Fractional Order Controller Design For Inverted Pendulum On A Cart System (POAC)

Abhaya Pal Singh
Research Scholar
Symbiosis International University
Symbiosis Institute of Technology, Pune, India
abhaya.aps@gmail.com

Himanshu Agarwal
Symbiosis International University
Symbiosis Institute of Technology, Pune
India
himanshu.agrawal@sitpune.edu.in

Pallavi Srivastava
Department of Electrical Engineering
SKN Sinhgad Institute of Technology & Science
Lonavala, Pune, India
pallavi.aps@gmail.com

Abstract: In this paper we propose a Fractional Order Controller for an under-actuated system (POAC). This system has two degrees-of-freedom and one control input. The model of the system is derived using Euler Lagrange method and the stabilization is achieved with constraints using Fractional Order Controller. To validate the control law, simulation are performed and compared with Integer Order Controller.

Key–Words: Coefficient of friction, Fractional order controller (FOC), FOPID, IOC, PID, POAC, Underactuated robotic system.

1 Introduction

Underactuated systems have fewer actuators than degrees of freedom of the system. Underactuated systems appear in ample of applications including Robotics, Aerospace, Marine, Flexible, Mobile, and Locomotive Systems etc. Many interesting methods and results have been presented for underactuated systems, the control of these systems still remains an open problem and can be found in Faruk Kazi (2008), Ioannis Sarras (2010), YangQuan Chen (2003), YangQuan Chen (2004), YangQuan Chen (2004), YangQuan Chen (2004), Jinsong Liang (2004), Jinsong Liang (2005), Said Djennoue (2013) and Igor Podlubny (1999). Important issues are: how can nonlinear control models be formulated for such systems and what are their controllability and stabilizability properties. These issues are thoroughly addressed in Mahmut Reyhanoglu,Arjan van der Schaft,N. Harris McClamroch and Ilya Kolmanovsky (1999). Numerous robotic tasks associated with underactuation have been studied in the literature C. Chevallereau, J.W. Grizzle and C.H. Moog (2004).

The Pendulum On a Cart (POAC) is a very good example for control engineers to study performance of a control law for underactuated mechanical system. The POAC is a highly nonlinear system which means that standard linear techniques cannot use to model the nonlinear dynamics of the system. Its underactuation property makes control more challenging. The POAC is an intriguing system from the control view due to their nonlinearity.

Here, the problem is to balance a pole on a mobile platform that can move in only two directions, to the left or to the right. In this paper we apply Fractional Order PID Controller (FOPID) to balance the POAC system and comparing the results with Integer Order PID Controller. In order to obtain control, the inverted pendulum dynamics should be linearised. The application of these control techniques to two or three stage inverted pendulum may result in a very critical design of controller and difficult stabilization Ahmad (2006/2007).

Fractional order control has been applied with success in rigid robots Subhransu Padhee, Abhinav Gautam, Yaduvir Singh, and Gagandeep Kaur (2011). In Manue F. Silva, J. A. Tenreiro Machado and A. M. Lopes. (2003-04) fractional control is applied to Hexapod Robot. In Manue F. Silva, J. A. Tenreiro Machado. (2006) a fractional order position/force algorithm is proposed for legged robot and control algorithms have superior performance. Fractional order PID controller is also used in Genetic Algorithm to tune the parameters of the Fractional Order PID controller Subhransu Padhee, Abhinav Gautam, Yaduvir Singh, and Gagandeep Kaur (2011). Iterative optimization method has been used in Ines Tejado (2013) for optimum setting for a fractional PID controller. Fractional order PID controller using state space theory is also developed in Pritesh Shah (2013). Observer design discussed in Pallavi Srivastava (2013) can also be extend to fractional observer design.

Temperature control arises in many engineering fields. For temperature control, On/Off control, proportional control, and traditional PID controller is used. Using FOPID controller as in Hyo-Sung Ahn, Varsha Bhambhani and YangQuan Chen (2008) a accurate temperature profile tracking of the spatially distributed heat flow was achieved. Fractional order controller is a good controller Abhaya Pal Singh (2012), Abhaya Pal Singh (2012). There are many systems that can be modeled to its fractional equivalent Abhaya Pal Singh (2013).

FOC is a generalization of the Integer Order Control (IOC) to a real or complex order J. L. Adams, T. T. Hartley and C. F. Lorenzo (2006). The real order generalization is as follows:

$$D^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ \int_0^t d\tau^{-\alpha} & \text{if } \alpha < 0 \end{cases}$$

where $\alpha \in \Re$
There are different definitions of FOC . Some of them for derivative are:-
a) Riemann-Liouville (RL):-

$$D^{\alpha} f(t) = \frac{d^m}{dt^m}[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{(\alpha+1-m)}} d\tau] \tag{1}$$

$m \in Z^+, (m-1) < \alpha \leq m$
b) Grunwald-Letnikov (GL):-

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\frac{t-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(t-mh) \tag{2}$$

c) Caputo:-

$$D^{\alpha} f(t) = [\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{(\alpha+1-m)}} d\tau] \tag{3}$$

The Caputo and Riemann-Liouville formulation have the same view when the initial conditions are zero.

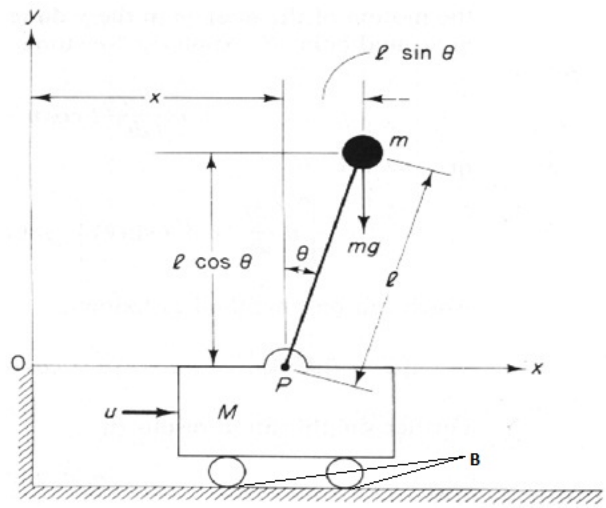

Figure 1: Inverted Pendulum on a Cart (Ogata)

In this paper, the system is modeled and the and governing equation of motion is derived. Using governing equations the controller is designed to achieve better control. Fractional order as well as Integer order controllers are designed to check the visibility of Fractional order control towards the better control.

2 Modeling

2.1 Modeling of the System:-

The inverted pendulum mounted on a cart as shown in Fig.(1), the defining nonlinear equations can be derived as follows. First we assume that the rod is mass less and that the cart mass and the pendulum mass are denoted as M and m, respectively. An external x-directed force on the cart is $u(t)$, B is the coefficient of friction and a gravity force acts on the pendulum mass at all times, where $x(t)$ represents the cart position and $\theta(t)$ is the tilt angle referenced to the vertically upward direction.

Here we use Lagrangian model for analysis. In this we have to find the kinetic energy and potential energy. The Lagrangian will be the difference of the kinetic energy and potential energy.

$$\mathcal{L} = T - V \tag{4}$$

where, T = Kinetic Energy (KE), V = Potential Energy (PE)

$$V_{total} = V_{cart} + V_{pendulum} \tag{5}$$

$$V_{total} = mgl \cos\theta, \tag{6}$$

and,

$$T_{total} = T_{cart} + T_{pendulum} \tag{7}$$

$$T_{total} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta) \tag{8}$$

Here, M = mass of the cart(kg), m= mass of pendulum (kg), g= acceleration due to gravity (m/s^2), h= height of the pendulum ($meter$) =$l\cos\theta$, x = displacement of cart ($meter$) and B= damping coefficient (kg/s).

The Lagrangian will be;

$$\mathcal{L} = T_{total} - V_{total} \tag{9}$$

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta) - mgl\cos\theta \tag{10}$$

$$R = \frac{1}{2}B\dot{x}^2 \tag{11}$$

where R: Rayleigh's dissipation function.

Put equation (10 & 11) into Euler-Lagrange equation with Damping below,

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{x}}) - \frac{\partial\mathcal{L}}{\partial x} + \frac{\partial R}{\partial\dot{x}} = u \tag{12}$$

and

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{13}$$

We get,

$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 + B\dot{x} = u \tag{14}$$

$$l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = 0 \tag{15}$$

After separating the values of $\ddot{x}$ and $\ddot{\theta}$ we get;

$$\ddot{\theta} = \frac{(u\cos\theta + ml\sin\theta\cos\theta\dot{\theta}^2 - (M+m)g\sin\theta) - B\dot{x}\cos\theta}{ml\cos^2\theta - (M+m)l} \tag{16}$$

$$\ddot{x} = \frac{(u + ml\sin\theta\dot{\theta}^2) - mg\sin\theta\cos\theta - B\dot{x}}{M + m - m\cos^2\theta} \tag{17}$$

We cast (16) and (17) into standard non-linear state-space representation as:-

$$\frac{d}{dt}(\underline{z}) = \underline{f}(\underline{z}, u, t) \tag{18}$$

Let, $z_1 = x$
$z_2 = \dot{x} = \dot{z_1}$
$z_3 = \theta$
$z_4 = \dot{\theta} = \dot{z_3}$

Resulting nonlinear state space representation is expressed as follows,

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ \frac{(u+ml\sin\theta\dot{\theta}^2)-mg\sin\theta\cos\theta-B\dot{x}}{M+m-m\cos^2\theta} \\ z_4 \\ \frac{(u\cos\theta+ml\sin\theta\cos\theta\dot{\theta}^2-(M+m)g\sin\theta)-B\dot{x}\cos\theta}{ml\cos^2\theta-(M+m)l} \end{pmatrix} \tag{19}$$

We now linearize the above state-space by taking Jacobian at $(\underline{z_0}, u_0) = (0, 0)$ J. R. White (1997).

The linearized form of the system becomes,

$$\frac{d}{dt}(\delta\underline{z}) = J_{=\underline{z}}(\underline{z_0}, u_0)\delta\underline{z} + J_{=u}(\underline{z_0}, u_0)\delta u \tag{20}$$

After solving the above we get the linearised matrix as,

$$\frac{d}{dt}(\delta\underline{z}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-B}{M} & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{B}{Ml} & \frac{(M+m)g}{Ml} & 0 \end{pmatrix}\delta\underline{z} + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{pmatrix}\delta u \tag{21}$$

And the output matrix as,

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{pmatrix} \tag{22}$$

3 Controller design

Here in this section, two cases of fractional order controller is discussed. The first one is fractional order PID controller (FOPID) design without considering the damping of the system and the

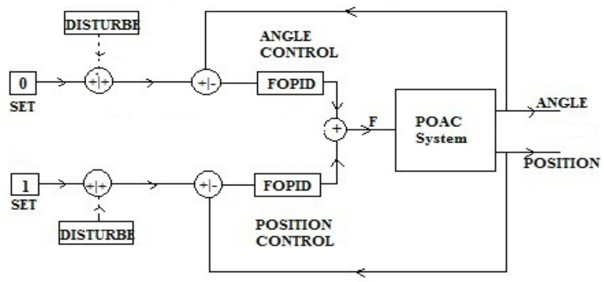

Figure 2: POAC system's block diagram.

second one is fractional order PID+Damping controller (FOPID+Damping) design with considering the damping of the system.

The integer order PID controller (IOPID) has the transfer function as,

$$C(s)_{pid} = k_p + \frac{k_i}{s} + k_d s \tag{23}$$

and the fractional order PID controller (FOPID) has the transfer function as,

$$C(s)_{fopid} = k_p + \frac{k_i}{s^{\alpha}} + k_d s^{\beta} \tag{24}$$

where, $0 < \alpha < 1$, and $0 < \beta < 1$

Including integer order PID controller (IOPID), three cases are presented here for designing of controllers and there comparison is also shown in this section.

- Traditional PID controller (IOPID).
- Fractional PID (FOPID)controller .
- Fractional PID with Damping (FOPID+Damping) controller.

Fig.(2) shows the block diagram of the whole controlled system's working principle. SET block indicates the desired points at which the system get to stable. Disturbance is the external input which can be applied either on cart or on pendulum. The controller will react with the input and the feedback and control the POAC system.

For the purpose of simulation, we consider following system parameters, $m = 0.2\ kg$, $M = 0.5\ kg$, $l = 1$ $meter$, $g = 9.8\ m/s^2$ and $B = 1\ kg/s$.

Clearly in equation (24), selecting $\alpha = \beta = 1$, a classical PID controller can be obtained. The options of $\alpha = 1$, $\beta = 0$, $\alpha = 0$,$\beta = 1$, respectively corresponds to traditional PI and PD controllers. Traditional (classical) PID controller are the special type of the fractional $PI^{\alpha}D^{\beta}$ controller and expected that the controller $PI^{\alpha}D^{\beta}$ may enhance the systems control performance J.Samardzic, M.P.Lazarevic, B. Cvetkovic (2011).

Since this kind of controller has five parameters to tune $(K_p, K_d, K_i, \alpha, \beta)$, up to five design specifications for the controlled system can be met, this is, two more than in the case of a traditional PID controller, where $\alpha = 1$ and $\beta = 1$. It is essential to study which specifications are more interesting as far as performance and robustness are concerned. All these constraints will be taken into account in the tuning technique in order to take advantage of the introduction of the fractional orders C. A. Monje, YangQuan Chen, B. M. Vinagre, Dingy Xue, Vicente Feliu. (2010).

We propose FOPID controller and FOPID+Damping controller for the state-space model of the system under consideration. We first propose optimal conventional PID controller and then exploit the freedom in FOPID to improve the performance. We first consider the state-space model obtained in equation (21) and (22), to select the parameters using integer PID tuning. Here, MATLAB simulation is used for PID, FOPID and FOPID+Damping controller's design. The optimal values of the parameters are as follows:

Table 1: Optimal values of parameters (Pendulum's angle control)

Controller	α	β	k_p	k_d	k_i
FOPID	0.95	0.9	-55	-8	-25
PID	1	1	-55	-8	-10
FOPID+Damping	0.95	0.9	-60	-8	-25

Table 2: Optimal values of parameters (Cart's position control)

Controller	α	β	k_p	k_d	k_i
FOPID	.009	.8	-1.8	-5	-0.8
PID	1	1	-1.7416	-3.192	-0.0128
FOPID+Damping	.009	.8	-1.7416	-6	-0.0128

The transfer function of fractional order PID controller and fractional order PID+Damping controller can be easily obtain from the optimal values summa-

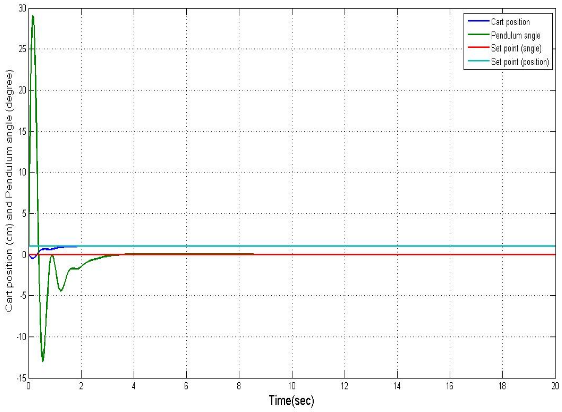

Figure 3: Fractional order controlled response.

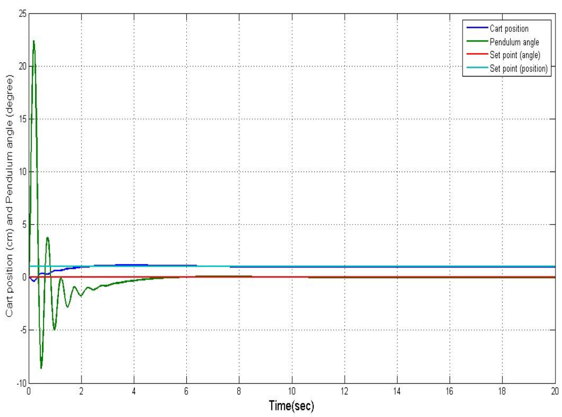

Figure 4: Integer order controlled response.

rized in the table (1) and table (2), and putting these values in equation (24).

The aim of this system is to make the cart position and pendulum angle to be stable at desired (set) points. The set points are pendulums angle at 0^o and position of cart at 1 cm. Fig.(3) shows the Fractional order PID response, Fig.(4) shows the Integer Order PID response and Fig.(5) shows the Fractional order PID+Damping response for the system under consideration with the objective of controlling the cart position and stabilizing pendulum angle.

From Fig.(6) we may conclude that there is some oscillation and settling time is large in the Integer Order PID (IOPID) controller for cart's position and that can be compensated by using the Fractional Order PID (FOPID) controller and will be better to use with damping. It was observed that there is an steady state error present when we design IOPID as compare to

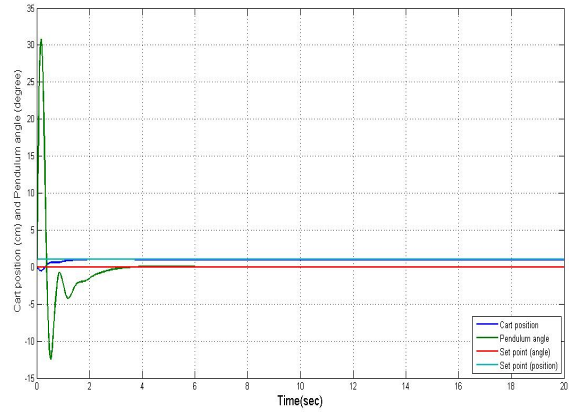

Figure 5: Fractional order controlled response with damping.

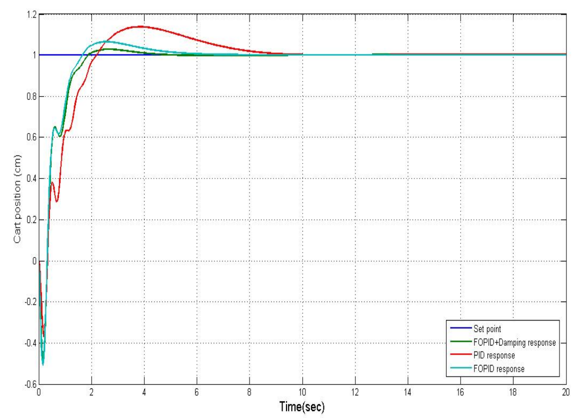

Figure 6: Cart's position control comparison.

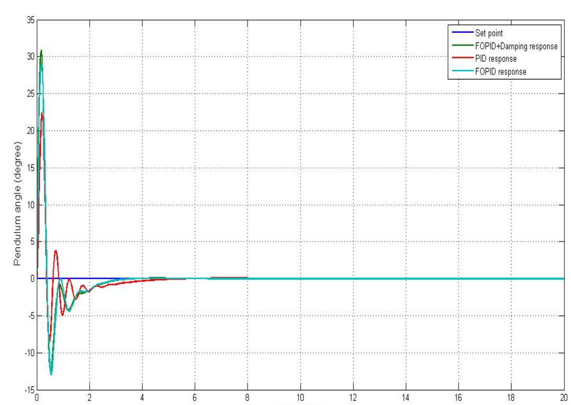

Figure 7: Pendulum's angle control comparison.

FOPID in Abhaya Pal Singh (2012).

From Fig.(7), we observe there is oscillation in Integer Order PID controller for pendulum's angle as compared to Fractional Order PID controller and also with FOPID+Damping.

Simulations show that the fractional order PID controller performs better compared to integer order PID controller and if FOPID is combined with damping then it shows its best result. It is summarized in Table (3)

Table 3: Summery from the three controllers

$\frac{Controllers\longrightarrow}{Specification\downarrow}$	PID	FOPID	FOPID+ Damping
Oscillations (pendulums angle)	High	Compensated	Compensated
Rise time (cart position)	Low	Better than PID	Better than FOPID
Settling time (cart position)	Low	Better than PID	Better than FOPID
Rise time (pendulums angle)	Low	Better than PID	Better than FOPID
Settling time (pendulums angle)	Low	Better than PID	Better than FOPID

4 Conclusion

In this paper, the main goal was to design a fractional order controller and fractional order controller with damping, which stabilizes the underactuated system, considered here a POAC system. This paper's approach successfully stabilized the POAC system using Fractional Order (FO) PID controller. The objective was set to control the angle of the pendulum and the position of the cart.

Also we have compared the results of FOPID controller and Integer Order PID controller. Simulation results reveals that the FOPID controller is better for stabilizing the underactuated robotic system and if FOPID is combined with damping then this controller shows its best result.

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