Design of Unknown Input Functional Observers for Delayed Singular Systems with State Variable Time Delay

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Abstract: In this paper, we propose a new time-domain design of an Unknown Input functional Observer (UIFO) for delayed singular systems with known and unknown inputs. The order of this observer is equal to the dimension of the vector to be estimated. Constant and variable delays act on the known input vectors and a variable state delay is also taken into account. The proposed approach is based on the unbiasedness of the estimation error and Lyapunov-Krasovskii stability theorem. The observer optimal gain satisfies a sufficient condition of the observer stability dependent on the state delay. This condition is expressed in term of Linear Matrix Inequalities (LMIs) formulation. The proposed approach is tested on a numerical example.

Key–Words: LMI, observer, stability, singular system, unknown input, variable time delay.

1 Introduction

Singular models are of great interest. Despite their complicated structure, these models known also as generalized mathematical representations, are the most adopted in researches for physical system description particularly when singular systems are concerned [18].

Functional observing problems is of great interest ([3, 6–8, 10]). In fact, it's equivalent to find an observer that estimates a linear combination of the states of a considered systems using the input and output measurement. This functional state can be used on control purposes ([9, 16]).

In addition, linear modeling with delayed state interests a large class of physical processes. It highlights the input propagation delay through the dynamics of the system to reach the output which clearly affect by the system stability [17]. Therefore, we review some techniques of stability detection based on Lyapunov-Krasovskii stability theorem in order to ensure the stability of the observer dynamics in spite of delays [15].

Observer design theorem for delayed systems has been investigated over the last decade and several design techniques have been proposed ([3–5, 14]). In addition the observers for systems with unknown input are of great interest in the fault detection and the control of systems in presence of disturbances [5, 6]. However, there is less literature about observer design with unknown input for singular systems with variable state delay [13].

In this paper and based on [7], a time domain method of a functional observer design for delayed singular systems with unknown input is proposed. In fact, we aim to reconstruct a functional state independently from the considered constant time delay acting on the known input vector and the unknown input vector. The observer design is based on a sufficient condition dependent on the state variable time delay and based on the Lyapunov-Krasovskii stability theorem [15].

This paper is organized as follows. Section 2 gives assumptions used through this paper and formulates the functional observer problem to be solved. Section 3, presents the contribution of the paper by giving the design procedure of a functional observer in the time domain. Using the unbiasedness condition, the problem is transformed into a matrix inequalities. LMI approach is then applied and the observer optimal gain is given as a solution of an LMI condition depending on the variable state delay. The fourth section summarizes the UIFO design steps. Section 5 gives a numerical example to illustrate our approach and section 6 concludes the paper.

2 Problem Formulation

Let's consider the following continuous-time linear time-delay singular system described by:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-\tau_1(t)) \\ &\quad +\sum_{j=1}^{q_v} B_{vj} u(t-\tau_{2j}(t)) \\ &\quad +\sum_{j=1}^{q_d} B_{d_j} u(t-\tau_{3_j}) \\ &\quad +Bu(t) + E_1 v(t) && \text{(1a)} \\ y(t) &= Cx(t) && \text{(1b)} \\ m(t) &= Lx(t) && \text{(1c)} \\ x(t_0) &= \phi_0 && \text{(1d)} \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $m(t) \in \mathbb{R}^{m_z}$ is the functional of the state to be estimated, $y(t) \in \mathbb{R}^p$ is the output vector, $u(t) \in \mathbb{R}^q$ is the known input vector, and $v(t) \in \mathbb{R}^r$ is the unknown input.
E, A, A_d, B, C, $B_{vj_{(1\le j\le q_v)}}$, $B_{dj_{(1\le j\le q_d)}}$, E_1 and L are known matrices of appropriate dimensions. ϕ_0 is the initial state, $\tau_1(t) \in \mathbb{R}_+$ is the state variable delay, $\tau_{2j}(t)$ is the known input variable delay and $\tau_{3j} \in \mathbb{R}_+$ is the known input constant delay. Note that q_v and q_d are positive integers.
The variable state delay satisfies the following condition:

$$0 \le \tau_1(t) \le \tau^*, \quad \forall t \in \mathbb{R}_+ \tag{2}$$

with $\tau^* \in \mathbb{R}_+$
In the sequel, we suppose that :

Hypothesis 1 *[7]*

1. $rank\ (E) = r_1 \le n$

2. $rank \begin{bmatrix} E \\ C \end{bmatrix} = n$

The main objective of this paper is to design in time domain an unknown input functional observer for delayed singular linear systems. The considered systems is affected by a bounded variable time delay acting on the state vector and by a variable and a constant delay associated both to the known input vector $u(t)$.

3 UIFO Time Domain Design

Under hypothesis 1, there exists a non singular matrix,

$$S = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \tag{3}$$

such that,

$$a_0 E + b_0 C = I_n \tag{4}$$

$$c_0 E + d_0 C = 0_{p\times n} \tag{5}$$

The seeked functional observer for system (1) is of the form :

$$\begin{aligned} \dot{z}(t) &= Nz(t) + N_d z(t-\tau_1(t)) \\ &\quad +\sum_{j=1}^{q_v} H_{vj} u(t-\tau_{2j}(t)) \\ &\quad +\sum_{j=1}^{q_d} H_{dj} u(t-\tau_{3j}) \\ &\quad +Hu(t) + D_1 y(t) \\ &\quad +D_2 y(t-\tau_1(t)) && \text{(6a)} \\ \hat{m}(t) &= z(t) + My(t) && \text{(6b)} \end{aligned}$$

with

$$M = L(b_0 + E_2 d_0) \tag{7}$$

where z is the state of the observer and $\hat{m}(t) \in \mathbb{R}^n$ is the estimate of the functional $m(t)$. Matrices N, N_d, $H_{vj_{(1\le j\le q_v)}}$, $H_{dj_{(1\le j\le q_d)}}$, H, D_1, D_2 and M will be determined in the sequel using LMI approach.

3.1 UIFO conditions of time-delay singular systems

The estimation error $e(t)$ can be given from (1c) and (6b), using (4) and (5) as :

$$\begin{aligned} e(t) &= m(t) - \hat{m}(t) && \text{(8a)} \\ &= L(I_n - b_0 C - E_2 d_0 C)x(t) \\ &\quad -z(t) && \text{(8b)} \\ &= GEx(t) - z(t) && \text{(8c)} \end{aligned}$$

with

$$G = L(a_0 + E_2 c_0) \tag{9}$$

Purpose : Given the singular system (1) and the functional observer (6), we aim to design the observer matrices N, N_d, H, $H_{vj_{(1\le j\le q_v)}}$, $H_{dj_{(1\le j\le q_d)}}$, D_1, D_2 and E_2 so that $\hat{m}$ converges asymptotically to m, so :

$$\lim_{t\to+\infty} e(t) = 0 \tag{10}$$

where $e(t)$ is given by (8a).
To do so, we propose the following theorem :

Theorem 1 *The functional observer (6) is an UIFO for singular model (1) if and only if the following conditions are satisfied :*

i) $\dot{e}(t) = Ne(t) + N_d e(t - \tau_1(t))$ *is asymptotically stable.*

ii) $GA - NGE - D_1 C = 0$

iii) $GA_d - N_d GE - D_2 C = 0$

iv) $GE_1 = 0$

v) $H = GB$

vi) $H_{vj} = GB_{vj}$, $1 \leq j \leq q_v$

vii) $H_{dj} = GB_{dj}$, $1 \leq j \leq q_d$

▲

Proof 1 *The derivative of (8c) is given as follows :*

$$\dot{e}(t) = GE\dot{x}(t) - \dot{z}(t) \tag{11}$$

By replacing $E\dot{x}(t)$ *and* $\dot{z}(t)$ *by their expressions in (1) and (6a) respectively, relation (11) becomes :*

$$\begin{aligned}\dot{e}(t) = \; & Ne(t) + N_d e(t - \tau_1(t)) + GE_1 v(t) \\ & + \sum_{j=1}^{q_v} (GB_{vj} - H_{vj}) u(t - \tau_{2j}(t)) \\ & + \sum_{j=1}^{q_d} (GB_{dj} - H_{dj}) u(t - \tau_{3j}) \\ & + (GA_d - N_d GE - D_2 C) x(t - \tau_1(t)) \\ & + (GA - NGE - D_1 C) x(t) \\ & + (GB - H) u(t)\end{aligned} \tag{12}$$

with the initial condition $e_0 = m_0 - \hat{m}_0$*. So, the unknown input functional observer (6) will estimate asymptotically the real functional of the state* $m(t)$*, for any initial conditions, any* $u(t)$*,* $u(t-\tau_{2j}(t))$*,* $u(t-\tau_{3j})$ *and independently of the unknown input* $v(t)$*, if and only if conditions i) - vii) are satisfied.* ■

3.2 UIFO time domain design

By replacing G by its expression given by (9) in conditions ii) - iv) of theorem 1 and according to (4) and (5), we have :

$$\begin{aligned} La_0 A &= NLa_0 E + F_1 C - LE_2 c_0 A && (13) \\ La_0 A_d &= N_d La_0 E + F_2 C - LE_2 c_0 A_d && (14) \\ La_0 E_1 &= -LE_2 c_0 E_1 && (15) \end{aligned}$$

where

$$F_1 = D_1 - NLE_2 d_0 \tag{16}$$

$$F_2 = D_2 - N_d LE_2 d_0 \tag{17}$$

Equations (13) - (15) can be written in the following matrix form :

$$X\Sigma = \Theta \tag{18}$$

where,

$$X = \begin{bmatrix} N & N_d & F_1 & F_2 & -LE_2 \end{bmatrix} \tag{19}$$

$$\Sigma = \begin{bmatrix} La_0 E & 0 & 0 \\ 0 & La_0 E & 0 \\ C & 0 & 0 \\ 0 & C & 0 \\ c_0 A & c_0 A_d & c_0 E_1 \end{bmatrix} \tag{20}$$

$$\Theta = \begin{bmatrix} La_0 A & La_0 A_d & La_0 E_1 \end{bmatrix} \tag{21}$$

Note that a general solution of (18), exists if and only if

$$rank \begin{bmatrix} \Sigma \\ \Theta \end{bmatrix} = rank(\Sigma) \tag{22}$$

So under condition (22), we can have :

$$X = \Theta\Sigma^+ - Z(I - \Sigma\Sigma^+) \tag{23}$$

where Σ^+ is the generalized inverse of the matrix Σ and Z is an arbitrary matrix of appropriate dimensions, that will be determined in the sequel using LMI approach. I is the identity matrix of appropriate dimension.

The unknown matrix N in (19) can be given by:

$$N = X \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{24}$$

By replacing (23) in (24), we obtain :

$$N = \Theta\Sigma^+ \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - Z(I - \Sigma\Sigma^+) \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{25}$$

Let's consider :

$$A_{11} = \Theta\Sigma^+ \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

and

$$B_{11} = (I - \Sigma\Sigma^{+}) \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{27}$$

Then,

$$N = A_{11} - ZB_{11} \tag{28}$$

Similarly for matrix N_d, we obtain :

$$N_d = A_{22} - ZB_{22} \tag{29}$$

where

$$A_{22} = \Theta\Sigma^{+} \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{30}$$

and

$$B_{22} = (I - \Sigma\Sigma^{+}) \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

At this stage, and based on theorem 1 and Lyapunov-Krasovskii stability theorem, one can get the gain matrix Z which parametrizes the observer matrices, as proposed in theorem 2.

Theorem 2 *:*
The proposed observer (6) is an UIFO for singular system (1), where the delay $\tau_1(t)$ satisfying (2), if there exist $P = P^T > 0$, $X_0 = X_0^T > 0$ and $X_1 = X_1^T > 0$ such that :

$$\Psi = \begin{pmatrix} Q & \tau^* P N_d & \tau^* P N_d \\ \tau^* N_d^T P & -\tau^{*2} X_0 & 0 \\ \tau^* {N_d}^T P & 0 & -\tau^{*2} X_1 \end{pmatrix} < 0 \tag{32}$$

with

$$Q = \alpha + \alpha^T + \tau^{*2} N^T X_0 N + \tau^{*2} {N_d}^T X_1 N_d \tag{33}$$

and

$$\alpha = (N + N_d)^T P \tag{34}$$

▲

Note that τ^* is given by (2).

Proof 2 *The chosen Lyapunov functional is :*

$$V(t) = V_1(t) + \tau_1(t) V_2(t) + \tau_1(t) V_3(t) \tag{35}$$

whith

$$V_1(t) = e(t)^T P e(t) \tag{36}$$

$$V_2(t) = \int_0^{\tau_1(t)} \int_{t-\theta}^{t} e(s)^T N^T X_0 N e(s) ds d\theta \tag{37}$$

$$V_3(t) = \int_0^{\tau_1(t)} \int_{t-\theta}^{t} e(s)^T N_d^T X_0 N_d e(s) ds d\theta \tag{38}$$

Where P, X_0 and X_1 are symmetric positive definite matrices of dimension n.

According to condition (2), $V(t) \geq 0$, $\forall t \in \mathbb{R}$ and knowing that $e(s)^T N^T X_0 N e(s)$ and $e(s)^T N_d^T X_1 N_d e(s)$ are positive scalars, we can write :

$$0 \leq V(t) \leq \overline{V}(t) \tag{39}$$

with

$$\overline{V}(t) = V_1(t) + \tau^* V_{21}(t) + \tau^* V_{31}(t) \tag{40}$$

$$V_{21}(t) = \int_0^{\tau^*} \int_{t-\theta}^{t} e(s)^T N^T X_0 N e(s) ds d\theta \tag{41}$$

$$V_{31}(t) = \int_0^{\tau^*} \int_{t-\theta}^{t} e(s)^T N_d^T X_0 N_d e(s) ds d\theta \tag{42}$$

So according to (39), if $\underset{t\to+\infty}{lim} \overline{V}(t) = 0$ then $\underset{t\to+\infty}{lim} V(t) = 0$.
To prove that $\overline{V}(t) \approx 0$ when $t \to +\infty$, we can prove that $\dot{\overline{V}}(t)_{t\to+\infty} < 0$ (See [1]).

The derivative of the functional $\overline{V}(t)$ is :

$$\dot{\overline{V}}(t) = \dot{V}_1(t) + \tau^* \dot{V}_{21}(t) + \tau^* \dot{V}_{31}(t) \tag{43}$$

By applying the Leibniz's transformation on (6), we have:

$$\begin{aligned} \dot{e}(t) &= (N + N_d) e(t) - N_d N \int_{-\tau_1(t)}^{0} e(t+\theta) d\theta \\ &\quad - N_d N_d \int_{-2\tau_1(t)}^{-\tau_1(t)} e(t+\theta) d\theta \end{aligned} \tag{44}$$

so :

$$\begin{aligned}\dot{V}_1(t) &= e(t)^T[(N+N_d)^TP+P(N+N_d)]e(t)\\ &\quad -\int_{-\tau_1(t)}^{0} e(t+\theta)^T(NN_d)^TPe(t)d\theta\\ &\quad -e(t)^TPN_dN\int_{-\tau_1(t)}^{0} e(t+\theta)d\theta\\ &\quad -\int_{-2\tau_1(t)}^{-\tau_1(t)} e(t+\theta)^T(NN_d)^TPe(t)d\theta\\ &\quad -e(t)^TPN_dN\int_{-2\tau_1(t)}^{-\tau_1(t)} e(t+\theta)d\theta \qquad (45)\end{aligned}$$

We have :

$$\begin{aligned}\dot{V}_{21}(t) &= \int_0^{\tau^*}[e(t)^TN^TX_0Ne(t)\\ &\quad -e(t-\theta)^TN^TX_0Ne(t-\theta)]d\theta \qquad (46)\end{aligned}$$

so,

$$\begin{aligned}\dot{V}_{21}(t) &= \tau^*e(t)^TN^TX_0Ne(t)\\ &\quad -\int_0^{\tau^*} e(t-\theta)^TN^TX_0Ne(t-\theta)]d\theta \qquad (47)\end{aligned}$$

Let's

$$\epsilon(t-\theta)=-Ne(t-\theta)\in\mathbb{R}^n \qquad (48)$$

so we write :

$$\begin{aligned}\dot{V}_{21}(t) &= \tau^*e(t)^TN^TX_0Ne(t)\\ &\quad -\int_0^{\tau^*} \epsilon(t-\theta)^TX_0\epsilon(t-\theta)]d\theta \qquad (49)\end{aligned}$$

and,

$$\begin{aligned}\dot{V}_{31}(t) &= \int_0^{\tau^*}[e(t)^TN_d^TX_1N_de(t)\\ &\quad -e(t-\theta)^TN_d^TX_1N_de(t-\theta)]d\theta \qquad (50)\end{aligned}$$

Let's :

$$\epsilon_d(t-\theta)=-N_de(t-\theta)\in\mathbb{R}^n \qquad (51)$$

so,

$$\begin{aligned}\dot{V}_{31}(t) &= \tau^*e(t)^TN_d^TX_1N_de(t)\\ &\quad -\int_0^{\tau^*} \epsilon_d(t-\theta)^TX_1\epsilon_d(t-\theta)d\theta \qquad (52)\end{aligned}$$

Uniform asymptotic stability of (40) implies that :

$$\lim_{t\to+\infty}\dot{\bar{V}}(t)\le 0 \qquad (53)$$

As θ is bounded, the quantities $\epsilon(t-\theta)$ and $\epsilon_d(t-\theta)$, respectively, given by (48) and (51) satisfy :

$$\lim_{t\to+\infty}\epsilon(t-\theta)=\lim_{t\to+\infty}\epsilon(t) \qquad (54)$$

and,

$$\lim_{t\to+\infty}\epsilon_d(t-\theta)=\lim_{t\to+\infty}\epsilon_d(t) \qquad (55)$$

and consequently,

$$\begin{aligned}&\lim_{t\to+\infty}(\int_0^{\tau^*}\epsilon(t-\theta)^TX_0\epsilon(t-\theta)d\theta)\\ &\qquad =\tau^*\lim_{t\to+\infty}\epsilon(t)^TX_0\epsilon(t) \qquad (56)\end{aligned}$$

and,

$$\begin{aligned}&\lim_{t\to+\infty}(\int_0^{\tau^*}\epsilon_d(t-\theta)^TX_1\epsilon_d(t-\theta)d\theta)\\ &\qquad =\tau^*\lim_{t\to+\infty}\epsilon_d(t)^TX_1\epsilon_d(t) \qquad (57)\end{aligned}$$

We set the variable's changes:

$$\begin{aligned}\gamma &= \lim_{t\to+\infty}\epsilon(t) \qquad (58)\\ \nu &= \lim_{t\to+\infty}\epsilon_d(t) \qquad (59)\end{aligned}$$

The equations (56) and (57) can be written as :

$$\begin{aligned}&\lim_{t\to+\infty}(\int_0^{\tau^*}\epsilon(t-\theta)^TX_0\epsilon(t-\theta)d\theta)\\ &\qquad =\tau^*\gamma^TX_0\gamma \qquad (60)\end{aligned}$$

and,

$$\begin{aligned}&\lim_{t\to+\infty}(\int_0^{\tau^*}\epsilon_d(t-\theta)^TX_1\epsilon_d(t-\theta)d\theta)\\ &\qquad =\tau^*\nu^TX_1\nu \qquad (61)\end{aligned}$$

We suppose that $\xi=\lim_{t\to+\infty}e(t)$, we have :

$$\begin{aligned}\lim_{t\to+\infty}\dot{\bar{V}}(t) &= [\xi^T[(N+N_d)^TP+P(N+N_d)]\xi\\ &\quad +\tau^{*2}\xi^TN^TX_0N\xi\\ &\quad +\tau^{*2}\xi^T{N_d}^TX_1N_d\xi]\\ &\quad +[\tau_1(t)\gamma^TN_d^TP\xi+\tau_1(t)\xi^TPN_d\gamma]\\ &\quad +[\tau_1(t)\nu^TN_d^TP\xi+\tau_1(t)\xi^TPN_d\nu]\\ &\quad +[-\tau^{*2}\gamma^TX_0\gamma-\tau^{*2}\nu^TX_1\nu] \qquad (62)\end{aligned}$$

and since $\gamma^T N_d^T P\xi$ is a scalar term, we can write

$$\tau_1(t)\gamma^T N_d^T P\xi + \tau_1(t)\xi^T P N_d\gamma = 2\tau_1(t)\gamma^T N_d^T P\xi \quad (63)$$

Let us define :

$$\rho = P N_d \gamma \quad (64)$$

and K a positive definite matrix such that :

$$\rho = K\xi \quad (65)$$

Since that for any vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ we have:

$$\pm 2a^T b \leq a^T K^{-1} a + b^T K b \quad (66)$$

so,

$$2\tau_1(t)\rho^T\xi \leq \tau_1(t)(\rho^T K^{-1}\rho + \xi^T K\xi) \quad (67)$$

Since $\rho^T K^{-1}\rho \in \mathbb{R}_+$, $\xi^T K\xi \in \mathbb{R}_+$ and $0 < \tau(t) \leq \tau^*$

then,

$$\tau_1(t)(\rho^T K^{-1}\rho + \xi^T K\xi) \leq \tau^*(\rho^T K^{-1}\rho + \xi^T K\xi) \quad (68)$$

Which results in :

$$\tau_1(t)(\rho^T K^{-1}\rho + \xi^T K\xi) \leq \tau^*(\rho^T\xi + \xi^T\rho) \quad (69)$$

Similarly:

$$\tau_1(t)\nu^T N_d^T P\xi + \tau(t)\xi^T P N_d\nu \leq \tau^*\nu^T N_d^T P\xi + \tau^*\xi^T P N_d\nu \quad (70)$$

Therefore:

$$\begin{aligned}\lim_{t\to+\infty}\dot{\overline{V}}(t) \leq\ & [\xi^T[(N+N_d)^T P + P(N+N_d)]\xi \\ & +\tau^{*2}\xi^T N^T X_0 N\xi \\ & +\tau^{*2}\xi^T {N_d}^T X_1 N_d\xi] \\ & +[\tau^*\gamma^T N_d^T P\xi + \tau^*\xi^T P N_d\gamma] \\ & +[\tau^*\nu^T N_d^T P\xi + \tau^*\xi^T P N_d\nu] \\ & +[-\tau^{*2}\gamma^T X_0\gamma - \tau^{*2}\nu^T X_1\nu] \end{aligned} \quad (71)$$

which can be written as :

$$\lim_{t\to+\infty}\dot{\overline{V}}(t) \leq \begin{bmatrix} \xi^T & \gamma^T & \nu^T \end{bmatrix} \Psi \begin{bmatrix} \xi \\ \gamma \\ \nu \end{bmatrix} \quad (72)$$

where Ψ is given by (32).

According to condtition (53) if $\lim_{t\to+\infty}\dot{\overline{V}}(t) \leq 0$ then $\Psi \leq 0$ which satisfies theorem 2. ■

To avoid the quadratic form present in equation (32) of theorem 2, we propose a congruence transformation of Ψ using the Schur lemma ([2]). Once equivalence is established, we can express the observer optimal gain Z as a solution of a LMI.

In fact, the chosen Lyapunov functional given by (35) can be modified by choosing $X_0 = X_1 = P$ and theorem 2 is equivalent to :

Theorem 3 *The proposed observer (6) is an UIFO for singular system (1), where the delay $\tau_1(t)$ satisfying (2), if there exist $P = P^T > 0$ and Y such as the symmetrical matrix Π is negative :*

$$\Pi = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} & \beta_{16} \\ * & \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} & \beta_{26} \\ * & * & \beta_{33} & \beta_{34} & \beta_{35} & \beta_{36} \\ * & * & * & \beta_{44} & \beta_{45} & \beta_{46} \\ * & * & * & * & \beta_{55} & \beta_{56} \\ * & * & * & * & * & \beta_{66} \end{pmatrix} \quad (73)$$

with

$$\begin{aligned} \beta_{11} &= \beta_{22} = \beta_{33} = \beta_{56} = -P && (74a) \\ \beta_{12} &= \beta_{13} = \beta_{15} = \beta_{16} = \beta_{23} = \beta_{24} \\ &= \beta_{35} = \beta_{36} = 0_{n\times n} && (74b) \\ \beta_{14} &= \tau^* P A_{11} - \tau^* Y B_{11} && (74c) \\ \beta_{25} &= \beta_{26} = P && (74d) \\ \beta_{34} &= \beta_{45} = \beta_{46} = \tau^* P A_{22} - \tau^* Y B_{22} && (74e) \\ \beta_{44} &= A_{11}^T P - B_{11}^T Y^T + P A_{11} - Y B_{11} \\ &+ A_{22}^T P - B_{22}^T Y^T + P A_{22} - Y B_{22} && (74f) \\ \beta_{55} &= \beta_{66} = -(\tau^{*2} + 1)P && (74g) \end{aligned}$$

The observer gain Z is given by :

$$Z = P^{-1} Y \quad (75)$$

▲

Proof 3 *Matrix Ψ can be written as :*

$$\Psi = M - S^T H^{-1} S \quad (76)$$

with

$$S = \begin{pmatrix} \tau^* N & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_n & I_n \\ \tau^* N_d & 0_{n\times n} & 0_{n\times n} \end{pmatrix} \quad (77a)$$

$$H = -\begin{pmatrix} P^{-1} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & P^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & P^{-1} \end{pmatrix} \quad (77b)$$

$$M = \begin{pmatrix} \alpha + \alpha^T & \tau^* P N_d & \tau^* P N_d \\ * & -\beta_{55} & -P \\ * & * & -\beta_{66} \end{pmatrix} \quad (77c)$$

According to the Schur lemma, $\Psi < 0$ and $H < 0$ if and only if the :

$$\Omega = \begin{pmatrix} H & S \\ S^T & M \end{pmatrix} < 0 \tag{78}$$

We apply a congruence transformation to Ω such as :

$$\Pi = T^T \Omega T < 0 \tag{79}$$

with T is a non singular matrix given by :

$$T = \begin{pmatrix} P & 0 & \dots & & \dots & 0 \\ 0 & P & \ddots & & & \vdots \\ \vdots & \ddots & P & \ddots & & \vdots \\ \vdots & & \ddots & I_n & \ddots & \vdots \\ \vdots & & & \ddots & I_n & 0 \\ 0 & \dots & \dots & \dots & 0 & I_n \end{pmatrix} \tag{80}$$

then by replacing N and N_d by their expressions given respectively by (28) and (29) in Π, theorem 3 holds. ■

Once Z is calculated using (75), all observer matrices can be given by :

$$F_1 = A_{33} - ZB_{33} \tag{81}$$

where

$$A_{33} = \Theta\Sigma^+ \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} \tag{82}$$

$$B_{33} = (I - \Sigma\Sigma^+) \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} \tag{83}$$

The matrix F_2 is given by :

$$F_2 = A_{44} - ZB_{44} \tag{84}$$

where

$$A_{44} = \Theta\Sigma^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix} \tag{85}$$

$$B_{44} = (I - \Sigma\Sigma^+) \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix} \tag{86}$$

and,

$$LE_2 = -A_{55} + ZB_{55} \tag{87}$$

where

$$A_{55} = \Theta\Sigma^+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} \tag{88}$$

$$B_{55} = (I - \Sigma\Sigma^+) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} \tag{89}$$

4 UIFO design steps Summary

Step 1) Verify hypothesis 1.
Step 2) Get the non singular matrix S verifying (5) and (6).
Step 3) Compute matrices Σ and Θ from (19) and (20).
Step 4) Verify the regular condition (21).
Step 5) Solve the LMI (73) to obtain P and Y.
Step 6) Compute the matrix Z from (75).
Step 7) Compute N, N_d, F_1, F_2 and LE_2 using equations (28), (29) and (81)-(89).
Step 8) Compute M using equation (7).
Step 9) Compute D_1 and D_2 using (16) and (17).
Step 10) Get matrices H, $H_{vj_{(1\leq j\leq q_v)}}$ and $H_{dj_{(1\leq j\leq q_d)}}$ using, respectively, conditions $v)$, $vi)$ and $vii)$ from theorem 1.

So, all observer matrices are known.

5 Numerical example

Let's consider system (1), where $q_v = 1$, $q_d = 1$ and,

$$E = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},\ A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_d = \begin{pmatrix} 0.5 & 0 \\ 1 & -1 \end{pmatrix},\ E_1 = \begin{pmatrix} 1 \\ 0.279 \end{pmatrix}$$

$$B_v = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ B_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} -0.5 & 5.23 \\ 3 & -0.87 \end{pmatrix},$$

The variable state delay is a sinusoid such :

$$\tau_1(t) = \frac{1}{4}(sin(20\pi t) + 1)$$

Obviously, we have $0 \leq \tau_1(t) \leq \tau^* = 0.5$, $\forall t \in \mathbb{R}$. The constant known input delay is evaluated at

$$\tau_{31} = 1s.$$

The variable input delay $\tau_{21}(t)$ has the following form:

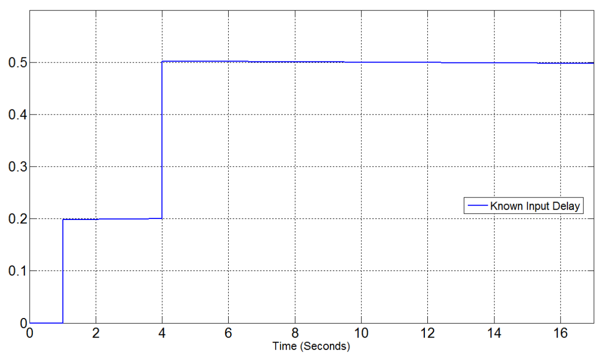

Figure 1: The Input Delay

We have $rank \begin{pmatrix} E \\ C \end{pmatrix} = 2$, so hypothesis 2 is verified.

According to equations (4) and (5), we have:

$$a_0 = \begin{pmatrix} 0.2 & 0.4 \\ -0.2 & -0.4 \end{pmatrix},\ b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$c_0 = \begin{pmatrix} 1 & -0.5 \end{pmatrix},\ d_0 = 0.$$

The resolution of the LMI (73) gives :

$$P = \begin{pmatrix} 30.2351 & 44.7667 \\ 4.7667 & 66.2824 \end{pmatrix},$$

$$Y = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$$

where

$$Y_1 = \begin{pmatrix} 9.9282 & 14.6993 & 7.1634 \\ 14.6994 & 21.7644 & 10.5945 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 10.5945 & 0.0007 & -0.0022 & -0.0004 \\ 15.6943 & -0.0005 & 0.0015 & 0.0003 \end{pmatrix},$$

Computing Z from equation (75) gives :

$$Z = 10^8 \times \begin{pmatrix} Z_1 & Z_2 \end{pmatrix}$$

where

$$Z_1 = \begin{pmatrix} 0.1044 & -0.0705 & 2.3217 \\ -0.0705 & 0.0476 & -1.5680 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} -1.5680 & 0.2958 & -0.9160 & -0.181 \\ 1.0590 & -0.1998 & 0.6187 & 0.1222 \end{pmatrix},$$

So, the functional observer matrix values are given as follows :

$$N = \begin{pmatrix} -126.1805 & -181.3236 \\ 84.9996 & 122.1363 \end{pmatrix}$$

$$N_d = \begin{pmatrix} -15.5045 & -24.5570 \\ 10.3118 & 16.3489 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -6.1162 \\ 4.1308 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 3.3295 \\ -2.2487 \end{pmatrix},$$

$$H = \begin{pmatrix} 0.9289 \\ -0.6274 \end{pmatrix},$$

$$H_v = \begin{pmatrix} 0.9289 \\ -0.6274 \end{pmatrix}$$

$$H_d = \begin{pmatrix} -3.3295 \\ 2.2487 \end{pmatrix}$$

$$LE_2 = \begin{pmatrix} 2.0749 \\ -1.4014 \end{pmatrix},$$

Next simulations are carried out using a known input given by Figure 2.

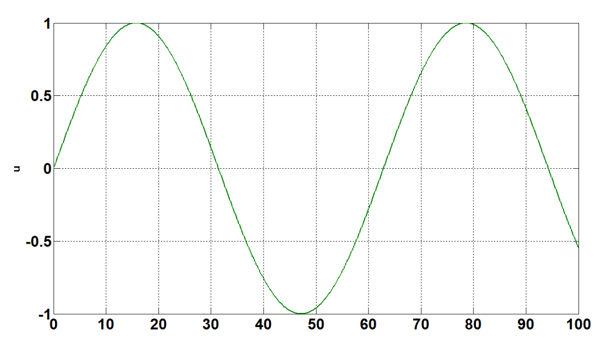

Figure 2: The known input : $u(t)$

Figures 3 and 4 show the real and the estimated components of the state functional and Figures 5 and 6 illustrate the evolution of the state error components.

We note the observer independence of the unknown input and so the effectiveness of the proposed approach.

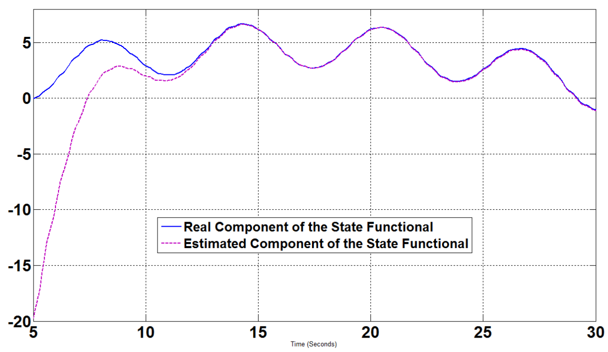

Figure 3: First States Component

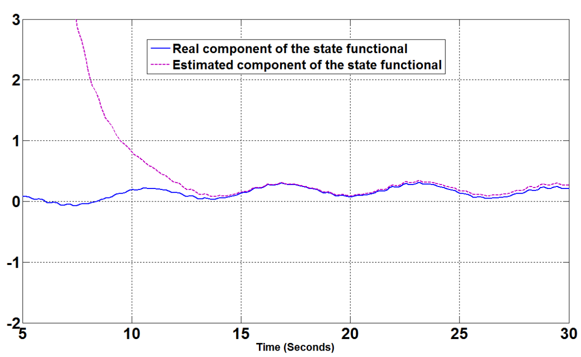

Figure 4: Second States Component

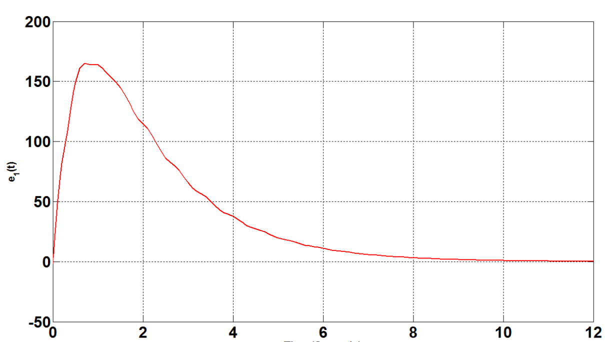

Figure 5: The First Component of the Estimation Error

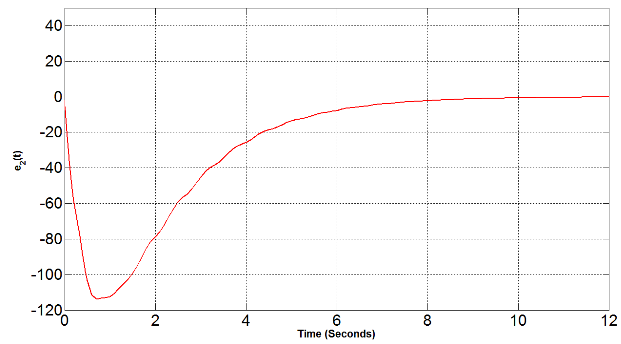

Figure 6: The Second Compoent of the Estimation Error

6 Conclusion

In this paper, a new time-domain design is proposed of an unknown input functional observer for singular systems with variable time delays acting on both state vector and known input vector. A constant time delay is also considered in the known input vector. First, we ensured the unbiasedness of the estimation error. Then, the observer gain that parametrizes all functional observer matrices is an optimal solution of LMIs conditions dependent on the bounded state delay. So, the estimation error converges for any constant time delay, any known input and independently from the unknown input. The application of the proposed design procedure on a numerical example shows the effectiveness of the proposed approach.

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