Dynamics Optimal Portfolios with CIR Interest Rate under a Heston Model

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Abstract: This paper studies the dynamic portfolios with the Cox-Ingersoll-Ross(CIR) interest rate under a Heston model, which aims at maximizing the expected utility of the terminal wealth. In the model, the manager can invest his weatlh to a zero-coupon bond, a riskless asset and a stock. By applying dynamic programming principle, the explicit solutions of optimal portfolio strategy for constant relative risk aversion(CRRA) utility are achieved successfully. Finally, a numerical example is presented to characterize the dynamic behavior of optimal portfolio strategy.

Key–Words: CIR interest rate; Heston model; CRRA utility; the optimal portfolio; dynamic programming;

1 Introduction

In recent years, some people realized that portfolio can disperse risk and increase revenues and has become a hot topic. Nowadays, there are a lot of literatures in studying the problem and the portfolio selection theory has been applied to the investment and consumption problems, DC pension fund and insurance fund management problems, and so on. For example, Markowitz [1] first arouse this problem in 1952 and provided a theoretical foundation for modern portfolio selection analysis. Li and Ng [2] considered the mean-variance formulation in multi-period framework and first presented an embedding technique to obtain the analytical solution of the efficient strategy and efficient frontier. Zhou and Li [3] studied a continuous-time mean-variance portfolio selection model that is formulated as a bicriteria optimization problem and the problem was seen as a class of auxiliary stochastic linear-quadratic (LQ) problems. Vigna and Haberman [4] analysed the financial risk in a DC pension scheme and found an optimal investment strategy. Gao [5] studied the portfolio optimization of DC pension fund under a CEV model and obtain the closed-form solution of the optimal investment strategy in power and exponential utility case. Li et al.[6] studied the optimal investment problem for utility maximization with taxes, dividends and transaction costs under the CEV model and obtained explicit solutions for the logarithmic, exponential and quadratic utility functions.

However, two aspects are worthy to be further explored based on the above-mentioned literatures. On the one hand, the articles above are all derived in the case that the interest rate is constant. In fact, the interest rate is always changing with time, which can reflect the change of interest rates of the market, and there are some term structure models to describe it such as the Vasicek model [7] and the CIR model[8]. So some studies about the portfolio selection problems with stochastic interest rate occurred. For instance, Korn and Kraft [9] used a stochastic control approach to deal with portfolio selection problems with stochastic interest rate and proved a verification theorem. Deelstra et al.[10] studied the optimal investment problems for DC pension fund in a continuous-time framework and assumed that the interest rates follow the affine dynamics, including the CIR model and the Vasicek model. Chang et al.[11] studied an asset and liability management problem with stochastic interest rate in which the interest rate was assumed to be an affine interest rate model. Chang et al. [12] investigated an investment and consumption problem with stochastic interest rate, in which interest rate was assumed to follow the Ho-Lee model and be correlated with stock price and derived optimal strategies for power and logarithm utility function. Gao [13] investigated the portfolio problem of a pension fund management in a complete financial market with stochastic interest rate. Chang and Lu [14] studied an asset and liability management problem with CIR interest rate dynamics and obtained the closed form solutions to the optimal investment strategies by applying dynamic programming principle and variable change technique. Boulier et al. [15] obtained the optimal strategy for DC pension management with stochastic interest rate.

On the other hand, these articles above are all assumed that the volatility is constant. In real world, there exists volatility risk and it is necessarty for us to investigate contiuous-time dynamic portfolio under the volatility risk. One can refer to the work of Heston [16]. Kraft [17] solved the portfolio problem under Heston model and presented a verification result. Li et.al. [18] considered the optimal time-consistent investment and reinsurance for an insurer under Heston model and presented economic implications and numerical sensitivity analysis.

In recent years, some scholars are concerned with the optimal investment problems under stochastic interest rate and stochastic volatility. Liu [19] explicitly solved dynamic portfolio choice problem with stochastic interest rate and stochastic volatility and presented three special applications. However, in this paper, the interest rate, the stock returns and volatility are all a function of the Markov diffusion factor. Li and Wu [20] studied an optimal investment problem where the stochastic interest rate was a CIR model and the volatility was a Heston model. However, in this article above, there is no correlation between interest rate dynamic and stock price dynamic. Chang and Rong [21] studied an investment and consumption problem with stochastic interest rate and stochastic volatility on the basis of the study of Li and Wu and obtained the optimal investment and consumption strategies. Guan and Liang [22] studied the optimal management of a DC pension plan in a stochastic interest rate and stochastic volatility framework and maximized the expectation of the CRRA utility over a guarantee, then derived the optimal strategies of the problem.

This paper intends to find the optimal investment strategy of the optimal portfolio for the financial market, in which interest rate is assumed to follow Cox-Intersoll-Ross(CIR) model while the stock price is supposed to be the Heston's stochastic volatility model. In addition,the financial market consists of three assets: a riskless asset, a zero-coupon bond and a stock. The objective of the manager is to maximize the expected utility of the terminal wealth. By using the principle of stochastic dynamic programming, we derive a complicated non-linear second-order partial differential equation. We assume that the risk preference of the investor satisfy CRRA utility and obtain the explicit solutions for the optimal investment strategies by using variable change technique. Finally, we present a numerical example to investigate more closely the dynamic behavior of the optimal portfolio strategy.

The rest of the paper is organized as followed. In section 2, we introduce the financial market and the wealth process. In section 3, we bring out the optimization criterion and derive the HIB equation. In section 4, we introduce the CRRA utility function and derive the explicit solution of the optimal portfolio. In section 5, we present a numerical application to demonstrate the result and section 6 concludes the paper.

2 The model

In this section, we introduce the financial market.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval [0,T]. The uncertainty involved by the financial market is described by three standard Brownian motions $W_r(t)$, $W_v(t)$ and $W_s(t)$, with $t \in [0, T]$, defined on a complete probability space (Ω, $\mathcal{F}$,P), where P is the real-word probability. The filtration $\mathcal{F}$={$\mathcal{F}_t$} is a right continuous filtration of $\sigma - algebras$ on this space that represents the information structure generated by the Brownian motions. $E[.]$ stands for the expected value.

2.1 The financial market

We assume that the market is composed of three financial assets, which the manager can buy or sell continuously. The first asset is the riskless asset. We denote the price at time t by S_0(t), which evolves the following equation:

$$dS_0(t) = r(t)S_0(t)dt, \tag{1}$$

In the one factor CIR model, the interest rate state variable is the short rate itself, which satisfies

$$dr(t) = (k_1 - k_2 r(t))dt + \sigma_r \sqrt{r(t)}dW_r(t), \tag{2}$$

where k_1, k_2 and σ_r are constants.

The second asset is the zero-coupon bond with maturity T, whose price at time t is denoted by $P(t, T)$, which is described by the following stochastic differential equation(referring to Liu [19]):

$$\frac{dP(t,T)}{P(t,T)} = (r(t)+b\lambda_1\sigma_s r(t))dt+b\sigma_r\sqrt{r(t)}dW_r(t), \tag{3}$$

where b, λ_1 and σ_s are constants. This is derived by Cox, Ingersoll and Ross [8]. The bond return has a risk premium $b\lambda_1 r(t)$ that changes with time t both implicitly through the dependence on $r(t)$ and explicitly through the dependence on b.

The third asset is a stock, whose price is denoted by S_1(t). Because of its self randomness, the impact of the interest rate and the volatility on the price of the stock, we assume $S_1(t)$ follows (referring to Liu [19]):

$$\frac{dS_1(t)}{S_1(t)} = (r(t) + \lambda_s v(t) + \lambda_v \sigma_s \lambda_1 r(t))dt + \sqrt{v(t)}dW_s(t) + \lambda_v \sigma_r \sqrt{r(t)}dW_r(t), \tag{4}$$

where the volatility $v(t)$ satisfies the Heston model:

$$dv(t) = (k_v - K_v v(t))dt + \sigma_v \sqrt{v(t)}dW_v(t), \tag{5}$$

with k_v,K_v and σ_v being positive constants.

Here we assume that there is no correlation between the Brownian motions $W_s(t)$ and $W_r(t)$ and between $W_v(t)$ and $W_r(t)$. The correlation between $W_s(t)$ and $W_v(t)$ is ρ.

2.2 Wealth process

Once the assets available to the investor have been described, we now model the dynamic investment.

Let $X(t)$ denote the wealth of the investor at time $t \in$ [0,T], $\pi_s(t)$ and $\pi_B(t)$ denote the amount invested in the stock and the zero-coupon bond, respectively. Thus, $\pi_0(t) = X(t) - \pi_s(t) - \pi_B(t)$ denotes the amount invested in the riskless asset. The dynamics of the wealth process is given by:

$$dX(t) = (X(t) - \pi_s(t) - \pi_B(t))\frac{dS_0(t)}{S_0(t)} + \pi_s(t)\frac{dS_1(t)}{S_1(t)} + \pi_B(t)\frac{dP(t,T)}{P(t,T)}. \tag{6}$$

Taking into account (1)-(5), the evolution of pension wealth can be rewritten as:

$$\begin{aligned} dX(t) = &\Big(X(t)r(t) + \pi_s(t)\lambda_s V(t) \\ &+ \pi_s(t)\lambda_v\sigma_s\lambda_1 r(t) + \pi_B(t)b\sigma_s\lambda_1 r(t)\Big)dt \\ &+ \pi_s(t)\sqrt{V(t)}dW_s(t) \\ &+ \Big(\pi_s(t)\lambda_v(t)\sigma_r\sqrt{r(t)} \\ &+ \pi_B(t)b\sigma_r\sqrt{r(t)}\Big)dW_r(t). \end{aligned} \tag{7}$$

3 The optimal control

In this section, we provide the optimal control program and derive the Hamilton-Jacobi-Bellman(HJB) equation.

3.1 The optimization criterion

Definition 1 *(Admissible Strategy) An investment strategy $\pi(t) = (\pi_s(t), \pi_B(t))$ is said to be admissible if the following conditions are satisfied.*

(i) $\pi_s(t)$ and $\pi_B(t)$ are all $\mathcal{F}_t$- measurable.

(ii) $E\Bigg(\int_0^T \Bigg(\pi_s^2(t)v(t) + (\pi_s(t)\lambda_v(t)\sigma_r + \pi_B(t)b\sigma_r)^2 r(t)\Bigg)dt\Bigg) < +\infty$

(iii) the SDE(7) has a unique solution according to $\forall\pi(t) = (\pi_s(t), \pi_B(t))$.

Assume that the set of all admissible strategies is denoted by Π. Under the wealth process denoted by (7), the investor looks for an optimal strategy $\pi_s^*(t)$ and $\pi_B^*(t)$ maximizing the expected utility of the terminal wealth, i.e.:

$$\max_{\pi(t)\in\Pi} E(U(X(T))), \tag{8}$$

where u(.) is strictly concave and satisfies the Inada conditions $u'(+\infty) = 0$ and $u'(0) = +\infty$. T is the horizon for the fund investment. In this paper, we consider one common utility function, i.e. the CRRA utility function. It is given by

$$U(x) = \frac{x^p}{p}, \quad (p<1, p \neq 0). \tag{9}$$

3.2 The optimization program

Based on the classical tools of stochastic optimal control, we define the value function:

$$H(t,r,v,x) = \max_{\pi(t)\in\Pi} E[U(X(T))|X(t) = x, r(t) = r, v(t) = v]), 0<t<T \tag{10}$$

The maximum principle leads to the following Hamilton-Jacobi-Bellman(HJB) equation:

$$\begin{aligned} \sup_{\pi(t)\in\Pi} &\{H_t + \big(x(t)r(t) + \pi_s(t)\lambda_s v(t) \\ &+ \pi_s(t)\lambda_v\sigma_s\lambda_1 r(t) + \pi_B(t)b\sigma_s\lambda_1 r(t)\big)H_x \\ &+ \frac{1}{2}\big(\pi_s^2(t)v(t) + (\pi_s(t)\lambda_v\sigma_r\sqrt{r(t)} \\ &+ \pi_B(t)b\sigma_r\sqrt{r(t)})^2\big)H_{xx} + (k_1 - k_2 r(t))H_r \\ &+ \frac{1}{2}\sigma_r^2 r(t)H_{rr} + (k_v - K_v v(t))H_v \\ &+ \big(\pi_s(t)\lambda_v\sigma_r^2 r(t) + \pi_B(t)b\sigma_r^2 r(t)\big)H_{xr} \\ &+ \frac{1}{2}\sigma_r^2 v(t)H_{vv} + \rho\sigma_v\pi_s(t)v(t)H_{xv}\} = 0, \end{aligned} \tag{11}$$

with $H(T,r,v,x)=U(x)$, where $H_t,\ H_v,\ H_x,\ H_r,\ H_{xx},\ H_{rr},\ H_{vv},\ H_{xr},\ H_{xv}$ denote partial derivatives of first and second orders with $t,\ r,\ v,\ x$.

Then we differentiate (11) with respect to $\pi_s(t)$ and $\pi_B(t)$ and obtain two equations :

$$\begin{aligned}&\lambda_s v H_x+\lambda_v\sigma_s\lambda_1 rH_x+vH_{xx}\pi_s(t)\\&+(\pi_s(t)\lambda_v\sigma_r\sqrt{r}+\pi_B(t)b\sigma_r\sqrt{r})\lambda_v\sigma_r\sqrt{r}H_{xx}\\&+\lambda_v\sigma_r^2 rH_{xr}+\rho\sigma_v vH_{xv}=0,\end{aligned}\tag{12}$$

$$\begin{aligned}&b\lambda_1\sigma_s rH_x+(\pi_s(t)\lambda_v\sigma_r\sqrt{r}\\&+\pi_B(t)b\sigma_r\sqrt{r})b\sigma_r\sqrt{r}H_{xx}+b\sigma_r^2 rH_{xr}=0.\end{aligned}\tag{13}$$

The first order maximizing conditions for the optimal strategy $\pi_s(t)$ and $\pi_B(t)$ can be derived by solving Eq(12) and (13):

$$\pi_B^*(t)=\frac{(\lambda_s\lambda_v\sigma_r^2-\lambda_1\sigma_s)H_x}{b\sigma_r^2H_{xx}}+\frac{\rho\sigma_v\lambda_v H_{xv}}{bH_{xx}}-\frac{H_{xr}}{bH_{xx}},\tag{14}$$

$$\pi_s^*(t)=-\frac{\lambda_s H_x+\rho\sigma_v H_{xv}}{H_{xx}}.\tag{15}$$

Putting (14) and (15) in Eq(11), a partial differential equation (PDE) for the value function can be simplified as the following equation:

$$\begin{aligned}&H_t+xrH_x-\left(\frac{\lambda_s^2 v}{2}+\frac{\lambda_1^2\sigma_s^2 r}{2\sigma_r^2}\right)\frac{H_x^2}{H_{xx}}\\&-\frac{\rho\sigma_v\lambda_s vH_xH_{xv}}{H_{xx}}+\frac{-2\lambda_1\sigma_s r+\lambda_1\sigma_s\sigma_r r}{H_{xx}}H_xH_{xr}\\&-\frac{\rho^2\sigma_v^2 vH_{xv}^2}{2H_{xx}}-\frac{\sigma_r^2 rH_{xr}^2}{2H_{xx}}+(k_1-k_2r)H_r\\&+\frac{1}{2}\sigma_r^2 rH_{rr}+(k_v-K_v)H_v+\frac{1}{2}\sigma_r^2 vH_{vv}=0.\end{aligned}\tag{16}$$

Now the problem turns to solving Eq(16) for the value function and replace it into the above two equations (14) and (15) in order to obtain the optimal portfolios.

4 Solution to the optimization problem

In this section, we adopt CRRA utility function and conjecture a solution to the equation (16) with the following form:

$$H(t,r,v,x)=\frac{x^\eta}{\eta}f(t,r,v),\quad \eta<1,\eta\neq 0\tag{17}$$

and its boundary condition is $f(T,r,v)=1$.

The following partial derivatives are derived according to Eq(17):

$$\begin{aligned}&H_t=\frac{x^\eta}{\eta}f_t,\quad H_x=x^{\eta-1}f,\\&H_{xx}=(\eta-1)x^{\eta-2}f,\quad H_v=\frac{x^\eta}{\eta}f_v,\\&H_{xv}=x^{\eta-1}f_v,\quad H_{xr}=x^{\eta-1}f_r,\\&H_r=\frac{x^\eta}{\eta}f_r,\quad H_{rr}=\frac{x^\eta}{\eta}f_{rr},\\&H_{vv}=\frac{x^\eta}{\eta}f_{vv},\end{aligned}\tag{18}$$

where $f_r,\ f_v,\ f_t$ are the first order derivatives of f respect to r, v and t respectively and the rest represent the second order derivatives about them.

Introducing the above derivatives into (16), we derive:

$$\begin{aligned}&\left(\eta r-\frac{\lambda_s^2 v\eta}{2(\eta-1)}-\frac{\lambda_1^2\sigma_s^2 r\eta}{2\sigma_r^2(\eta-1)}\right)f+f_t\\&+\left(-\frac{2\eta\lambda_1\sigma_s r}{\eta-1}+\frac{\eta\lambda_1\sigma_s\sigma_r r}{\eta-1}+k_1-k_2r\right)f_r\\&+\left(k_v-(K_v+\frac{\eta\rho\sigma_v\lambda_s}{\eta-1})v\right)f_v-\frac{\eta\sigma_r^2 rf_r^2}{(\eta-1)f}\\&-\frac{\eta\rho^2\sigma_v^2 vf_v^2}{2(\eta-1)f}+\frac{1}{2}\sigma_r^2 rf_{rr}+\frac{1}{2}\sigma_r^2 vf_{vv}=0.\end{aligned}\tag{19}$$

In order to solve the equation, we conjecture $f(t,r,v)$ as the following form:

$$f(t,r,v)=e^{D_1(t)+D_2(t)r+D_3(t)v},\tag{20}$$

where $D_1(T)=0,\quad D_2(T)=0,\quad D_3(T)=0.$

Then,

$f_t=(D_1'(t)+D_2'(t)r+D_3'(t)v)f,$

$f_v=D_3(t)f,\quad f_r=D_2(t)f,$

$f_{rr}=D_2^2(t)f,\quad f_{vv}=D_3^2(t)f.$

Thus,we derive:

$$\begin{aligned}&rf\Bigg(D_2'(t)+(\eta-\frac{\eta\lambda_1^2\sigma_s^2}{2(\eta-1)\sigma_r^2})-\frac{\sigma_r^2}{2(\eta-1)}D_2^2(t)\\&+(-2\frac{\eta}{\eta-1}\lambda_1\sigma_s+\lambda_1\sigma_s\sigma_r\frac{\eta}{\eta-1}-k_2)D_2(t)\Bigg)\\&+vf\Bigg(D_3'(t)-\frac{\eta}{2(\eta-1)}\lambda_s^2-(\frac{\eta}{\eta-1}\rho\sigma_v\lambda_s+K_v)D_3(t)\\&+(\frac{1}{2}\sigma_r^2-\frac{\eta}{2(\eta-1)}\rho^2\sigma_v^2)D_3^2(t)\Bigg)\\&+f\Bigg(D_1'(t)+k_1D_2(t)+k_vD_3(t)\Bigg)=0,\end{aligned}\tag{21}$$

which can be simplified as:

$$rfI_1(t) + vfI_2(t) + fI_3(t) = 0, \tag{22}$$

in which

$$\begin{aligned} I_1(t) = & D_2'(t) + \left(\eta - \frac{\eta\lambda_1^2\sigma_s^2}{2(\eta-1)\sigma_r^2}\right) \\ & - \frac{\sigma_r^2}{2(\eta-1)}D_2^2(t) + \left(-2\frac{\eta}{\eta-1}\lambda_1\sigma_s \right. \\ & \left. + \lambda_1\sigma_s\sigma_r\frac{\eta}{\eta-1} - k_2\right)D_2(t), \end{aligned} \tag{23}$$

$$\begin{aligned} I_2(t) = & D_3'(t) - \frac{\eta}{2(\eta-1)}\lambda_s^2 \\ & - \left(\frac{\eta}{\eta-1}\rho\sigma_v\lambda_s + K_v\right)D_3(t) \\ & + \left(\frac{1}{2}\sigma_r^2 - \frac{\eta}{2(\eta-1)}\rho^2\sigma_v^2\right)D_3^2(t), \end{aligned} \tag{24}$$

$$I_3(t) = D_1'(t) + k_1 D_2(t) + k_v D_3(t). \tag{25}$$

In order to make the equation (21) established constantly, the only need is to make the coefficients of rf ,vf and f to be zero, that is: $I_1(t) = 0, \quad I_2(t) = 0$ and $I_3(t) = 0$.

From what we have studied, it is clear that $I_1(t) = 0$ and $I_2(t) = 0$ are the general Riccati equations. Now we turn to solving the three equations.

As for $I_1(t) = 0$, remember

$$\begin{aligned} \Delta_1 &= b_1^2 - 4a_1c_1 \\ &= (4\lambda_1^2\sigma_s^2 + \lambda_1^2\sigma_s^2\sigma_r)\frac{\eta^2}{(\eta-1)^2} \\ &+ (4k_2\lambda_1\sigma_s - 4\lambda_1^2\sigma_s^2\sigma_r + 2\sigma_r^2)\frac{\eta}{\eta-1} \\ &+ k_2^2 - 2k_2\lambda_1\sigma_s\sigma_r + \frac{\eta}{(\eta-1)^2}\lambda_1^2\sigma_s^2 \end{aligned} \tag{26}$$

where

$a_1 = \frac{\sigma_r^2}{2(\eta-1)}$,

$b_1 = \frac{1}{\eta-1}[2\eta\lambda_1\sigma_s - \lambda_1\sigma_s\sigma_r\eta + k_2(\eta-1)]$,

$c_1 = \frac{\eta\lambda_1^2\sigma_s^2}{2(\eta-1)\sigma_r^2} - \eta$.

as the discriminant of the quadratic function

$\frac{1}{2(\eta-1)}\sigma_r^2 D_2^2(t) + \frac{1}{\eta-1}\Big(2\eta\lambda_1\sigma_s - \lambda_1\sigma_s\sigma_r\eta + k_2(\eta-1)\Big)D_2(t) + \frac{\eta\lambda_1^2\sigma_s^2}{2(\eta-1)\sigma_r^2} - \eta = 0.$

It is obvious that $I_1(t)$ has different solutions depending on whether $\Delta_1>0, \quad \Delta_1 = 0$ and $\Delta_1 < 0$. Now we let $\Delta_1 > 0$. Then, the quadratic function has two different roots denoted by m_1 and m_2 such that:

$a_1(D_2(t) - m_1)(D_2(t) - m_2) = D_2'(t)$,

in which

$m_1 = \frac{-b_1+\sqrt{\Delta_1}}{2a_1}, \quad m_2 = \frac{-b_1-\sqrt{\Delta_1}}{2a_1}$.

Now the problem turns to solving the differential equation

$$\frac{1}{m_1 - m_2}\left(\frac{1}{D_2(t) - m_1} - \frac{1}{D_2(t) - m_2}\right)dD_2(t) = adt. \tag{27}$$

Then, we integral Eq(27) with respect to t from t to T. With a view of the boundary condition above, we derive

$$\begin{aligned} D_2(t) &= \frac{m_2 - m_2 e^{a_1(m_1-m_2)(T-t)}}{1 - \frac{m_2}{m_1}e^{a_1(m_1-m_2)(T-t)}} \\ &= \frac{m_2m_1 - m_1m_2e^{a_1(m_1-m_2)(T-t)}}{m_1 - m_2e^{a_1(m_1-m_2)(T-t)}}. \end{aligned} \tag{28}$$

For equation $I_2(t) = 0$, we have the discriminant $\Delta_2 = K_v^2 + 2\frac{\eta}{\eta-1}\rho\sigma_v\lambda_s K_v + \frac{\eta}{\eta-1}\lambda_s^2\sigma_r^2$

Under the condition $\Delta_2 > 0$, we assume the roots as m_3 and m_4. As we all know, Eq(24) is a equation similar to Eq(23). Thus, we use the same technique as Eq(23) and obtain the explicit solution as follows:

$$\begin{aligned} D_3(t) &= \frac{m_4 - m_4 e^{a_2(m_3-m_4)(T-t)}}{1 - \frac{m_4}{m_3}e^{a_2(m_3-m_4)(T-t)}} \\ &= \frac{m_3m_4 - m_3m_4e^{a_2(m_3-m_4)(T-t)}}{m_3 - m_4e^{a_2(m_3-m_4)(T-t)}}, \end{aligned} \tag{29}$$

with $a_2 = \frac{\eta}{2(\eta-1)}\rho^2\sigma_v^2 - \frac{1}{2}\sigma_r^2$,

$b_2 = \frac{\eta}{\eta-1}\rho\sigma_v\lambda_s + K_v$,

$c_2 = \frac{\eta\lambda_s^2}{2(\eta-1)}, \quad m_3 = \frac{-b_2+\sqrt{\Delta_2}}{2a_2}$,

$m_4 = \frac{-b_2-\sqrt{\Delta_2}}{2a_2}$.

As for $I_3(t) = 0$, there is $D_1'(t) = -k_1D_2(t) - k_vD_3(t)$, we integral both the sides with respect to t from t to T and obtain

$$D_1(t) = k_1\int_t^T D_2(t)dt + k_v\int_t^T D_3(t)dt. \tag{30}$$

From the equations above, we can derive that

$$\frac{H_x}{H_{xx}} = \frac{x}{\eta-1}, \quad \frac{H_{xv}}{H_{xx}} = \frac{xD_2(t)}{\eta-1}, \quad \frac{H_{xr}}{H_{xx}} = \frac{xD_3(t)}{\eta-1}. \tag{31}$$

From what has been discussed above, substituting them into Eq(14) and (15), we are ready to state the following theorem.

Theorem 2 *The optimal portfolio strategy under the stochastic interest rate and the stochastic volatility framework with the CRRA utility function is given by:*

$$\pi_B^*(t) = \frac{(\lambda_s\lambda_v\sigma_r^2 - \lambda_1\sigma_s)}{b\sigma_r^2}\frac{1}{\eta-1}X(t) + \frac{\rho\sigma_v\lambda_v}{b}\frac{D_2(t)}{\eta-1}X(t) - \frac{1}{b}\frac{D_3(t)}{\eta-1}X(t), \tag{32}$$

$$\pi_s^*(t) = -\frac{\lambda_s + \rho\sigma_v D_3(t)}{\eta-1}X(t) \tag{33}$$

Remark 3 *From the equation(23), we note that $D_2(t)$ depends on η, λ_1, σ_s, σ_r, k_2. From equation (24), we can find that $D_3(t)$ is related to η, λ_s, σ_v, σ_r, ρ, K_v. Besides, $D_1(t)$ is relevant to k_1, k_v, $D_2(t)$ and $D_3(t)$, that is to say that $D_1(t)$ depends on η, λ_1, σ_s, σ_r, k_2, λ_s, σ_v, ρ, K_v, k_1, k_v.*

Remark 4 *According to Theorem 2, we find that the optimal amount invested in the zero-coupon bond depends on λ_s, ρ, σ_v, λ_v, σ_r, λ_1, σ_s, b, η, k_2 and K_v, but it doesn't depend on k_1 and k_v. However, the fact is that the value of k_1 has effect on the dynamic of interest rate, which greatly affect the price of zero-coupon bond. It is surprised us.*

Remark 5 *The optimal amount invested in the stock relies on λ_s, ρ, σ_v, η, σ_r and K_v, but it isn't related to the parameters λ_v, b, λ_1, σ_s, k_2, k_1, k_v.*

5 Numerical analysis

In this section, we provide a numerical example to illustrate the properties of the optimal strategy derived in the previous section. Here we take most of the parameters in Deelstra et al.(2003). Throughout this section, unless otherwise stated, we assume that the basic parameters are given by $k_1 = 0.018712$, $k_2 = 0.2339$, $\rho = 0.5$, $b = 0.7$, $\lambda_1 = 1$, $\lambda_s = 1.5$, $\lambda_v = 0.018712$, $\sigma_1 = 0.18$, $\sigma_s = 0.15$, $\sigma_r = 0.95$, $\sigma_v = 0.36$, $k_v = 1.2$, $K_v = 0.4$, $\eta = -2$, $t = 0$. Consider that the initial investment amount with $x_0 = 100$ and the maturity time with $T = 1$. With the data provided above, we can test and verify that $\Delta>0$, then the analysis would be instructive and valuable.

Now the figures below give some analysis on the optimal portfolios.

First, Fig.1 gives us the trends how the wealth invested in the three assets change with time t on the condition that the other coefficients are decided in advance. As we can see from the picture, there is a positive relationship between the optimal investment

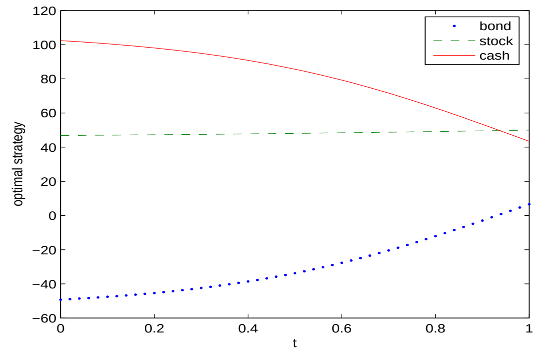

Figure 1: The effect of t on the optimal investment strategy

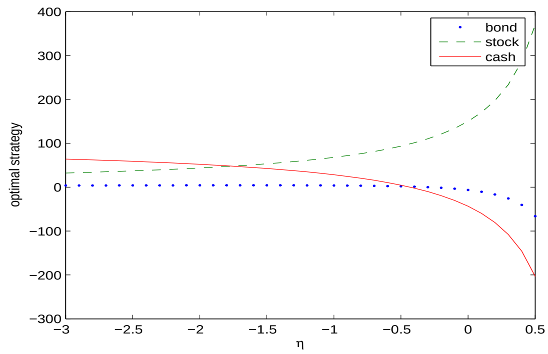

Figure 2: The effect of η on the optimal investment strategy

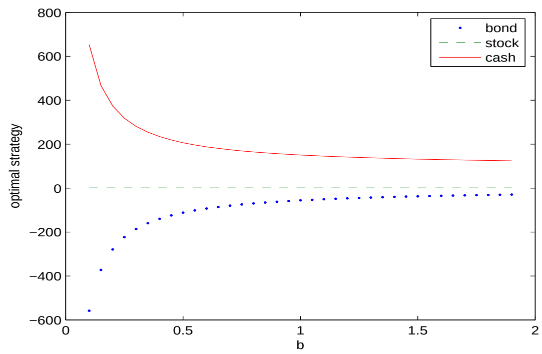

Figure 3: The effect of b on the optimal investment strategy

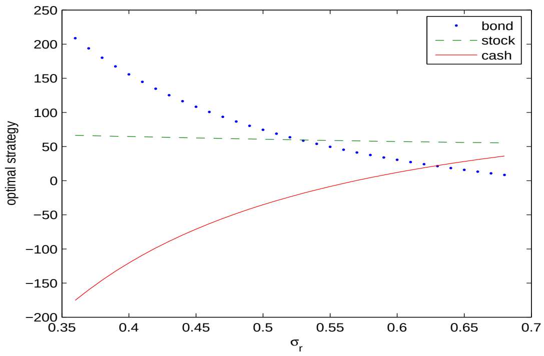

Figure 4: The effect of σ_r on the optimal investment strategy

portfolio in bond and t. That is, as t runs, so does the optimal amount invested in bond. However, the optimal amount invested in stock almost remains unchanged, and the optimal strategy in cash decreases as time goes by. This indicates that as time t goes on, the investor are told to more position in bond and shorter position in cash.

In addition, Figure 1 tells us that the amount invested in bond is negative at the beginning, which indicates that bond is short-selling, and as time increases, the investor invests some of its wealth into bond.

Figure 2 illustrates that how the parameter η of CRRA utility function affects the optimal investment strategy $\pi_B^*(t)$, $\pi_s^*(t)$ and $\pi_0^*(t)$. Figure 2 shows that $\pi_B^*(t)$ decreases with respect to the parameter η. In other words, for a larger η, the amount invested in stock is larger. As we know, the degree of risk aversion for investors is $1-\eta$, that is to say that as η increases, the amount invested in stock will increase. Besides, the part of $\pi_B^*(t)$ will become less and that part of the cash almost stay invariant.

Figure 3 shows the relationship between the parameter b and the optimal investment strategy. From Figure 3, the amount that the investor invest in the stock remains to be 5, which indicates that the parameter b has no effect on the amount invested in the stock. However, as b increases, the optimal amount invested in the cash decreases severely at first and towards smooth to a constant around 130. As we can see from Figure 3, the amount invested in bond is less than zero, which inflects that the investor needs short-selling the bond.

Figure 4 shows us the relationship of σ_r and the optimal investment strategy. From the figure, we find that the amount invested in stock remains to a constant around 65 and the amount of bond decreases as σ_r increases. That is, the interest rate has little influence on the optimal investment for stock.

From Figure 5, we can know that the relationship between λ_1 and the optimal investment value $\pi_B(t)$ and $\pi_s(t)$. That is, the investor invests all his money to cash to avoid risk at first, however, the amount invested in cash decreases as λ_1 increases. In addition, there is a fixed amount to invest to stock and the amount of bond is less than 0, which indicates that the investor chooses the way of short-selling for bond to reach the optimal portfolios.

Figure 6 illustrates the influence of λ_s on the optimal investment strategy. As we can see from the figure, the amount invested in cash decreases as λ_s increases, at the same time, the investment strategy for stock increases. However, $\pi_B(t)$ is less than 0 and almost stays unchanged.

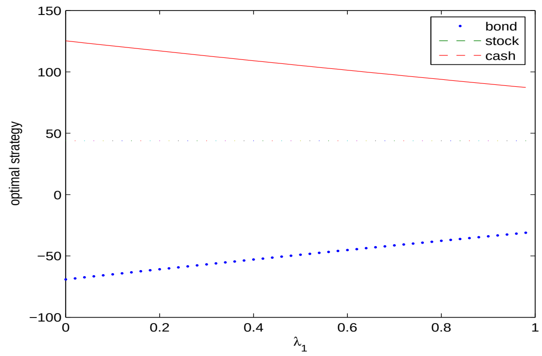

Figure 5: The effect of λ_1 on the optimal investment strategy

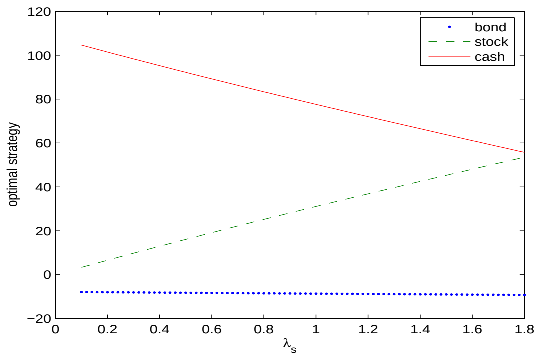

Figure 6: The effect of λ_s on the optimal investment strategy

6 Conclusions

In this paper, we consider the dynamic portfolios with the CIR interest rate under a Heston model. Our objective aims at maximizing the expected utility of the terminal wealth. The investor has to deal with the risk of both interest rate and volatility. The interest rate obeys the CIR model and the volatility of the stock is stochastic and follows the Heston's SV model. Here the market consists of three assets, i.e. a riskless asset, a bond and a stock. Under the CRRA utility function, we derive the optimal investment strategies. From the numerical analysis, we can conclude that the optimal strategy of stock is only related to λ_s, ρ, η, λ_s, σ_v, σ_r, K_v. Besides, the optimal amount invested in bond is irrelevant to k_1 and k_v.

As far as we know, there are some limits in our study: (i) in order to obtain the explicit solutions, we only consider the special utility function; (ii) we only quote the CIR interest rate model and do not study the optimal portfolios with affine interest rate; (iii) we only consider the simplest but important stochastic volatility model i.e. Heston model; (iv) we only consider the dynamics asset allocations but not consider the pension fund investment problems and investment and reinsurance problems. In our future works, we will relax these limits and extend them in the more general market environments.

However, in the context of these limitations, our paper also has its value: (i) we obtain the explicit solutions for the optimal asset allocation problem with CIR interest rate under a Heston model; (ii) we analyze the optimal portfolios via some numerical examples, and at the same time we interpret its economic meanings in real market.

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