

For control protocol (2), if $t_{k+1} - t_k$ is not fixed as above, then protocol (2) becomes a non-periodic sampling-data protocol. For this case, we need the following assumption:

Assumption 3. *The time-varying sampling interval is selected in turn from the set of $\{h, 2h\}$, where $h > 0$.*

Remark 1. *In this paper, we only consider the case that all the intervals are selected in turn from the set $\{h, 2h\}$. For the set of $\{l_1 h, l_2 h\}$ or $\{l_1 h, l_2 h, \cdots, l_m h\}$, similar analysis can be obtained.*

When h' is selected in turn from the set of $\{h, 2h\}$, similar to the periodic sampling case, from (2), we get

$$z(k+1) = \Phi(t) z(k), \tag{6}$$

where $z(k)$ is defined as above, $\Phi(t) \in \{\Phi_0, \Phi_1\}$,

$$\Phi_0 = I_n \otimes H_1 - L \otimes G_1, H_1 = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} \frac{h^2\alpha + h\beta}{2} & 0 & \frac{-h\beta}{2} & 0 \\ h\alpha + \beta & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \Phi_1 = I_n \otimes H_2 - L \otimes G_2, H_2 = \begin{bmatrix} 1 & 2h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 2h^2\alpha + h\beta & 0 & h\beta & 0 \\ 2h\alpha + \beta & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to obtain from (6) that

$$\begin{aligned} & z(1) = \Phi_0 z(0) \\ & z(2) = \Phi_1 z(1) = (\Phi_1 \Phi_0) z(0) \\ & z(4) = (\Phi_1 \Phi_0) z(2) \\ & \vdots \\ & z(2r) = (\Phi_1 \Phi_0) z(2(r-1)). \end{aligned}$$

Define $y(r) = z(2r)$, then

$$y(r+1) = (\Phi_1 \Phi_0) y(r), \tag{7}$$

where

$$\begin{aligned} \Phi_1 \Phi_0 &= (I_n \otimes H_2 - L \otimes G_2)(I_n \otimes H_1 - L \otimes G_1) \\ &= I_n \otimes H_2 H_1 - L \otimes (H_2 G_1 + G_2 H_1) \\ &\quad + L^2 \otimes G_2 G_1. \end{aligned}$$

Remark 2. *It thus follows from (7) that system (1) achieves containment control if and only if system (7) achieves containment control.*

3 Containment control analysis in a sampled-data setting

In this section, we will establish the containment control problem of system (1) with only sampled position information available in a periodic and non-periodic sampling-data setting, respectively.

3.1 Containment control in a periodic sampled-data setting

First, for the periodic sampling control protocol (2), we give the following theorem.

Theorem 1. *For system (1) under control protocol (2), the states of the followers will asymptotically converge to the convex hull formed by the states of the dynamic leaders if and only if Assumption 1 always holds and h, α, β satisfy $\max_{\lambda \in \Lambda(L_{\mathcal{FF}})} \lambda < \frac{4\beta - 2h\alpha}{h\beta(h\alpha + 2\beta)}$. Furthermore, the final positions of the followers are given by $-L_{\mathcal{FF}}^{-1} L_{\mathcal{FR}} x_{\mathcal{R}}(t)$, where $x_{\mathcal{R}}(t)$ is the position of the leaders.*

Proof. (Sufficiency) From Lemma 1, note that $L_{\mathcal{FF}}$ is nonsingular, $L_{\mathcal{FF}}^{-1}$ exists and every entry of $L_{\mathcal{FF}}^{-1}$ is nonsingular. Define

$$\overline{w}(k) = z_{\mathcal{F}}(k) + (L_{\mathcal{FF}}^{-1} L_{\mathcal{FR}} \otimes I_4) z_{\mathcal{R}}(k). \tag{8}$$

It is obvious to get that containment control for system (1) can be guaranteed by $\overline{w}(k) \to 0$, as $k \to +\infty$. Furthermore, from (6) and (7),

$$\begin{aligned} \overline{w}(k+1) &= z_{\mathcal{F}}(k+1) + (L_{\mathcal{FF}}^{-1} L_{\mathcal{FR}} \otimes I_4) z_{\mathcal{R}}(k+1) \\ &= (I_{n-m} \otimes H_1 - L_{\mathcal{FF}} \otimes G_1) \overline{w}(k). \end{aligned} \tag{9}$$

Hence, the achievement of containment control for system (1) can be further guaranteed by the Schur stability of system (9), i.e., all the eigenvalues of the matrix $I_{n-m} \otimes H_1 - L_{\mathcal{FF}} \otimes G_1$ should lie in the unit circle.

Let $\lambda_1, \cdots, \lambda_{n-m}$ denote the eigenvalues of $L_{\mathcal{FF}}$. According to matrix theory in Huang (1984), there exists an invertible matrix W such that

$W^{-1}L_{\mathcal{FF}}W = diag\{\lambda_1, \cdots, \lambda_{n-m}\}$. Then, we have

$$\begin{aligned} & (W^{-1} \otimes I_4)(I_{n-m} \otimes H_1 - L_{\mathcal{FF}} \otimes G_1)(W \otimes I_4) \\ = \; & I_{n-m} \otimes H_1 - diag\{\lambda_1, \cdots, \lambda_{n-m}\} \otimes G_1 \\ = \; & diag\{\begin{bmatrix} 1 - \frac{h\alpha^2 + h\beta}{2}\lambda_1 & h & \frac{h\beta\lambda_1}{2} & 0 \\ -(h\alpha+\beta)\lambda_1 & 1 & \beta\lambda_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ & \cdots, \begin{bmatrix} 1 - \frac{h\alpha^2 + h\beta}{2}\lambda_{n-m} & h & \frac{h\beta\lambda_{n-m}}{2} & 0 \\ -(h\alpha+\beta)\lambda_{n-m} & 1 & \beta\lambda_{n-m} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\} \\ & i = 1, \cdots, n-m. \end{aligned}$$

Obviously, the eigenvalues of $(I_{n-m} \otimes H - L_{\mathcal{FF}} \otimes G)$ can be obtained by solving $a_i(s) = 0, i = 1, \cdots, n-m$, where

$$\begin{aligned} & a_i(s) \\ = \; & det(sI_4 - (H_1 - \lambda_i \otimes G_1)) \\ = \; & det\begin{bmatrix} s - 1 + \frac{h\alpha^2 + h\beta}{2}\lambda_i & -h & -\frac{h\beta\lambda_i}{2} & 0 \\ (h\alpha+\beta)\lambda_i & s-1 & -\beta\lambda_i & 0 \\ -1 & 0 & s & 0 \\ 0 & -1 & 0 & s \end{bmatrix} \\ = \; & s[s^3 - s^2(2 - \frac{h\alpha^2 + h\beta}{2}\lambda_i) \\ & + s(1 + h^2\alpha\lambda_i - \frac{h\alpha^2\lambda_i}{2}) - \frac{h\beta\lambda_i}{2}] \\ & i = 1, \cdots, n-m. \end{aligned}$$

Noticing that they have the same form, we can analyze them uniformly. Let $\lambda \in \Lambda^+(L_{\mathcal{FF}})$ represent the eigenvalues of $L_{\mathcal{FF}}$. Then, we only need to determine the Schur stability of the following polynomial:

$$\begin{aligned} a(s) \;=\; & s[s^3 - s^2(2 - \frac{h\alpha^2 + h\beta}{2}\lambda) && (10) \\ & + s(1 + h^2\alpha\lambda - \frac{h\alpha^2\lambda}{2}) - \frac{h\beta\lambda}{2}]. && (11) \end{aligned}$$

From the above polynomial, it is easy to obtain that $s_1 = 0$. Let $f(s) = s^3 - s^2(2 - \frac{h\alpha^2 + h\beta}{2}\lambda) + s(1 + h^2\alpha\lambda - \frac{h\alpha^2\lambda}{2}) - \frac{h\beta\lambda}{2}$. Therefore, (10) is Schur stable if and only if $f(s)$ is simultaneously Schur stable. By applying the bilinear transformation $s = \frac{\sigma+1}{\sigma-1}$, $f(s)$ can be rewritten as a new polynomial

$$r(\sigma) = b_0\sigma^3 + b_1\sigma^2 + b_2\sigma + b_3, \qquad (12)$$

where $b_0 = h^2\alpha\lambda, b_1 = 2h\beta\lambda, b_2 = 4 - h^2\alpha\lambda - 2h\beta\lambda, b_3 = 4$.

Then, the Schur stability of the polynomial $f(s)$ is equivalent to the Hurwitz stability of the polynomial (12). Since $b_0 > 0$ always holds, by Lemma 2, the polynomial (12) is Hurwitz stable if and only if the following conditions hold:

(a) $b_1, b_2, b_3 > 0$,

(b) $b_1b_2 - b_0b_3 > 0$.

For (a), it follows that

$$\begin{cases} b_1 = 2h\beta\lambda > 0, \\ b_2 = 4 - h^2\alpha\lambda - 2h\beta\lambda > 0, \\ b_3 = 4 > 0. \end{cases} \qquad (13)$$

For (b), $b_1b_2 - b_0b_3 > 0$, we can get

$$h\beta\lambda(4 - h^2\alpha\lambda - 2h\beta\lambda) - 2h^2\alpha\lambda > 0. \qquad (14)$$

By solving inequality (14), we can obtain

$$\lambda < \frac{4\beta - 2h\alpha}{h\beta(h\alpha + 2\beta)}. \qquad (15)$$

Through simple calculations, the inequality (15) can guarantee that (13) is valid. According to the above analysis, we can get that $\max_{\lambda \in \Lambda(L_{\mathcal{FF}})} \lambda < \frac{4\beta - 2h\alpha}{h\beta(h\alpha + 2\beta)}$ holds.

(Necessity) Suppose that the Assumption 1 is not satisfied in the directed graph $\mathcal{G}$. Therefore, there exists at least one follower (labelled as k), satisfying that all leaders have no directed path to it. It follows that the states of follower k is independent of the states of the leaders. Obviously, follower k cannot asymptotically converge to the convex hull formed by the states of the leaders. □

3.2 Containment control in a non-periodic sampled-data setting

In the sequel, we aim to establish the containment control criteria for the non-periodic sampled-data setting.

System (7) can be further rewritten as

$$\begin{cases} y_{\mathcal{F}}(k+1) = (I_{n-m} \otimes H_2H_1 - L_{\mathcal{FF}} \otimes (H_2G_1 + G_2H_1) + L^2_{\mathcal{FF}} \otimes G_2G_1)y_{\mathcal{F}}(k) \\ \qquad + (L_{\mathcal{FF}}L_{\mathcal{FR}} \otimes G_2G_1 - L_{\mathcal{FR}} \otimes (H_2G_1 + G_2H_1))y_{\mathcal{R}}(k), \\ y_{\mathcal{R}}(k+1) = (I_m \otimes H_2H_1)y_{\mathcal{R}}(k). \end{cases} \qquad (16)$$

Define

$$\overline{m}(k) = y_{\mathcal{F}}(k) + (L^{-1}_{\mathcal{FF}}L_{\mathcal{FR}} \otimes I_2)y_{\mathcal{R}}(k). \qquad (17)$$

Furthermore, we can get that

$$\overline{m}(k+1) = y_{\mathcal{F}}(k+1) + (L_{\mathcal{F}\mathcal{F}}^{-1}L_{\mathcal{F}\mathcal{R}} \otimes I_2)y_{\mathcal{R}}(k+1). \tag{18}$$

By (16)-(18), we get

$$\begin{aligned}\overline{m}(k+1) &= (I_{n-m} \otimes H_2H_1 - L_{\mathcal{F}\mathcal{F}} \otimes (H_2G_1 + G_2H_1) \\ &\quad + L_{\mathcal{F}\mathcal{F}}^2 \otimes G_2G_1)\overline{m}(k). \end{aligned} \tag{19}$$

Similar to the proof of Theorem 1, from (18), it is easy to get that the Schur stability of (16) can guarantee the achievement of containment control of system (1) under the non-periodic sampling protocol (2). Hence, in the sequel, we only need to guarantee that all the eigenvalues of the matrix $I_{n-m} \otimes H_2H_1 - L_{\mathcal{F}\mathcal{F}} \otimes (H_2G_1 + G_2H_1) + L_{\mathcal{F}\mathcal{F}}^2 \otimes G_2G_1$ lie in the unit circle.

By matrix theory, there exits an invertible matrix W such that $W^{-1}L_{\mathcal{F}\mathcal{F}}W = diag\{\lambda_1, \cdots, \lambda_{n-m}\}$. Then, $W^{-1}L_{\mathcal{F}\mathcal{F}}^2W = diag\{\lambda_1^2, \cdots, \lambda_{n-m}^2\}$,

$$\begin{aligned} &(W^{-1} \otimes I_4)(I_{n-m} \otimes H_2H_1 - L_{\mathcal{F}\mathcal{F}} \otimes \\ &(H_2G_1 + G_2H_1) + L_{\mathcal{F}\mathcal{F}}^2 \otimes G_2G_1)(W \otimes I_4) \\ = &\ I_{n-m} \otimes H_2H_1 - diag\{\lambda_1, \cdots, \lambda_{n-m}\} \otimes \\ &(H_2G_1 + G_2H_1) + diag\{\lambda_1^2, \cdots, \lambda_{n-m}^2\} \otimes G_2G_1 \\ = &\ diag\begin{bmatrix} P & 3h - h^2\lambda_i(2h\alpha+\beta) & \frac{5}{2}h\beta\lambda_i - \frac{1}{2}h^2\beta\lambda_i^2(h\alpha+\beta) & 0 \\ Q & 1 - h\lambda_i(2h\alpha+\beta) & \beta\lambda_i - 2h\lambda_i^2(h\alpha\beta+\beta^2) & 0 \\ 1 - \frac{1}{2}h\lambda_i(h\alpha+\beta) & h & \frac{1}{2}h\beta\lambda_i & 0 \\ -h\alpha\lambda_i - \beta\lambda_i & 1 & \beta\lambda_i & 0 \end{bmatrix}. \\ &i = 1, \cdots, n-m, \end{aligned}$$

where

$P = 1 - \frac{1}{2}h\lambda_i(9h\alpha + 5\beta) + \frac{1}{2}h^2\lambda_i^2(2h^2\alpha^2 + 3h\alpha\beta + \beta^2)$,

$Q = -3h\alpha\lambda_i - \beta\lambda_i + \frac{1}{2}h\lambda_i^2(2h^2\alpha^2 + 3h\alpha\beta + \beta^2)$.

Then, similar to the proof process of Theorem 1, we can get the following containment control criterion.

Theorem 2. *For system (1) under the non-period sampled interval of control protocol (2), the states of the followers will asymptotically converge to the convex hull formed by the states of the dynamics leaders if and only if the Assumption 1 holds and h, α, β satisfy the following conditions:*

$$\begin{cases} 6h\beta\lambda_i + 4h^2\alpha\lambda_i - 3h^3\alpha\beta\lambda_i^2 - 2h^4\alpha^2\lambda_i^2 > 0, \\ 4 - 9h^2\alpha\lambda_i - 2h^2\beta^2\lambda_i^2 > 0, \\ 4 - 6h\beta\lambda_i - 4h^2\alpha\lambda_i + 2h^2\beta^2\lambda_i^2 \\ +3h^3\alpha\beta\lambda_i^2 + 2h^4\alpha^2\lambda_i^2 > 0, \\ 24h\beta\lambda_i - 20h^2\alpha\lambda_i - 8h^3\alpha\beta\lambda_i^2 \\ -12h^3\beta^3\lambda_i^3 - 8h^4\alpha^2\lambda_i^2 - 26h^4\alpha\beta^2\lambda_i^3 \\ +6h^5\alpha\beta^3\lambda_i^4 + 4h^6\alpha^2\beta^2\lambda_i^4 > 0, \\ h > 0, \alpha > 0, \beta > 0. \end{cases} \tag{20}$$

Furthermore, the final positions of the followers are given by $-L_{\mathcal{F}\mathcal{F}}^{-1}L_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t)$, where $x_{\mathcal{R}}(t)$ is the position of the leaders, λ_i is the nonzero eigenvalues of the Laplacian matrix $L_{\mathcal{F}\mathcal{F}}$, $i = 1, 2, \cdots, n-m$, respectively.

Remark 3. *In Theorems 1-2, we not only establish necessary and sufficient containment control conditions, but also give the final states of the followers, namely, $x_{\mathcal{F}}(t) \to -L_{\mathcal{F}\mathcal{F}}^{-1}L_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t)$. Furthermore, from the proof of these two theorem, we can see that the velocity of followers will also asymptotically converge to the convex hull formed by the velocity of leaders.*

Remark 4. *Theorems 1-2 not only prove that the topology structure, the eigenvalues of Laplacian matrix, sampling interval, and gain parameters play an important role in the achievement of containment control, but also give the specific range size of these parameters.*

4 Numerical simulations

This section provides two examples to illustrate our main theoretical results. For system (1) with four followers ($i = 1, 2, 3, 4$) and two leaders ($i = 5, 6$), the corresponding communication topology is shown in Fig. 1 below.

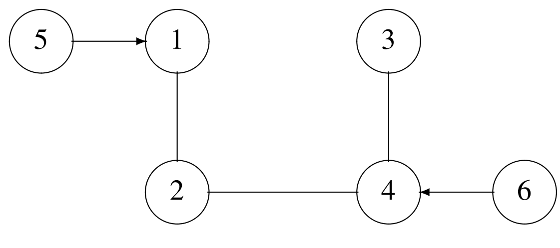

Fig.1 The topology graph of agents.

The Laplacian matrix L =

$$\begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_{\mathcal{F}\mathcal{F}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}.$$

It is easy to compute that the four eigenvalues of $L_{\mathcal{F}\mathcal{F}}$ are $\lambda_1 = 0.382, \lambda_2 = 1, \lambda_3 = 2.618, \lambda_4 = 4$, respectively. Obviously, $max\lambda_i = 4$.

Example 4.1: (Periodic-sampling case) Let the initial states of the position be $[1, -2, -5, 11, 0, 0]^T$ and velocity be $[8, -4, -1, 2, 5\sin(1), 10\sin(1)]^T$, respectively. $h = 0.3, \alpha = 1, \beta = 1$ satisfy (15), since, $\frac{4\beta - 2h\alpha}{h\beta(h\alpha + 2\beta)} = 4.9 > 4$. Thus, by Theorem 1, system (1) can achieve containment control. The position and velocity trajectories shown in Fig. 2, also verify system (1) can achieve containment control. However, $h = 0.3, \alpha = 3, \beta = 1$ cannot satisfy (15), since, $\frac{4\beta - 2h\alpha}{h\beta(h\alpha + 2\beta)} = 2.5 < 4$. Fig. 3 shows that system (1) under (2) cannot achieve containment control with the same initial values.

Example 4.2: (Non-periodic-sampling case) For system (1) with communication topology graph as shown in the Fig. 1. Let the initial states of the position be $[1, -10, -5, 20, 5, 5]^T$ and velocity be $[8, -15, -6, 7, 5\sin(1), 10\sin(1)]^T$, respectively. By calculating, $h = 0.1, \alpha = 1, \beta = 1$ satisfy (20). Fig. 4 shows that all followers will asymptotically converge to the convex hull formed by the states of the dynamics leaders. However, Fig. 5 shows that system (1) under control protocol (2) cannot achieve containment control by choosing $h = 0.5, \alpha = 1, \beta = 1$ with the same initial values.

5 Conclusions

This paper investigated the containment control problem of second-order multi-agent control system by assuming that only the sampled position information can be obtained. The topology between the leaders and the followers is assumed to be directed graphs, while the topology among the followers is assumed to be undirected. Necessary and sufficient containment control conditions are established, which depends on the eigenvalues of Laplacian matrix, sampling interval, and scaling parameter. Then, we attempt to resolve the containment control problem of multi-agent

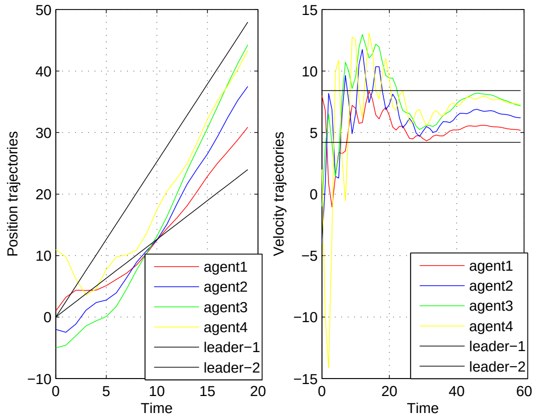

Fig. 2 $h = 0.3, \alpha = 1, \beta = 1$, position and velocity trajectories

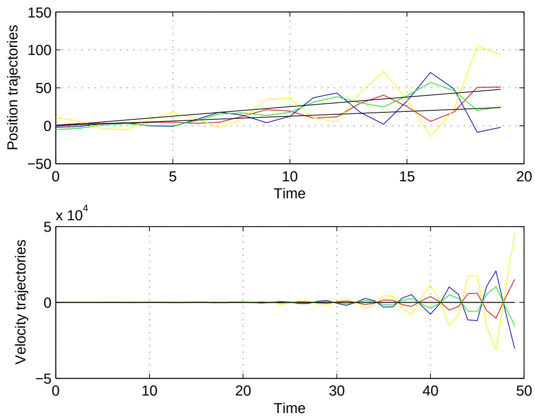

Fig. 3 $h = 0.3, \alpha = 3, \beta = 1$, position and velocity trajectories

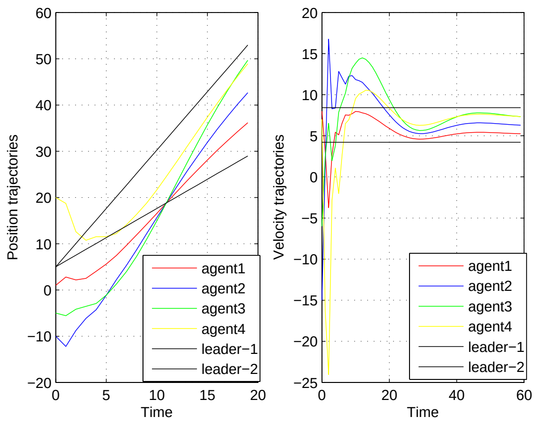

Fig. 4 $h = 0.1, \alpha = 1, \beta = 3$, position and velocity trajectories

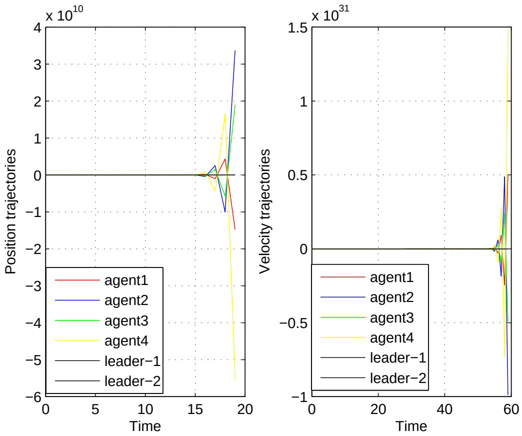

Fig. 5 $h = 0.5, \alpha = 1, \beta = 1$, position and velocity trajectories

system in a non-periodic sampling setting. Finally, experimental results are provided to illustrate the effectiveness of the theoretical results. Future work will focus on more complex containment control problem, for example, the containment control with switching topologies.

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References:

[1] Ren W and Cao YC (2009) Distributed coordnation of multi-agent networks: emergent problems, models and issues. *Springer*.

[2] Cao Y, Li Y, Ren W and Chen Y (2010) Distributed coordnation of networked fractional-order systems. *IEEE Transactions on System, Man, and Cybernetics - Part B: Cybernetics*, Vol.40, No.2, pp. 362-370.

[3] Cao Y, Ren W and Li Y (2009) Distributed discrete-time coordnated tracking with a time-varying reference state and limited communication. *Automatica*, Vol.45, No.5, pp. 1299-1305.

[4] Wang S and Xie D (2012) Consensus of second-order multi-agent systems via sampled control: undirected fixed topology case. *IET Control Theory and Applications*, Vol.6, No.7, pp. 893-909.

[5] Cheng YL and Xie D (2013) Co-ordinated Average Consensus of Multi-agent Systems via Sampled Control. *WSEAS Transactions on Systems and Control*, Vol.8, No.1, pp. 21-32.

[6] Cao YC, Stuart D, Ren W and Meng ZY (2011) Distributed containment control for multiple autonomous vehicls with double-integrator dynamics: algorithms and experiments. *IEEE Transactions on Control Systems Technology*, Vol.19, No.4, pp. 929-938.

[7] Liu HY, Xie GM and Wang L (2012) Necessary and sufficient conditions for containment control of networked multi-agent systems. *Automatica*, Vol.48, No.7, pp. 1415-1422.

[8] Li JZ, Ren W and Xu SY (2012) Distributed containment control with multiple dynamic leaders for double-integrator dynamics using only position measurements. *IEEE Transactions on Automatic Control*, Vol.57, No.6, pp. 1553-1559.

[9] Zhao Y and Duan ZS (2013) Distributed finite-time containment control for multi-agent systems with multiple dynamic leaders. *25th Chinese Control and Decision Conference,* pp. 298-303.

[10] Cao YC and Ren W (2009) Containment control with multiple stationary or dynamic leaders under a directed interaction graph. *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference,* pp. 3014-3019.

[11] Li ZK, Ren W, Liu XD and Fu MY (2013) Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders. *International Journal of Robust and Nonlinear control*, Vol.23, No.5, pp. 534-547.

[12] Wen GH, Hu GQ, Zuo ZQ and Zhao Y (2012). Distributed containment control of linear multi-agent systems using output information. *12th International Conference on Control, Automation, Robotics, Vision,* pp. 1483-1488.

[13] Cao YC, Ren W and Egerstedt M (2012) Distributed containment control with multiple stationary or dynamic leaders in fixed and swithing directed networks. *Automatica*, Vol.48, No.7, pp. 1586-1597.

[14] Zheng YS and Wang L (2013). Containment control of heterogeneous multi-agent systems. *International Journal of Control*, DOI: 10.10080/00207179.2013.814074.

[15] Alefeld G and Schneider N (1982) On square roots of M-matrices. *Linear Algebra Appl,* Vol.42, pp. 119-132.

[16] Mei J, Ren Wa and Ma GF (2011) Distributed containment control for multiple nonlinear systems with identical dynamics. *Proceedings of the 30th Chinese Control Conference,* pp. 6544-6549.

[17] Horn RA and Johnson CR (1985) Matrix analysis. *Cambridge University Press: Cambridge*, Vol.56, No.5, pp. 1226-1231.

[18] Gao YP, Wang L, Xie GM and Wu B (2009) Consensus of multi-agent systems based on sampled-data control. *International Journal of Control*, Vol.82, No.12, pp. 2193-2205.

[19] Ma Q, Xu SY and Lewis FL (2013) Second-order consensus for directed multi-agent systems with sampled data. *International Jouranal of Robust and Nonlinear Control*, DOI: 10.1002/rnc.3010.

[20] Huang L (1984) Linear algebra in systems and control theory. *Beijing: Science Press.*