Portfolio Selection with the Extended CIR Model in the Utility Framework*

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Abstract: This paper studies a continuous-time dynamic portfolio selection problem with multiple risky assets in the utility framework, where we assume that the financial market is composed of one risk-free asset, multiple risky assets and one zero-coupon bond, and short rate is driven by the CIR model. By using dynamic programming principle and solving corresponding Hamilton-Jacobi- Bellman(HJB) equation, we obtain the optimal portfolios for power utility and exponential utility. In addition, we obtain the closed-form solutions to the optimal portfolios under Hyperbolic Absolute Risk Aversion (HARA) utility, which covers power utility, exponential utility and logarithm utility as special cases.

Key–Words: the CIR model; portfolio selection; dynamic programming principle; utility criterion; HARA utility;

1 Introduction

Portfolio selection theory is to study how investors distributes their money to a variety of assets so as to arrive at the goal of increasing returns and reducing the risks, which is an important research aspect in the financial system engineering. In recent years, more and more scholars have begin to pay more increasing attentions to portfolio selection problems with stochastic interest rates. Nowadays, many research results have been published on this topic. For more detailed discussions, these interested readers can refer to the works of Stanton [1], Deelstra et al. [2], Grasselli [3], Gao [4], Chang et al. [5], Chang and Lu [6] and so on. Later, some scholars found out that both interest rate and volatility should be stochastic in the real-world environments. In addition, they thought that it was very necessary to introduce stochastic interest rate and stochastic volatility into portfolio selection models and the corresponding results would be very practical. Some results were achieved on this topic, such as Liu [7], Li and Wu [8], Noh and Kim [9], Chang and Rong [10], Liu et al. [11] and so on. But these models were studied under some specific utility criterion, for example, power utility, exponential utility or logarithm utility.

In the utility theory, HARA utility covers power utility, exponential utility and logarithm utility as special cases. Therefore, studying portfolio selection problems under HARA utility and obtaining the optimal portfolios will be of theoretical values and application prospect. However, owing to the complicated structure of HARA utility, there were seldom works on portfolio selection models with HARA preference in the existing literatures, except Jung and Kim [12], Chang et al. [13], Chang and Rong [14]. To our knowledge, portfolio selection problems with the extended CIR model in the utility framework have not been reported.

In this paper, inspired by the work of Ferland and Watier [15], we devote ourselves to studying a

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continuous-time dynamic portfolio selection problem with the extended CIR model in the utility framework. By using the principle of stochastic dynamic programming and variable change technique, we achieve the explicit expressions of the optimal portfolios for specific and general utility function. Compared with the work of Ferland and Watier [15], our paper has three main contributions: (i) we use stochastic optimal control theory to study a portfolio selection problem with the extended CIR model under utility criterion, while Ferland and Watier [15] applied backward stochastic differential equation theory to discuss this problem under mean-variance criterion; (ii) under HARA utility, we directly conjecture the form of the value function and solve the original HJB equation by using variable change technique, which is completely different from the works of Jung and Kim[12], Chang et al.[13], Chang and Rong [14]; (iii) we don't obtain only the closed-form solutions of the optimal portfolio under power utility and exponential utility, but also those under HARA utility.

The remainder of the paper is organized as follows. Section 2 presents the financial market and utility criterion. Section 3 derives the HJB equation for the value function. Section 4 and Section 5 study the optimal portfolios under power utility, exponential utility and HARA utility respectively. Section 6 concludes the paper.

2 The model

Throughout the paper, the transpose of a matrix or a vector is denoted by $(\cdot)'$ and the norm of a vector is denoted by $\|x\|$, namely $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. $E(\cdot)$ is the mathematical expectation of a random variable, and T is the finite fixed investment horizon. $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$ is a given complete probability space. Assume that the financial market is composed of $n+2$ assets: one risk-free asset, n risk assets and one zero-coupon bond.

The first asset is a risk-free asset (e.g., a bank account), whose price process at time t is denoted by $S_0(t)$, then $S_0(t)$ satisfies

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t)$ is the short rate.

In this paper, we assume that short rate $r(t)$ is dynamic and is described by the following interest rate term structure:

$$\begin{aligned} &dr(t) = (a - br(t))dt - k\sqrt{r(t)}dW_r(t), \\ &r(0) = r_0 > 0, \end{aligned} \tag{2}$$

where a, b, k are positive constants satisfying $2a \geqslant k^2$. It is well known that $r(t) > 0$ for all $t \geqslant 0$ under the condition of $2a \geqslant k^2$. $W_r(t)$ is a one-dimensional well-defined and independent adapted Brownian motion on given filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$.

The risky assets are n stocks. Assumed that the price of the ith risky asset is denoted by $S_i(t)$ at time t, $i = 1, 2, \cdots, n$, $t \in [0, T]$. Considering the affect of $r(t)$ on $S_i(t)$, we let $S_i(t)$ satisfy(referring to Ferland and Watier [15]):

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= r(t)dt + \sum_{j=1}^{n} \sigma_{ij}\left(dW_j(t) + \lambda_j dt\right), \\ &\quad + \sigma_{ir}\sqrt{r(t)}\left(dW_r(t) + \lambda_r\sqrt{r(t)}dt\right), \\ S_i(0) &= s_i > 0, \end{aligned} \tag{3}$$

where $\lambda_s = (\lambda_1, \lambda_2, \cdots, \lambda_n)'$, $\sigma_s = (\sigma_{ij})_{n\times n}$, $\sigma_r = (\sigma_{1r}, \sigma_{2r}, \cdots, \sigma_{nr})'$. $W_s(t) = (W_1(t), \cdots, W_n(t))'$ is a n-dimensional well-defined and independent adapted Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$, and is independent of $W_r(t)$.

The last asset is a zero-coupon bond with maturity T, whose price process at time t is denoted by $B(t, T)$, $t \in [0, T]$. In the stochastic interest rate environment, the zero-coupon bond is not a risk-free asset but a risk asset, and the price process is mainly affected by short rate dynamics $r(t)$. So we can assume that $B(t, T)$ meets the following stochastic differential equation(SDE)(referring to Ferland and Watier [15]):

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r(t)dt + \sigma_B(t)(dW_r(t) \\ &\quad + \lambda_r\sqrt{r(t)}dt), B(T,T) = 1, \end{aligned} \tag{4}$$

where $\sigma_B(t) = kh(t)\sqrt{r(t)}$, and $h(t)$ is given by

$$\begin{aligned} h(t) &= \frac{2\left(1 - e^{m(T-t)}\right)}{m - (b - \lambda_r k) + e^{m(T-t)}(m + b - \lambda_r k)}, \\ m &= \sqrt{(b - \lambda_r k)^2 + 2k^2}. \end{aligned}$$

Suppose that investor's initial fund is $x_0 > 0$. And $\pi_i(t)$ and $\pi_B(t)$ are the amount invested in the ith stock and zero-coupon bond respectively, $i = 1, 2, \cdots, n$. The investor's wealth at time t is denoted by $X(t)$, then the amount invested in risk-free asset is $\pi_0(t) = X(t) - \sum_{i=1}^{n} \pi_i(t) - \pi_B(t)$, $t \in [0, T]$. Let $\pi_s(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_n(t))'$, then under the investment strategy $\pi_s(t)$ and $\pi_B(t)$, the wealth process of the investor satisfies:

$$\begin{aligned} dX(t) &= \Big(X(t)r(t) + \pi_B(t)\sigma_B(t)\lambda_r\sqrt{r(t)}\, dt \\ &+\pi_s'(t)(\sigma_s\lambda_s + \sigma_r\lambda_r r(t))\Big) + \pi_s'(t)\sigma_s dW_s(t) \\ &+ \Big(\pi_s'(t)\sigma_r\sqrt{r(t)} + \pi_B(t)\sigma_B(t)\Big)\, dW_r(t), \\ X(0) &= x_0 > 0, \end{aligned} \tag{5}$$

Definition 1 *(Admissible strategy) The portfolios $\pi_s(t)$ and $\pi_B(t)$ are said to be admissible if the following conditions are satisfied:*

(i) $\pi_s(t)$ and $\pi_B(t)$ are $\mathscr{F}_t$–progressively measurable, and satisfy the conditions

$$\int_0^T \Big(\|\pi_s(t)\|^2 + \pi_B^2(t)\Big)dt < \infty;$$

(ii) $$E\bigg(\int_0^T \Big(\pi_s'(t)\sigma_r\sqrt{r(t)} + \pi_B(t)\sigma_B(t)\Big)^2 dt + \int_0^T \|\pi_s'(t)\sigma_s\|^2 dt\bigg) < \infty;$$

(iii) the SDE (5) has a unique solution corresponding to any $\pi_s(t)$ and $\pi_B(t)$;

Assume that the set of all admissible investment strategies $\pi_s(t)$ and $\pi_B(t)$ is denoted by Γ. Our goal is to maximize the expected utility of the terminal wealth. Mathematically, the objective function can be rewritten as follows:

$$\underset{\pi_B\in\Gamma,\pi_s\in\Gamma}{Maximize}\ E\left[U(X(T))\right], \tag{6}$$

where $U(x)$ is the utility function, which satisfied the conditions: $\frac{dU(x)}{dx} > 0$ and $\frac{d^2U(x)}{dx^2} < 0$.

In this paper, we fucus on the following three utility functions:

$$\begin{aligned} &(i) \quad U(x) = \frac{x^\eta}{\eta}, \quad \eta < 1, \quad \eta \neq 0; \\ &(ii) \quad U(x) = -e^{-\beta x}, \quad \beta > 0; \\ &(iii) \quad U(x) = \frac{1-p}{qp}\bigg(\frac{q}{1-p}x + \gamma\bigg)^p. \end{aligned}$$

3 The HJB equation

In this section we see the problem (6) as a stochastic optimal control problem, and derive the HJB equation of the value function by applied dynamic programming principle.

We define the value function $H(t, r, x)$ as:

$$\begin{aligned} &\underset{\pi_B\in\Gamma,\pi_s\in\Gamma}{Maximize}\ H(t,r,x) \\ &= E\left[U(X(T))|X(t) = x, r(t) = r\right], \end{aligned}$$

with boundary condition given by $H(T, r, x) = U(x)$.

According to dynamic programming principle, $H(t, r, x)$ can be seen as a continuous solution of the following HJB equation.

$$\begin{aligned} \sup_{\pi_B\in\Gamma,\pi_s\in\Gamma} &\big\{H_t + (rx + \pi_B(t)\sigma_B(t)\lambda_r\sqrt{r} \\ &+ \pi_s'(t)(\sigma_s\lambda_s + \sigma_r\lambda_r r))H_x + \frac{1}{2}((\pi_s'(t)\sigma_s)^2 \\ &+ (\pi_s'(t)\sigma_r\sqrt{r} + \pi_B(t)\sigma_B(t))^2)H_{xx} \\ &+ (a - br)H_r + \frac{1}{2}(k\sqrt{r})^2 H_{rr} \\ &- k\sqrt{r}(\pi_s'(t)\sigma_r\sqrt{r} + \pi_B(t)\sigma_B(t))H_{xr}\big\} = 0, \end{aligned} \tag{7}$$

where $H_t, H_x, H_{xx}, H_r, H_{rr}, H_{rx}$ are the first-order and second-order partial derivatives with respect to t, x, r respectively.

By using the first-order condition of the optimality principle, we get the optimal value:

$$\begin{aligned} \pi_s^*(t) &= -(\sigma_s')^{-1}\lambda_s\frac{H_x}{H_{xx}}, \\ \pi_B^*(t) &= \frac{(\lambda_s'\sigma_s^{-1}\sigma_r - \lambda_r)\sqrt{r}}{\sigma_B(t)} \cdot \frac{H_x}{H_{xx}} \\ &+ \frac{k\sqrt{r}}{\sigma_B(t)} \cdot \frac{H_{xr}}{H_{xx}}. \end{aligned} \tag{8}$$

Putting (8) in (7), we obtain a partial differential equation for the value function $H(t, r, x)$:

$$\begin{aligned} &H_t + rxH_x + (a - br)H_r \\ &+ \frac{1}{2}k^2 r H_{rr} - \frac{1}{2}\Big(\|\lambda_s\|^2 + \lambda_r^2 r\Big)\frac{H_x^2}{H_{xx}} \\ &+ k\lambda_r r\frac{H_x H_{xr}}{H_{xx}} - \frac{1}{2}k^2 r\frac{H_{xr}^2}{H_{xx}} = 0. \end{aligned} \tag{9}$$

4 Optimal portfolios under specific utility

In this section, we devote ourselves to investigating the optimal portfolios in the power and exponential utility cases.

4.1 Power utility

According to the boundary condition given by $H(T, r, x) = U(x) = \frac{x^\eta}{\eta}$, the solution of the equation (9) can be conjectured as:

$$H(t,r,x)=\frac{x^{\eta}}{\eta}f(t,r),\quad f(T,r)=1.$$

Then, first-order and second-order partial derivatives of $H(t,r,x)$ with respect to t,x,r are as follows.

$$\begin{aligned}&H_t=\frac{x^{\eta}}{\eta}f_t,\quad H_x=x^{\eta-1}f,\\&H_{xx}=(\eta-1)x^{\eta-2}f,\quad H_r=\frac{x^{\eta}}{\eta}f_r,\\&H_{rr}=\frac{x^{\eta}}{\eta}f_{rr},\quad H_{xr}=x^{\eta-1}f_r.\end{aligned}\tag{10}$$

Putting (10) in (9), we obtain

$$\begin{aligned}&\frac{x^{\eta}}{\eta}\left(f_t+\left(\eta r-\frac{\eta}{2(\eta-1)}\left(\|\lambda_s\|^2+\lambda_r^2r\right)\right)f\right.\\&+\left(a-br+\frac{\eta}{\eta-1}k\lambda_r r\right)f_r\\&\left.+\frac{1}{2}k^2rf_{rr}-\frac{\eta}{2(\eta-1)}k^2r\frac{f_r^2}{f}\right)=0.\end{aligned}$$

Eliminating the dependence on x, we get

$$\begin{aligned}&f_t+\left(\eta r-\frac{\eta}{2(\eta-1)}\left(\|\lambda_s\|^2+\lambda_r^2r\right)\right)f\\&+\left(a-br+\frac{\eta}{\eta-1}k\lambda_r r\right)f_r\\&+\frac{1}{2}k^2rf_{rr}-\frac{\eta}{2(\eta-1)}k^2r\frac{f_r^2}{f}=0.\end{aligned}\tag{11}$$

Lemma 2 *Assume that the solution of the equation (11) is of the form $f(t,r)=e^{B_1(t)+B_2(t)r}$, and its boundary condition is $B_1(T)=B_2(T)=0$, then when $\eta<min\left\{\frac{b^2}{(k\lambda_r-b)^2+2k^2},\ 1\right\}$ and $\eta\neq 0$, $B_1(t)$ and $B_2(t)$ are determined by (16) and (15) respectively.*

Proof. Putting $f(t,r)=e^{B_1(t)+B_2(t)r}$ in the equation (11), we get

$$\begin{aligned}&e^{B_1(t)+B_2(t)r}\left(\left(\dot{B}_2(t)+\left(\eta-\frac{\eta\lambda_r^2}{2(\eta-1)}\right)\right.\right.\\&\left.+\left(\frac{\eta k\lambda_r}{\eta-1}-b\right)B_2(t)-\frac{k^2}{2(\eta-1)}B_2^2(t)\right)r\\&\left.+\dot{B}_1(t)-\frac{\eta}{2(\eta-1)}\|\lambda_s\|^2+aB_2(t)\right)=0.\end{aligned}$$

Eliminating the dependence on r, we get

$$\begin{aligned}&\dot{B}_2(t)+\eta-\frac{\eta\lambda_r^2}{2(\eta-1)}+\left(\frac{\eta k\lambda_r}{\eta-1}-b\right)B_2(t)\\&-\frac{k^2}{2(\eta-1)}B_2^2(t)=0,\quad B(T)=0;\end{aligned}\tag{12}$$

$$\dot{B}_1(t)-\frac{\eta}{2(\eta-1)}\|\lambda_s\|^2+aB_2(t)=0,\quad A(T)=0.\tag{13}$$

We rewrite the equation (12) as:

$$\begin{aligned}\dot{B}_2(t)=&\frac{k^2}{2(\eta-1)}B_2^2(t)-\left(\frac{\eta k\lambda_r}{\eta-1}-b\right)B_2(t)\\&-\left(\eta-\frac{\eta\lambda_r^2}{2(\eta-1)}\right).\end{aligned}\tag{14}$$

After easy calculations, the discriminant for quadratic equation

$$\begin{aligned}&\frac{k^2}{2(\eta-1)}B_2^2(t)-\left(\frac{\eta k\lambda_r}{\eta-1}-b\right)B_2(t)\\&-\left(\eta-\frac{\eta\lambda_r^2}{2(\eta-1)}\right)=0\end{aligned}$$

is given by $\triangle_1=\frac{1}{1-\eta}\left(b^2-\eta\left((k\lambda_r-b)^2+2k^2\right)\right)$.

Considering the conditions $\eta<1$ and $\eta\neq 0$, when $\eta<\frac{b^2}{(k\lambda_r-b)^2+2k^2}$, we have $\triangle_1>0$, i.e. when $\eta<min\left\{\frac{b^2}{(k\lambda_r-b)^2+2k^2},\ 1\right\}$, the above quadratic equation has two different roots, which can be expressed as

$$m_{1,2}=\frac{\eta k\lambda_r-(\eta-1)b}{k^2}\pm\frac{(\eta-1)\sqrt{\triangle_1}}{k^2}.$$

The equation (14) can be changed into

$$\dot{B}_2(t)=\frac{k^2}{2(\eta-1)}(B_2(t)-m_1)(B_2(t)-m_2).$$

Integrating the above equation from t to T, we get

$$\begin{aligned}&\frac{1}{m_1-m_2}\int_t^T\left(\frac{1}{B_2(t)-m_1}-\frac{1}{B_2(t)-m_2}\right)dB_2(t)\\&=\frac{k^2}{2(\eta-1)}(T-t).\end{aligned}$$

Therefore, we have

$$B_2(t)=\frac{m_1m_2\left(1-\exp\left\{\frac{k^2}{2(\eta-1)}(m_1-m_2)(T-t)\right\}\right)}{m_1-m_2\cdot\exp\left\{\frac{k^2}{2(\eta-1)}(m_1-m_2)(T-t)\right\}}.\tag{15}$$

Introducing (15) into (13), we obtain

$$\begin{aligned}B_1(t)=&\left(am_2-\frac{\eta}{2(\eta-1)}\|\lambda_s\|^2\right)(T-t)\\&+\frac{2a(1-\eta)}{k^2}\ln\frac{m_1-m_2}{m_1-m_2\exp\{\frac{k^2}{2(\eta-1)}(m_1-m_2)(T-t)\}}.\end{aligned}\tag{16}$$

The proof is completed. ■

Considering (10) and Lemma 2, we get

$$\frac{H_x}{H_{xx}} = \frac{1}{\eta-1}x, \quad \frac{H_{xr}}{H_{xx}} = \frac{1}{\eta-1}B_2(t)x.$$

To sum up, the optimal investment strategy of the problem (6) under power utility can be formulated as follows.

Theorem 3 *If utility function is given by $U(x) = \frac{x^\eta}{\eta}$, $\eta < 1$ and $\eta \neq 0$, then under the condition of $\eta < min\left\{\frac{b^2}{(k\lambda_r-b)^2+2k^2},\ 1\right\}$ and $\eta \neq 0$, the optimal investment strategy of the problem (6) is expressed as*

$$\begin{aligned}\pi_s^*(t) =& \frac{1}{1-\eta}(\sigma_s')^{-1}\lambda_s X(t),\\ \pi_B^*(t) =& \frac{1}{\eta-1}\cdot\frac{(\lambda_s'\sigma_s^{-1}\sigma_r-\lambda_r)\sqrt{r(t)}}{\sigma_B(t)}X(t)\\ &+\frac{1}{\eta-1}\cdot\frac{k\sqrt{r(t)}}{\sigma_B(t)}B_2(t)X(t).\end{aligned} \tag{17}$$

where $B_2(t)$ is determined by (15).

From Theorem 3, we have the following conclusions:

(1) The optimal amount $\pi_s^*(t)$ invested in the stock at time t is correlated with market parameters $\eta, \sigma_s, \lambda_s$, and has nothing to do with the other market parameters;

(2) $\pi_s^*(t)$ is increasing with the parameter η, and is proportional to wealth process $X(t)$;

(3) The optimal amount $\pi_B^*(t)$ invested in the zero-coupon bond at time t has correlations with market parameters $\eta, \sigma_s, \lambda_s, b, k, \sigma_r, \lambda_r, r(t)$, and has nothing to do with the parameter a;

(iv) $\pi_B^*(t)$ is proportional to wealth process $X(t)$. When $\eta \to 0$, we have $B_2(t) = 0$. As a result, the optimal investment strategy of the problem (6) is reduced to:

$$\pi_s^*(t) = (\sigma_s')^{-1}\lambda_s X(t),$$

$$\pi_B^*(t) = \frac{(\lambda_r - \lambda_s'\sigma_s^{-1}\sigma_r)\sqrt{r}}{\sigma_B(t)}X(t).$$

As we all know, power utility is degenerated into logarithmic utility in the case of $\eta \to 0$. So we indeed get the optimal investment strategy of the problem (6) under logarithmic utility.

4.2 Exponential utility

Under exponential utility, the boundary condition of the equation (9) is given by $H(t,r,x) = U(x) = -e^{-\beta x}$. So the solution of the equation (9) can be supposed to be of the form

$$H(t,r,x) = -e^{-\beta x g(t,r)+h(t,r)},$$

$$g(T,r) = 1, \quad h(T,r) = 0.$$

Then, first-order and second-order partial derivatives of $H(t,r,x)$ with respect to t, x, r are following:

$$\begin{aligned}&H_t = H(-\beta x g_t + h_t), \quad H_x = H(-\beta g),\\ &H_{xx} = H(-\beta g)^2, \quad H_r = H(-\beta x g_r + h_r),\\ &H_{rr} = H(-\beta x g_r + h_r)^2 + H(-\beta x g_{rr} + h_{rr}),\\ &H_{xr} = H(-\beta g)(-\beta x g_r + h_r) + H(-\beta g_r).\end{aligned} \tag{18}$$

Introducing (18) into (9) yields

$$\begin{aligned}&H\Bigg\{-\beta x\bigg(g_t + rg + (a - br + k\lambda_r r)g_r\\ &+\frac{1}{2}k^2 r g_{rr} - k^2 r\frac{g_r^2}{g}\bigg)\\ &+h_t + \bigg(a + k\lambda_r r - br - k^2\frac{g_r}{g}r\bigg)h_r\\ &+\frac{1}{2}k^2 r h_{rr} - \frac{1}{2}\bigg(\|\lambda_s\|^2 + \lambda_r^2 r\bigg)\\ &+k\lambda_r r\frac{g_r}{g} - \frac{1}{2}k^2 r\frac{g_r^2}{g^2}\Bigg\} = 0.\end{aligned}$$

Comparing the coefficients both sides of the equation, we get

$$\begin{aligned}&g_t + rg + (a - br + k\lambda_r r)g_r\\ &+\frac{1}{2}k^2 r g_{rr} - k^2 r\frac{g_r^2}{g} = 0, \quad g(T,r) = 1;\end{aligned} \tag{19}$$

$$\begin{aligned}&h_t + \left(a + k\lambda_r r - br - k^2\frac{g_r}{g}r\right)h_r\\ &+\frac{1}{2}k^2 r h_{rr} - \frac{1}{2}\left(\|\lambda_s\|^2 + \lambda_r^2 r\right)\\ &+k\lambda_r r\frac{g_r}{g} - \frac{1}{2}k^2 r\frac{g_r^2}{g^2} = 0, \quad h(T,r) = 0.\end{aligned} \tag{20}$$

Lemma 4 *Suppose that the solution of the equation(19) is of the structure $g(t,r) = e^{C_1(t)+C_2(t)r}$, with boundary conditions $C_1(T) = 0$ and $C_2(T) = 0$, then $C_1(t)$ and $C_2(t)$ are determined by (24) and (23) respectively.*

Proof. Putting $g(t,r) = e^{C_1(t)+C_2(t)r}$ in the equation (19) yields

$$e^{C_1(t)+C_2(t)r}\left(\dot{C}_1(t) + aC_2(t) + r\left(\dot{C}_2(t) + 1\right.\right.$$

$$+(k\lambda_r - b)C_2(t) - \frac{1}{2}k^2C_2^2(t)\Big)\Big) = 0.$$

Eliminating the dependence on r, we get

$$\begin{aligned}&\dot{C}_2(t) + 1 + (k\lambda_r - b)C_2(t) - \frac{1}{2}k^2C_2^2(t) = 0,\\ &C_2(T) = 0;\end{aligned} \tag{21}$$

$$\dot{C}_1(t) + aC_2(t) = 0, \quad C_1(T) = 0. \tag{22}$$

Using the same solving process as the equation (12), we obtained

$$C_2(t) = \frac{m_3m_4(1 - \exp\left\{\frac{1}{2}k^2(m_3 - m_4)(T - t)\right\})}{m_3 - m_4\exp\left\{\frac{1}{2}k^2(m_3 - m_4)(T - t)\right\}}, \tag{23}$$

where

$$m_{3,4} = \frac{k\lambda_r - b}{k^2} \pm \frac{\sqrt{(k\lambda_r - b)^2 + 2k^2}}{k^2}.$$

Further, we obtained

$$\begin{aligned}C_1(t) = a\Bigg(&m_4(T - t)\\ &- \frac{2}{k^2}\ln\frac{m_3 - m_4}{m_3 - m_4\exp\{\frac{1}{2}k^2(m_3 - m_4)(T - t)\}}\Bigg).\end{aligned} \tag{24}$$

The proof ends. ■

Lemma 5 *Assume that the solution of the equation (20) is given by $h(t, r) = C_3(t) + C_4(t)r$, with boundary conditions: $C_3(T) = 0$ and $C_4(T) = 0$, then $C_3(t)$ and $C_4(t)$ are determined by (28) and (27) respectively.*

Proof. Introducing $h(t, r) = C_3(t) + C_4(t)r$ into (20), we arrive at

$$\begin{aligned}&\dot{C}_3(t) + aC_4(t) - \frac{1}{2}\|\lambda_s\|^2\\ &+ r\Big(\dot{C}_4(t) + (k\lambda_r - b - k^2C_2(t))C_4(t)\\ &- \Big(\frac{1}{2}k^2C_2^2(t) - k\lambda_rC_2(t) + \frac{1}{2}\lambda_r^2\Big)\Big) = 0.\end{aligned}$$

Comparing the coefficients both sides of the above equation

$$\begin{aligned}&\dot{C}_4(t) + (k\lambda_r - b - k^2C_2(t))C_4(t)\\ &= \frac{1}{2}k^2C_2^2(t) - k\lambda_rC_2(t) + \frac{1}{2}\lambda_r^2, \quad C_4(T) = 0,\end{aligned} \tag{25}$$

$$\dot{C}_3(t) + aC_4(t) - \frac{1}{2}\|\lambda_s\|^2 = 0, \quad C_3(T) = 0, \tag{26}$$

Solving the equations (25) and (26), we get

$$\begin{aligned}C_4(t) = &-e^{-\int_0^t(k\lambda_r - b - k^2C_2(s))ds}\\ &\times\int_t^T\frac{1}{2}(kC_2(t) - \lambda_r)^2e^{\int_0^t(k\lambda_r - b - k^2C_2(s))ds}dt,\end{aligned} \tag{27}$$

$$C_3(t) = a\int_t^TC_4(t)dt - \int_t^T\frac{1}{2}\|\lambda_s\|^2dt. \tag{28}$$

Therefore, we complete the proof. ■

Considering (18) and Lemma 4 and Lemma 5, we get

$$\frac{H_x}{H_{xx}} = -\frac{1}{\beta g},$$

$$\frac{H_{xr}}{H_{xx}} = xC_2(t) - \frac{C_4(t)}{\beta g} - \frac{C_2(t)}{\beta g}.$$

To sum up, the optimal investment strategy for the problem (6) under exponential utility can be formulated as follows.

Theorem 6 *If utility function is given by $U(x) = -e^{-\beta x}$, $\beta > 0$, then the optimal investment strategy of the problem (6) is*

$$\pi_s^*(t) = (\sigma_s')^{-1}\lambda_s\frac{1}{\beta g(t, r)}, \tag{29}$$

$$\begin{aligned}\pi_B^*(t) = &-\frac{(\lambda_s'\sigma_s^{-1}\sigma_r - \lambda_r)\sqrt{r(t)}}{\sigma_B(t)}\cdot\frac{1}{\beta g(t, r)}\\ &+\frac{k\sqrt{r(t)}}{\sigma_B(t)}\cdot\Big(X(t)C_2(t) - \frac{C_4(t)}{\beta g(t, r)} - \frac{C_2(t)}{\beta g(t, r)}\Big).\end{aligned} \tag{30}$$

where $g(t, r) = e^{C_1(t) + C_2(t)r}$, $C_2(t)$ and $C_4(t)$ is determined by (23) and (27) respectively.

From Theorem 6, we arrive at the following conclusions.

(i) The optimal amount $\pi_s^*(t)$ invested in the stock at time t is correlated with market parameters $\beta, a, b, k, \sigma_s, \lambda_s, \lambda_r, r(t)$, and has nothing to do with σ_r;

(ii) $\pi_s^*(t)$ is decreasing with the parameter β.

(iii) The optimal amount $\pi_B^*(t)$ invested in the zero-coupon bond at time t is connected with all market parameters.

5 Optimal portfolio under general utility

For general utility function, we are concerned with HARA utility, which covers power utility, exponential utility and logarithmic utility as special cases. According to the boundary condition

$$H(T,r,x)=U(x)=\frac{1-p}{qp}\left(\frac{q}{1-p}x+\gamma\right)^p,$$

we assume the solution of the equation (9) to be of form

$$H(t,r,x)=\frac{1-p}{qp}\left(\frac{q}{1-p}x+\gamma\varphi(t,r)\right)^p J^{1-p}(t,r),$$

$$\varphi(T,r)=1,\quad J(T,r)=1.$$

Then, first-order and second-order partial derivatives for $H(t,r,x)$ with respect to t,x,r are calculated as follows:

$$\begin{aligned}H_t=&\frac{1-p}{q}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-1}\gamma\varphi_t J^{1-p}\\&+\frac{(1-p)^2}{qp}\left(\frac{q}{1-p}x+\gamma\varphi\right)^p J_t J^{-p},\\H_x=&J^{1-p}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-1},\end{aligned}$$

$$\begin{aligned}H_r=&\frac{1-p}{q}\gamma\varphi_r J^{1-p}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-1}\\&+\frac{(1-p)^2}{qp}J^{-p}J_r\left(\frac{q}{1-p}x+\gamma\varphi\right)^p,\\H_{xx}=&(-q)J^{1-p}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-2},\end{aligned}$$

$$\begin{aligned}H_{rr}=&\frac{(1-p)^2}{-q}\gamma^2\varphi_r^2 J^{1-p}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-2}\\&+\frac{1-p}{q}\gamma\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-1}\\&\times[\varphi_{rr}J^{1-p}+2(1-p)J^{-p}J_r\varphi_r]\\&+\frac{(1-p)^2}{qp}\left(\frac{q}{1-p}x+\gamma\varphi\right)^p\\&\times[J_{rr}J^{-p}-pJ^{-p-1}J_r^2],\end{aligned}$$

$$\begin{aligned}H_{rx}=&(1-p)J^{-p}J_r\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-1}\\&+J^{1-p}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-2}(p-1)\gamma\varphi_r.\end{aligned}$$

Putting the above partial derivatives in (9), we get

$$\begin{aligned}&\frac{1-p}{qp}\left(\frac{q}{1-p}x+\gamma\varphi\right)^{p-1}pJ^{1-p}\gamma\Big(\varphi_t-r\varphi\\&+(a-br+k\lambda_r r)\varphi_r+\frac{1}{2}k^2r\varphi_{rr}\Big)\\&+\frac{1-p}{qp}\left(\frac{q}{1-p}x+\gamma\varphi\right)^p(1-p)J^{-p}\\&\times\left(J_t+\left(\frac{p}{1-p}r+\frac{p}{2(1-p)^2}(\|\lambda_s\|^2+\lambda_r^2r)\right)J\right.\\&\left.+\left(a-br-\frac{p}{1-p}k\lambda_r r\right)J_r+\frac{1}{2}k^2rJ_{rr}\right)=0.\end{aligned}$$

Eliminating the dependence on x, we obtain

$$\varphi_t-r\varphi+(a-br+k\lambda_r r)\varphi_r+\frac{1}{2}k^2r\varphi_{rr}=0. \quad (31)$$

$$\begin{aligned}&J_t+\left(\frac{p}{1-p}r+\frac{p}{2(1-p)^2}(\|\lambda_s\|^2+\lambda_r^2r)\right)J\\&+\left(a-br-\frac{p}{1-p}k\lambda_r r\right)J_r+\frac{1}{2}k^2rJ_{rr}=0.\end{aligned} \quad (32)$$

Lemma 7 *Assume that the solution of the equation (31) is of the form $\varphi(t,r)=e^{D_1(t)+D_2(t)r}$, and its boundary conditions are given by $D_1(T)=D_2(T)=0$, then $D_1(t)$ and $D_2(t)$ are determined by (36) and (35) respectively.*

Proof. Putting $\varphi(t,r)=e^{D_1(t)+D_2(t)r}$ in (31)

$$\begin{aligned}&e^{D_1(t)+D_2(t)r}\bigg(\Big(\dot{D}_2(t)+(k\lambda_r-b)D_2(t)\\&+\frac{1}{2}k^2D_2^2(t)-1\Big)r+\dot{D}_1(t)+aD_1(t)\bigg)=0.\end{aligned}$$

Eliminating the dependence on r, we get

$$\dot{D}_2(t)+\frac{1}{2}k^2D_2^2(t)+(k\lambda_r-b)D_2(t)-1=0. \quad (33)$$

$$\dot{D}_1(t)+aD_1(t)=0. \quad (34)$$

We write (33) as

$$\dot{D}_2(t)=-\frac{1}{2}k^2D_2^2(t)-(k\lambda_r-b)D_2(t)+1.$$

Using the same technique as the equation (14), we get

$$D_2(t)=\frac{n_1n_2\left(1-\exp\left\{-\frac{1}{2}k^2(T-t)(n_1-n_2)\right\}\right)}{n_1-n_2\exp\left\{-\frac{1}{2}k^2(T-t)(n_1-n_2)\right\}} \quad (35)$$

where

$$n_{1,2} = \frac{b - k\lambda_r \pm \sqrt{(b-k\lambda_r)^2 + 2k^2}}{k^2}.$$

Introducing (35) into (34), we have

$$D_1(t) = \frac{2a}{k^2} \ln \frac{n_1 - n_2}{n_1 - n_2 \exp\{-\frac{1}{2}k^2(T-t)(n_1-n_2)\}} + an_2(T-t). \tag{36}$$

The proof ends. ■

Lemma 8 *Suppose that the solution of the equation (33) is given by $J(t,r) = e^{D_3(t)+D_4(t)r}$, with boundary conditions: $D_3(T) = D_3(T) = 0$, then under the condition of $p < min\left\{\frac{b^2}{(k\lambda_r - b)^2 + 2k^2}, 1\right\}$ and $p \neq 0$, $D_3(t)$ and $D_4(t)$ are determined by (40) and (39) respectively.*

Proof. Introducing $J(t,r) = e^{D_3(t)+D_4(t)r}$ in the equation (32)

$$e^{D_3(t)+D_4(t)r}\left(\left(\dot{D}_4(t) - \left(b + \frac{p}{1-p}k\lambda_r\right)D_4(t) + \frac{1}{2}k^2 D_4^2(t) + \frac{p}{1-p} + \frac{1}{2}\lambda_r^2 \frac{p}{(1-p)^2}\right) r + \dot{D}_3(t) + aD_4(t) + \frac{1}{2}\|\lambda_s\|^2 \frac{p}{(1-p)^2}\right) = 0.$$

Eliminating the dependence on r, we get

$$\dot{D}_3(t) + aD_4(t) + \frac{1}{2}\|\lambda_s\|^2 \frac{p}{(1-p)^2} = 0, \quad D_3(T) = 0. \tag{37}$$

$$\dot{D}_4(t) + \frac{1}{2}k^2 D_4^2(t) - \left(b + \frac{p}{1-p}k\lambda_r\right)D_4(t) + \frac{p}{1-p} + \frac{1}{2}\lambda_r^2 \frac{p}{(1-p)^2} = 0, \quad D_4(T) = 0. \tag{38}$$

We rewrite (38) as

$$\dot{D}_4(t) = -\frac{1}{2}k^2 D_4^2(t) + \left(b + \frac{p}{1-p}k\lambda_r\right)D_4(t) - \frac{1}{2}\lambda_r^2 \frac{p}{(1-p)^2} - \frac{p}{1-p}.$$

The discriminant of quadratic equation

$$-\frac{1}{2}k^2 D_4^2(t) + \left(b + \frac{p}{1-p}k\lambda_r\right)D_4(t) - \frac{1}{2}\lambda_r^2 \frac{p}{(1-p)^2} - \frac{p}{1-p} = 0$$

is given by

$$\Delta_2 = \left(b + \frac{p}{1-p}k\lambda_r\right)^2 - 2k^2\left(\frac{1}{2}\lambda_r^2 \frac{p}{(1-p)^2} + \frac{p}{1-p}\right). = \frac{1}{1-p}\left(b^2 - p\left((k\lambda_r - b)^2 + 2k^2\right)\right)$$

Using the same analysis as Δ_1, we have $\Delta_2 > 0$ under the condition of $p < min\left\{\frac{b^2}{(k\lambda_r - b)^2 + 2k^2}, 1\right\}$ and $p \neq 0$. Meantime, the two different roots for the above quadratic equation are given by

$$n_{3,4} = \frac{b + \frac{p}{1-p}k\lambda_r}{k^2} \pm \frac{\sqrt{\Delta}_2}{k^2},$$

By the same technique as the equation (14) and (13), we obtain

$$D_4(t) = \frac{n_3 n_4\left(1 - \exp\left\{-\frac{1}{2}k^2(T-t)(n_3-n_4)\right\}\right)}{n_3 - n_4 \exp\left\{-\frac{1}{2}k^2(T-t)(n_3-n_4)\right\}}. \tag{39}$$

$$D_3(t) = \frac{2a}{k^2} \ln \frac{n_3 - n_4}{n_3 - n_4 \exp\{-\frac{1}{2}k^2(T-t)(n_3-n_4)\}} + \left(an_4 + \frac{1}{2}\|\lambda_s\|^2 \frac{p}{(1-p)^2}\right)(T-t). \tag{40}$$

The proof is completed. ■

Considering Lemma 7 and Lemma 8, we have

$$\frac{H_x}{H_{xx}} = -\frac{1}{q}\left(\frac{q}{1-p}x + \gamma\varphi\right),$$

$$\frac{H_{xr}}{H_{xx}} = \frac{p-1}{q}\left[D_4(t)\left(\frac{q}{1-p}x + \gamma\varphi\right) + \gamma\varphi_r\right].$$

In a word, the optimal investment strategy of the problem (6) under HARA utility can be formulated as follows.

Theorem 9 *If utility function is $U(x) = \frac{1-p}{qp}\left(\frac{q}{1-p}x + \gamma\right)^p$, $q > 0, p < 1, p \neq 0$, then under the condition of $p < min\left\{\frac{b^2}{(k\lambda_r - b)^2 + 2k^2}, 1\right\}$ and $p \neq 0$, the optimal investment strategies of the problem (6) are given by*

$$\pi_s^*(t) = \frac{\lambda_s}{q\sigma_s'}\left(\frac{q}{1-p}X(t) + \gamma\varphi\right)$$

$$\pi_B^*(t) = \frac{(\lambda_s'\sigma_s^{-1}\sigma_r - \lambda_r)\sqrt{r(t)}}{-q\sigma_B(t)}\left(\frac{q}{1-p}X(t) + \gamma\varphi\right)$$

$$+\frac{k\sqrt{r(t)}}{\sigma_B(t)}\cdot\frac{p-1}{q}\left[D_4(t)\left(\frac{q}{1-p}X(t)+\gamma\varphi\right)+\gamma\varphi_r\right]$$

where $D_4(t)$ are determined by (39), and

$$\varphi=\varphi(t,r)=e^{D_1(t)+D_2(t)r},$$

$$\varphi_r=\frac{\partial\varphi(t,r)}{\partial r}=D_2(t)e^{D_1(t)+D_2(t)r},$$

From Theorem 9, we can draw the following conclusions:

(i) The optimal amount $\pi_s^*(t)$ invested in the stock at time t is connected with market parameters $\lambda_s, p, q, \gamma, \sigma_s$, and has nothing to do with the other market parameters;

(ii) $\pi_s^*(t)$ is increasing with parameters p and γ, and decreasing with the parameter q;

(iii) The optimal amount $\pi_B^*(t)$ invested in the zero-coupon bond at time t is connected with all market parameters except a and b;

6 Conclusions

The CIR model is one of the most important mathematical models describing term structure of interest rate and can accurately fit the dynamic behavior of interest rate in actual investment environments. In this article, we get the closed-form solution of the optimal investment strategies under power utility, exponential utility and HARA utility by using dynamic programming principle and variable change technique.

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