On the Possible Use of Geophysical Cybernetics in Climate Manipulation (Geoengineering) and Weather Modification

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Abstract: - Geophysical cybernetics is a new research area that studies the geophysical processes and phenomena using cybernetic methods and approaches. A mathematical formulation of the problem of optimal control of the geophysical system is presented from the standpoint of geophysical cybernetics. Further, the essential features of the geophysical system as a control object are considered. The optimal control problem for the large scale atmospheric dynamics is considered and the necessary optimality conditions are derived.

Key-Words: - geophysical cybernetics, geophysical system, optimal control, atmospheric dynamics, weather modification, geoengineering.

1 Introduction

The ongoing climate change, which scientific community agrees is predominantly a man-made phenomenon [1], represents one of the biggest challenges facing humanity today. For a very long time, mainly unwittingly, human civilization has been influencing the environment and changing its properties. Nevertheless, the current rate of climate change has no precedent in the available geological records or even in the known history of our civilisation. At the turn of the century, geoengineering, a new research and technological area, was proposed as an instrument to reduce global warming via an intentional intervention in the earth's climate system. It is necessary to point out that many attempts and experiments to modify the atmospheric processes and weather phenomena have been made in the previous century. However, both geoengineering and weather modification are considered outside the scope of control theory. This gives rise to a number of very important problems, which are currently only formulated in general terms, such as problems related to the validation of the input and output variables, determination of the boundaries of both geoengineering activities and weather modification, statement of climate and weather manipulation goals as well as methods of achieving the objectives. Meanwhile in the late 1970s, a uniform methodology for manipulating atmospheric, hydrological and other geophysical processes was formulated on the basis of the ideas from cybernetics [2]. In this monograph the concept of geophysical cybernetics was introduced as the new area of research within the control theory. Geophysical cybernetics explores a self-regulating cybernetic system, in which the geophysical system is the control object, and the role of the controller is given to human society as a whole. The geophysical system is defined as a set of objects of inanimate nature [2]. Thus, the main components of the earth's climate system, the atmosphere and ocean, can be considered as subsystems of the geophysical system. Geophysical cybernetics, as an interdisciplinary research area, it is currently evolving on the basis of ideas and methods of optimal control theory, dynamical systems theory, technical cybernetics, geophysics, and other academic disciplines.

This paper has two main objectives. The first objective is to present a general formulation of the problem of optimal control of the geophysical system considering its uniqueness as a physical object with a number of specific features. The second objective is to consider the control problem of large-scale dynamical processes in the atmosphere and to derive the necessary optimality conditions for this problem.

2 Geophysical System as a Physical and Control Object

In the beginning of this section let us make the following definition: The geophysical system is an interactive system consisting of four major components: the atmosphere, the hydrosphere, the cryosphere, and the land surface, influenced by various external forcing factors.

The geophysical system is very difficult to control because it is a unique physical object, which possesses a number of specific features:

- The geophysical system is a complex interactive system with numerous positive and negative feedback mechanisms. Various physical and chemical interaction processes occur among the different components of the geophysical system on a wide range of space and time scales, making the system enormously complex;
- The components of the geophysical system are very different in their physical and chemical properties, structure and dynamics; they are linked together by fluxes of momentum, mass and energy;
- The geophysical system is also an open system but its impact on the external environment is negligible;
- Time scales of the physical processes occurring in the geophysical system vary over a wide range - from seconds (turbulent fluctuations) to tens and hundreds of years;
- Since the geophysical system is a global system its spatial spectrum of motions covers molecular to planetary scales;
- Geophysical processes oscillate due to both internal factors (natural oscillations) and external forcing (forced oscillations). Natural oscillations are due to internal instability of the geophysical system with respect to stochastic infinitesimal disturbances. Anthropogenic impacts on geophysical system, both intentional and unintentional, belong to the category of external forcing;
- Geophysical dynamical processes are strongly nonlinear and chaotic [3].

Certainly, geophysical system has a number of other specific features that make it a unique physical object, which is virtually impossible to study using laboratory simulations (with rare exceptions). Therefore, the main method of studying the geophysical system is mathematical modeling. It is important to note that the atmosphere is the most unstable and rapidly changing component of the geophysical system.

With respect to the problem of control of the geophysical system, it should be emphasized that the application of cybernetic approaches and techniques developed for the study and optimal control of systems in many scientific areas, ranging from engineering and sciences to economics and social sciences, is very difficult. This is due to the following factors:

- Geophysical processes are not sufficiently well identified as control objects; their mathematical models are insufficiently perfect, accurate and adequate;
- The geophysical system refers to a class of distributed parameter systems described by partial differential equations, making the mathematical models of geophysical processes quite complex. Synthesis of control systems of such objects requires the development of control theory, which was created mainly for objects (systems) with lumped parameters (e.g. [4-9]).

The formulation of optimal control problems includes the following: (a) a mathematical model of the geophysical system that describes its behavior under the influence of control actions and external forcing (disturbances); (b) a specification of the control objectives; (c) a control model that imposes constrains on the controls and the state variables of the geophysical system; and (d) specifications of boundary and initial conditions for the model equations.

In our previous monograph [10] an optimal control problem for the geophysical system has been considered in a conceptual probabilistic manner. In practice, a probabilistic approach can be used for the development of control strategies on the assumption that the geophysical system is in a steady state. However, geophysical flows and processes occurring in the geophysical system and its components are non-stationary. For this reason, deterministic mathematical models are mainly used for numerical modeling and prediction of the dynamics and evolution of the geophysical system and its processes (e.g. [11-15]).

3 The Geophysical System: a Non-Linear Dissipative Dynamical System

3.1 Preliminary notes

Dynamical systems theory represents a very powerful framework for developing mathematical models of the geophysical system and analyzing and exploring time evolution of its components and processes [16]. In the most general sense, a hypothetical dynamical system can be formally specified by its state vector the coordinates of which (state or dynamic variables) characterize exactly the state of a system at any moment, and a well-defined function (i.e. rule), which describes, given the current state, the evolution with time of state variables. There are two kinds of dynamical system: continuous time and discrete time. Continuous-time dynamical systems are commonly specified by a set of ordinary or partial differential equations and the problem of the evolution of state variables in time is then considered as an initial value problem. However, with respect to the geophysical system simulations, discrete-time deterministic systems are of particular interest, because, commonly, the solution of differential equations describing the evolution of the geophysical system and its subsystems can be only obtained numerically.

Suppose that the current state of a hypothetical dynamical system is defined by the set of *n* real variables $q_1, q_2, \dots, q_n$. The number *n* is referred as the dimension of the system. A particular state $q = (q_1, q_2, \dots, q_n)$ corresponds to a point in an *n*-dimensional space $Q \subseteq \mathbb{R}^n$, the so-called phase space of the system. Let $t_m \in \mathbb{Z}_+$ ($m = 0,1,2,\dots$) be the discrete time, and $f = (f_1, f_2, \dots, f_n)$ is a smooth vector-valued function defined in the phase space domain $Q \subseteq \mathbb{R}^n$. The function f describes the evolution of the system state from the moment of time $t_0 = 0$ to the state of the system at the moment of time t_m such that $f : Q \to Q$. Thus, a deterministic dynamical system with discrete time is specified by the following equations:

$$q(t_{m+1}) = f(q(t_m)), \; q(t_0) = q_0, \\ q = (q_1, q_2, \dots, q_n), \; m = 0,1,2,\dots \tag{1}$$

It is obvious, that the evolution operator f satisfies the following semigroup property: $f^{m+s} = f^m \cdot f^s$. Therefore the system state $q(t_m)$ at time t_m can be explicitly expressed through the initial conditions q_0

$$q(t_m) = f^m(u_0), \tag{2}$$

where $f^m(q_0)$ denotes a *m*-fold application of f to q_0. The sequence $\{q(t_m)\}_{m=0}^{\infty}$ is an orbit (trajectory) of the system (1) in its phase space $q \in Q$, which is uniquely defined by the initial values of state variables q_0 (initial conditions). The vector-function *f* is a nonlinear function of state variables. This nonlinearity arises from the numerous feedbacks existed between the different components of the geophysical system, external forcing caused by natural and anthropogenic processes, and the chaotic nature of dynamical processes occurring in geospheres. It is usually assumed that the solution to the system (1) is deterministic, i.e. q exists and is unique at a given q_0.

Mathematical models of the geophysical system belong to the class of dynamical dissipative systems. It is very important if we are interested in climate manipulations since dissipative systems, such as the atmosphere and ocean, possess a global attractor, which is a certain set in the model phase space.

3.2 Dynamical model of the geophysical system

Mathematical models of the geophysical system used for variety of purposes represent systems of partial differential equations based on fundamental laws of physics, chemistry and fluid dynamics. Model equations also take into consideration the specific properties of geophysical system as well as its cycles such the water cycle, the carbon cycle, and the nitrogen cycle. The model equations can be solved only numerically and, consequently, provide a discrete in space and time solution.

Let $\mathcal{D}_t = \mathcal{D} \times [0,T]$ be a bounded space-time domain in which the geophysical system is characterized by the state vector $\boldsymbol{\varphi}(\mathbf{r},t) \in \mathcal{Q}(\mathcal{D}_t)$, where $\mathcal{Q}(\mathcal{D}_t)$ is the infinite real space of sufficiently smooth state functions satisfying some problem-specific boundary conditions at the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$, $\mathbf{r} \in \mathcal{D} \subset \mathbb{R}^3$ is a vector of spatial variables and $t \in [0,T] = \mathcal{T} \subset \mathbb{R}^+$ is the time. The domain $\mathcal{D}$ could represent the earth's sphere, hemisphere or another limited area on the earth's surface. In general operator form the evolution of geophysical system in the domain $\mathcal{D}_t$ can be written as follows

$$\frac{\partial \boldsymbol{\varphi}(\mathbf{r},t)}{\partial t} = \mathcal{L}(\boldsymbol{\varphi}(\mathbf{r},t), \boldsymbol{\alpha}(\mathbf{r},t)), \\ \boldsymbol{\varphi}(\mathbf{r},t)\big|_{t=0} = \boldsymbol{\varphi}_0(\mathbf{r}), \tag{3}$$

where $\mathcal{L}$ is a nonlinear model operator, $\boldsymbol{\alpha}$ is the model parameter vector, and $\boldsymbol{\varphi}_0$ is a given vector-valued function defining the initial state estimate.

The space-time spectrum of the processes occurring in the geophysical system is extremely wide. Hence, state-of-the-art mathematical models are unable to realistically simulate all of these processes. Let τ be a characteristic time-scale of a certain geophysical process. In this case, an explicit description of the processes with time scales smaller than τ is altered on a parametric representation (i.e. small scale processes are parameterized). Some of these parameters can be considered as control variables within the cybernetic framework. So, by varying the control parameters, we can formally control the dynamics of geophysical system.

The state vector $\boldsymbol{\varphi}(\mathbf{r},t)$ has an infinite dimension because the geophysical system is a continuous medium. However, in the control theory, state vector is usually finite-dimensional. In order to obtain a system which contains a finite number of degrees of freedom, and then to use the methods of control theory developed for lumped systems, the equation (3) is projected into the subspace spanned by the orthogonal basis $\{\psi_i\}_{i=1}^{n}$ so that $\boldsymbol{\varphi}(\mathbf{r},t)$ can be represented as a normally convergent series

$$\boldsymbol{\varphi}(\mathbf{r},t)=\sum_{i=1}^{n}x_i(t)\psi_i(\mathbf{r}). \tag{4}$$

Substituting (4) into (3) and applying then the Galerkin-like procedure, one can obtain instead of equations (3) the lumped system that is described by the following set of ordinary differential equations

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{f}\left(\mathbf{x}(t),\mathbf{u}(t)\right),\ t\in\mathcal{T} \tag{5}$$

with the initial conditions

$$\mathbf{x}(t)\big|_{t=0}=\mathbf{x}_0, \tag{6}$$

where $\mathbf{x}\in X\subset\mathbb{R}^n$ is the state vector the components of which belong to the class of continuously differentiable functions $\mathcal{C}^1(\mathcal{T})$, $\mathbf{u}\in U\subset\mathbb{R}^m$ is a control vector the components of which belong to the class of piecewise continuous functions $\hat{\mathcal{C}}(\mathcal{T})$, and $\mathbf{f}\in\mathbb{R}^n$ is a nonlinear vector-valued function defined in the domain $X\times U\times\mathcal{T}$ that is continuous with respect to both **u** and **x**, continuously differentiable with respect to **x,** as well as piecewise continuous with respect to *t*, such that $\mathbf{f}:X\times U\times\mathcal{T}\rightarrow X$, and $\mathbf{x}_0$ is a given vector-valued function.

In the right-hand side of the equations (5), only control parameters are presented, while the uncontrolled parameters are omitted. We assume that control parameters depend on the state of the system, i.e. $\mathbf{u}(t)=\mathbf{g}(t,\mathbf{x}(t))$, which implies that equations (5) describe a closed-loop control system, representing the geophysical system.

Finite-difference and spectral methods are mainly used to solve numerically the equations (5).

4 Formulation of the Control Problem

It is important to note that the problem of optimal control of the geophysical system remains poorly studied both scientifically and technically due to its relative novelty and enormous complexity:

- Geophysical processes as spatially distributed objects require spatially distributed control actions. However, implementation of such controls is very poorly developed;
- Geophysical processes possess enormous energy potential. Implementation of comparable energy control actions is a very difficult issue. Therefore, the identification of sensitive points, which can be manipulated to produce the desired result, is a critical problem;
- The scale and the huge energy of geophysical processes impose very strict requirements for accuracy and reliability of control systems; even minor errors in the control can be disastrous;
- Processes occurring in the geophysical system are interconnected, so the changes in the dynamics of some processes can lead to uncontrollable consequences. This should be taken into account in the development of control systems for geophysical processes.

The inherent features of geophysical processes provide possible ways to control them (see [10] for more details). The nature itself provides also the ability to design physical foundations for control of geophysical processes based on the existing natural physical mechanisms that influence the behavior of geophysical system.

Let's assume the system (5) is controllable and control parameters belong to a set of admissible controls $\mathbf{u}\in\mathcal{U}\subset U$. It is important that the set of admissible controls $\mathcal{U}$ is defined on the basis of physical and technical feasibility taking into account the above-mentioned specific properties of the geophysical system as a control object. Further, suppose that controls belong to the class of piecewise continuous functions with values in U or Lebesgue measurable functions with values in *U*, then, according to the the classical Caratheodory's theorem [17], one can prove that the Cauchy problem (5) has a unique solution defined on a time interval in $\mathbb{R}^+$. However, we cannot a priori determine whether the geophysical system is controllable or not. Conclusion concerning controllability of the system can only be made by solving a specific problem.

The main objective of the problem considered is to synthesize the control law that ensures the achievement of the desired results. Since these results are expressed in terms of extremal problem, we are specifically interested in synthesis of an optimal control. Formally, the optimal control problem for the geophysical system is formulated as follows:

Find the control trajectory

$$\mathbf{u}^*:\ [0,T] \to \mathcal{U} \subset U \qquad (7)$$

and the system state orbit **x*** generated by **u***

$$\mathbf{x}^*:\ [0,T] \to \mathcal{X} \subset X \qquad (8)$$

such that the given objective functional (performance index) $\mathcal{J}$ reaches its extremum (minimum or maximum)

$$\mathcal{J}(\mathbf{x},\mathbf{u}) = \Phi_0\left(t_0, \mathbf{x}(t_0), T, \mathbf{x}(T)\right) + \int_0^T \Phi\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt \to extr, \qquad (9)$$

with an integrand $\Phi:\ X \times U \to \mathbb{R}$. The problem (7)-(9) includes a set X at which the functional $\mathcal{J}$ is defined, and constraints on the model state given by the subset $\mathcal{X}$ of a set X. Remind that the dynamic constraints are given by the equations (5).

The functional (9) is a classical cost functional, which is commonly used in the optimal control theory, and corresponds to the Bolza problem. Problems of Mayer and Lagrange are special cases of the Bolza problem. The formulation of an objective functional depends on the problem under consideration and there are no universal approaches how it can be specified.

The Hamiltonian function $H:\ \mathcal{X} \times \mathcal{U} \times \mathbb{R}^n \times \mathcal{T}$ associated with the optimal control problem (7)-(9) can be written in the following form

$$H\left(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t\right) = \mathcal{F}\left(\mathbf{x}(t), \mathbf{u}(t), t\right) + \boldsymbol{\lambda}^{\mathrm{T}}(t)\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^n$ is the costate vector.

There are several methods available for solving the problem (7)-(9): classical methods of the variational calculus, dynamical programming, the Pontryagin's maximum principle and other methods.

5 Stabilizing the Geophysical System around the Reference Orbit

Stabilization of the geophysical system around the reference trajectory in phase space represents a specific class of optimal control problems relevant to the control of geophysical processes. Stabilization of an orbit of the climate system to weaken the global warming is one of the examples of this class of problems. For this particular class of problems, the differential equations (5) describing the evolution of the geophysical system is linearized with respect to the natural (reference) trajectory $\mathbf{x}^e(t)$ caused by external natural unperturbed forcing $\mathbf{u}^e(t)$:

$$\frac{d\delta\mathbf{x}(t)}{dt} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}^* \\ \mathbf{u}=\mathbf{u}^*}} \cdot \delta\mathbf{x}(t) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}^* \\ \mathbf{u}=\mathbf{u}^*}} \cdot \delta\mathbf{u}(t)$$

with the initial conditions

$$\delta\mathbf{x}(t)\big|_{t=0} = 0,$$

where $\delta\mathbf{x}$ is the perturbation of natural trajectory of the geophysical system due to anthropogenic disturbances, $\delta\mathbf{u}$ is a control vector to ensure the stabilization of the natural trajectory, $\partial\mathbf{f}/\partial\mathbf{x}$ and $\partial\mathbf{f}/\partial\mathbf{u}$ are the Jacobian matrices. It is assumed that

$$\mathbf{u} = \mathbf{u}^e + \delta\mathbf{u}, \quad |\delta\mathbf{u}| << |\mathbf{u}^e|,$$

$$\mathbf{x} = \mathbf{x}^e + \delta\mathbf{x}, \quad |\delta\mathbf{x}| << |\mathbf{x}^e|.$$

Then we consider the following optimal control problem:

Find the control vector

$$\delta\mathbf{u}^*(t) \in \mathcal{U}$$

generating the correction of the natural orbit

$$\delta\mathbf{x}^*, \ \ \mathbf{x}^e + \delta\mathbf{x}^* \in \mathcal{X} \subset X$$

such that the cost functional $\mathcal{J}$ is minimized

$$\delta\mathbf{u}^* = \arg\min_{\delta\mathbf{u}\in\mathcal{U}} \mathcal{J}\left(\delta\mathbf{x}, \delta\mathbf{u}\right),$$

$$\mathcal{J}(\delta\mathbf{x}, \delta\mathbf{u}) = \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}}(T)\mathbf{G}\delta\mathbf{x}(T) + \frac{1}{2}\int_0^T \left[\delta\mathbf{x}^{\mathrm{T}}(t)\mathbf{W}\delta\mathbf{x}(t) + \delta\mathbf{u}^{\mathrm{T}}(t)\mathbf{Q}\delta\mathbf{u}(t)\right] dt,$$

where **W**(t) and **G** are weighting positive semi-definite $n \times n$ matrices, normalizing the energy of geophysical system per unit mass, **Q**(t) is a weighting positive define $m \times m$ matrix, normalizing the energy of control actions per unit mass.

The stabilization problem is solved, given the fact that the system travels along its natural trajectory that is subject to external natural forcing. The control goal is to keep $\delta\mathbf{x}(t)$ close to zero using control actions $\delta\mathbf{u}(t)$. The information on the geophysical system state $\mathbf{x}(t)$ is obtained by measurement devices and instruments followed by the processing of these information within the data assimilation system [18].

6 Control of Large-Scale Dynamics of the Atmosphere

In this section, we will consider the problem of optimal control of large-scale dynamics of the atmosphere focusing on extratropical wave motions having a planetary length scale. These quasi-barotropic atmospheric waves propagate quasi-horizontally and, therefore, their behavior can be simulated using the equivalent barotropic model of the atmosphere [19].

6.1 Atmospheric model

As a starting point, let us consider the vorticity equation written in normalized isobaric coordinates (x, y, ξ) in the following form [20]

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial(\zeta + f)}{\partial x} + v\frac{\partial(\zeta + f)}{\partial y} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \quad (10)$$

where u and v are the components of horizontal velocity vector, f is the Coriolis parameter, ζ is the vertical components of relative vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

The equation of vorticity (10) is derived from the horizontal momentum equations and may be rewritten as

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial(\zeta + f)}{\partial x} + v\frac{\partial(\zeta + f)}{\partial y} = \frac{f}{P}\frac{\partial \omega}{\partial \xi}, \quad (11)$$

where we have used the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{P}\frac{\partial \omega}{\partial \xi} = 0.$$

Here $\omega \equiv dp/dt$ is pressure vertical velocity, where p is pressure, and P is a pressure approximately equal to the surface pressure. At the so-called equivalent barotropic atmospheric level $\xi^* \approx 0.5$ the vorticity equation may be expressed in the form [19]

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial(\zeta + f)}{\partial x} + v\frac{\partial(\zeta + f)}{\partial y} = \frac{fA}{P}\omega_0, \quad (12)$$

where $A(\xi)$ is an empirical function used to describe the vertical variation of the horizontal wind speed, ω_0 is a pressure vertical velocity at the level $\xi = 1$ (i.e. at the earth's surface $z = 0$).

The large-scale atmospheric motions are quasi-geostrophic, therefore

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

where $\psi = \Phi/f$ is the geostrophic streamfunction, and Φ is the geopotential. Rewriting the vorticity equation (12) in terms streamfunction yields

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + f\right) = \frac{fA}{P}\omega_0, \quad (13)$$

where

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

is a Jacobian. By the definition [20]

$$\omega_0 = \frac{dp_0}{dt} = \frac{\partial p_0}{\partial t} + u_0\frac{\partial p_0}{\partial x} + v_0\frac{\partial p_0}{\partial y} - \rho_0 g w_0,$$

where $w_0 = dz/dt$ is the vertical velocity at the level $z = 0$ ($\xi = 1$), ρ_0 is the air density and g is the gravity acceleration. The subscript 0 refers to the level $z = 0$. On the lower boundary the kinematic boundary condition requires that $w_0 = 0$. Further, if the geostrophic approximation is used, it follows that $u_0(\partial p_0/\partial x) + v_0(\partial p_0/\partial y) = 0$. With these two conditions the value of ω_0 reduces to

$$\omega_0 = \partial p_0/\partial t.$$

Substitution of this expression into (13) gives:

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + f\right) = \frac{fA}{P}\frac{\partial p_0}{\partial t}. \quad (14)$$

Suppose that there is a correlation between $\partial p_0/\partial t$ and $\partial\Phi/\partial t$, where Φ is the geopotential at the level ξ^*, i.e. $\partial p_0/\partial t = R_c(\partial\Phi/\partial t)$, where R_c is a correlation coefficient. Since

$$\psi = \Phi/f$$

then

$$\partial p_0/\partial t = R_c f(\partial\psi/\partial t)$$

and the equation (14) becomes

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + f\right) = \frac{AR_c f^2}{P}\frac{\partial \psi}{\partial t}.$$

Coefficient $AR_c f^2/P$ has the dimension of an inverse square length. Denoting this coefficient by $1/L_0^2$, we obtain the following Helmholtz-type equation with respect to variable $\partial\psi/\partial t$:

$$\nabla^2\frac{\partial \psi}{\partial t} - \frac{1}{L_0^2}\frac{\partial \psi}{\partial t} = F, \quad (15)$$

where

$$F = -J\left(\psi, \nabla^2\psi + f\right).$$

When the equation (12) is equal to zero on the right-hand side we refer to it as a barotropic vorticity equation. Thus, (12) is a forced barotropic equation in which the forcing term is determined by the vertical motion at the lower boundary ω_0. It is known [18] that the natural forcing of large-scale waves in the atmosphere is mainly of orographic or thermal origin. These two effects are formally described by the right-hand term in the vorticity equation (12), i.e. by the vertical velocity ω_0. Thus, the variable ω_0 can be chosen as a control variable.

6.2 Optimal control problem formulation

Let Ω be the domain on the *xy* – plane bounded by Γ. We will consider in Ω the dynamics of large-scale atmospheric waves. Hence, the horizontal domain of the model covers only part of the earth. Let $[t_0, T]$ be a time domain on which the solution of equation (15) is defined. Let the boundary curve Γ be represented in parametric form by the equations

$$x = \gamma_1(\sigma), \quad y = \gamma_2(\sigma),$$

where γ_1 and γ_2 are piecewise continuous functions of the parameter σ differentiable on the intervals where they are continuous. Assuming the coordinate origin coincides with the North Pole, we shall obtain

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \qquad (16)$$

where (r, ϑ) are polar coordinates. We set the following conditions on the boundary Γ:

$$\left.\frac{\partial\psi}{\partial t}\right|_{\Gamma} = 0,$$

then we get the Dirichlet problem for the Helmholtz equation (15):

$$\begin{cases} \nabla^2 \dfrac{\partial\psi}{\partial t} - \dfrac{1}{L_0^2}\dfrac{\partial\psi}{\partial t} = F & \text{in } \Omega\times[t_0, T], \\ \dfrac{\partial\psi}{\partial t} = 0 & \text{on } \Gamma\times[t_0, T], \\ \psi(x, y, t_0) = \psi_0(x, y) & \text{at } t = t_0. \end{cases} \qquad (17)$$

Solving the equation (15), we get the tendency $\partial\psi/\partial t$ at time t. Then we can extrapolate ψ in time using a forward difference

$$\psi(t+\delta t) = \psi(t) + \delta t\left(\frac{\partial\psi}{\partial t}\right), \qquad (18)$$

where δt is the integration time step. Multiplying both sides of the forecasting equation and the boundary conditions by δt, we can rewrite the problem (17) on the time interval $[t_0, t_0+\delta t]$ as follows:

$$\begin{cases} \nabla^2 z - \dfrac{1}{L_0^2} z = G & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma, \\ \psi(x, y, t_0) = \psi_0(x, y) & \text{at } t = t_0, \end{cases} \qquad (19)$$

where $z = \delta t(\partial\psi/\partial t)$ and $G = F\delta t$. After solving the problem (19), we obtain the streamfunction ψ at the next time step, i.e. the forecast of geopotential field Φ, which characterizes the dynamics of large-scale atmospheric waves. Note that the forward difference (18) can be used only on the first time step and then on the next time steps a central difference

$$\psi(t-\delta t) = \psi(t+\delta t) + 2\delta t\left(\frac{\partial\psi}{\partial t}\right)$$

can be applied.

Consider the following control system on the time interval $[t_0, t_0+\delta t]$ with the state variable z and the control variable U:

$$\nabla^2 z - \frac{1}{L_0^2} z + U = G \text{ in } \Omega, \; z = 0 \text{ on } \Gamma. \qquad (20)$$

The right-hand side term G in this equation is calculated using the geostrophic streamfunction at the initial time t_0

$$G \equiv G_0 = -J\left(\psi_0, \nabla^2\psi_0 + f\right)\cdot\delta t$$

where

$$\psi_0 = \Phi_0 / f.$$

Let us assume that the control variable U satisfies the following condition:

$$U \in \mathcal{U},$$

where $\mathcal{U}$ is the set of all permissible controls. The control U, from a physical standpoint, is a measure of additional vertical velocity on the bottom of the atmosphere near the ground.

Let us introduce a cost functional:

$$\mathcal{J} = \frac{1}{2}\iint_{\Omega}\left[(z - z_0)^2 + \eta U^2\right] d\Omega, \qquad (21)$$

where the pair (z, U) satisfies the equation (20) and z_0 is the desired spatial distribution of z. The term ηU (with $\eta > 0$) is proportional to the consumed energy. The control problem is as follows: find the control $U^* \in \mathcal{U}$ generating the system state $z^* \in \mathcal{X}$ such that the cost functional (21) is minimized. It is important to note that $\mathcal{X}$ is a set of constraints on the state variable z.

The control problem (20) and (21), from a physical viewpoint, reflects the ability to manipulate phase speeds of large scale atmospheric waves by changing a vertical velocity on the lower boundary of the atmosphere. Thus, we assume that the system described by the Helmholtz equation (15) is controllable and control parameter belongs to the set of admissible controls $U \in \mathcal{U}$. The set $\mathcal{U}$ should be defined on the basis of physical and technical feasibility, taking into account the properties of the atmosphere as a physical object.

To solve the optimal control problem we should convert the equation (20) to a normal form by introducing new dependent and parametric variables [8, 21]. As a result we get the following equivalent system of partial differential equations:

$$\begin{cases} \dfrac{\partial z_1}{\partial x} - z_2 = 0, \\ \dfrac{\partial z_1}{\partial y} - z_3 = 0, \\ \dfrac{\partial z_2}{\partial x} - \chi_1 = 0, \\ \dfrac{\partial z_2}{\partial y} - \chi_2 = 0, \\ \dfrac{\partial z_3}{\partial x} - \chi_2 = 0, \\ \dfrac{\partial z_3}{\partial y} + \chi_1 + U - \dfrac{1}{L_0^2} z_1 - G = 0. \end{cases} \tag{22}$$

Here the new dependent variables z_2, z_3, χ_1 and χ_2 were introduced and the original z being denoted by z_1. The state variables now are z_1, z_3 and z_3, while χ_1 and χ_2 are the parametric variables, which play the role of additional controls. To obtain the necessary optimality conditions let us adjoin (22) to the functional (21) by introducing the unknown Lagrange multipliers λ_1, λ_2, λ_3, μ_1, μ_2 and μ_3

$$\mathcal{J} = \iint_\Omega \left\{ \frac{1}{2}\left[(z_1 - z_0)^2 + \eta U^2 \right] + \lambda_1 \left(z_2 - \frac{\partial z_1}{\partial x} \right) + \mu_1 \left(z_3 - \frac{\partial z_1}{\partial y} \right) + \lambda_2 \left(\chi_1 - \frac{\partial z_2}{\partial x} \right) + \mu_2 \left(\chi_2 - \frac{\partial z_2}{\partial y} \right) + \lambda_3 \left(\chi_2 - \frac{\partial z_3}{\partial x} \right) + \mu_3 \left(-\chi_1 - U + \frac{1}{L_0^2} z_1 + G - \frac{\partial z_3}{\partial y} \right) \right\} d\Omega.$$

Let us introduce the Hamiltonian function H

$$H = \frac{1}{2}\left[(z_1 - z_0)^2 + \eta U^2 \right] + \lambda_1 z_2 + \lambda_2 \chi_1 + \lambda_3 \chi_2 + \mu_1 z_3 + \mu_2 \chi_2 + \mu_3 \left(-\chi_1 - U + \frac{1}{L_0^2} z_1 + G \right). \tag{23}$$

The augmented cost functional $\mathcal{J}_A$ is then can be written as

$$\mathcal{J}_A = \iint_\Omega \left(H - \lambda_1 \frac{\partial z_1}{\partial x} - \lambda_2 \frac{\partial z_2}{\partial x} - \lambda_3 \frac{\partial z_3}{\partial x} - \mu_1 \frac{\partial z_1}{\partial y} - \mu_2 \frac{\partial z_2}{\partial y} - \mu_3 \frac{\partial z_3}{\partial y} \right) d\Omega.$$

By introducing vectors $z = (z_1, z_2, z_3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$, the functional $\mathcal{J}_A$ may be represented as

$$\mathcal{J}_A = \iint_\Omega \left(H - \lambda \frac{\partial z}{\partial x} - \mu \frac{\partial z}{\partial y} \right) d\Omega.$$

If we add and subtract the term

$$\left(\frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} \right) z$$

to the integrand of this equation, we shall obtain:

$$\mathcal{J}_A = \iint_\Omega \left[H + \left(\frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} \right) z \right] d\Omega - \iint_\Omega \left[\frac{\partial}{\partial x}(\lambda z) + \frac{\partial}{\partial y}(\mu z) \right] d\Omega. \tag{24}$$

By the use of Green's theorem, the second integral in (24) becomes a line integral, and then $\mathcal{J}_A$ takes the form

$$\mathcal{J}_A = \iint_\Omega \left[H + \left(\frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} \right) z \right] d\Omega + \int_\Gamma \left[\lambda \gamma_2'(\sigma) - \mu \gamma_1'(\sigma) \right] z \, d\sigma.$$

The first variation in $\mathcal{J}_A$ due to variations in the control and parametric variables is

$$\delta \mathcal{J}_A = \iint_\Omega \left[\left(\frac{\partial H}{\partial z} + \frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} \right) \delta z + \frac{\partial H}{\partial U_1} \delta U_1 \right] d\Omega + \int_\Gamma \left[\lambda \gamma_2'(\sigma) - \mu \gamma_1'(\sigma) \right] \delta z \, d\sigma, \tag{25}$$

where $U_1 = (U, \chi_1, \chi_2)$. We choose the functions λ and μ to cause the first term in parentheses in the double integral in (25) to vanish

$$\frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} = -\frac{\partial H}{\partial z}. \tag{26}$$

Since the function z_1 does not change along the boundary Γ ($z_1 = 0$), its variation $\delta z_1 = 0$ on Γ. For $z_2 = 0$ and $z_3 = 0$, which are not defined on the boundary Γ, we require the relation

$$\lambda \gamma_2'(\sigma) - \mu \gamma_1'(\sigma) = 0$$

to hold along Γ. Thus, taking this into account (16), we obtain the boundary conditions for the system (22)

$$\begin{cases} \lambda_2 \cos\vartheta + \mu_2 \sin\vartheta = 0, \\ \lambda_3 \cos\vartheta + \mu_3 \sin\vartheta = 0. \end{cases} \tag{27}$$

The first variation in $\mathcal{J}_A$ then becomes:

$$\delta \mathcal{J}_A = \iint_\Omega \left(\frac{\partial H}{\partial U_1} \delta U_1 \right) d\Omega. \tag{28}$$

Thus, a necessary condition for the extremality of $\mathcal{J}_A$ takes the form

$$\frac{\partial H}{\partial U_1} = 0 . \quad (29)$$

Equations (26) and (29) with the boundary conditions (19) and (27) represent the necessary conditions for optimum. For the Hamiltonian function H defined by the equation (23) the necessary conditions for an extremum of $\mathcal{J}_A$ are as follows:

$$\begin{cases} \frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = -(z_1 - z_0) - \frac{\mu_3}{L_0^2}, \\ \frac{\partial \lambda_2}{\partial x} + \frac{\partial \mu_2}{\partial y} = -\lambda_1, \\ \frac{\partial \lambda_3}{\partial x} + \frac{\partial \mu_3}{\partial y} = -\mu_1, \\ \eta U - \mu_3 = 0, \\ \lambda_2 - \mu_3 = 0, \\ \lambda_3 + \mu_2 = 0. \end{cases} \quad (30)$$

Thus, to find the control U that minimize the functional (21) we must solve the system of 12 partial differential equations (22) and (29) with 12 unknown variables: $z = (z_1, z_2, z_3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$, and $U_1 = (\chi_1, \chi_2, U)$, and with given boundary conditions:

$$\begin{cases} z_1|_\Gamma = 0, \\ \lambda_2|_\Gamma = \mu_2|_\Gamma = 0. \end{cases} \quad (31)$$

This problem is solved numerically. Let us consider a particular case of the optimal control problem (20) and (21) in which the constant η is equals to zero. In this case

$$\mu_3 = \lambda_2 = 0 ,$$

and the necessary conditions (30) for an extremum of $\mathcal{J}_A$ takes the form:

$$\begin{cases} \frac{\partial \lambda_1}{\partial x} + \frac{\partial \mu_1}{\partial y} = -(z_1 - z_0), \\ \frac{\partial \mu_2}{\partial y} = -\lambda_1, \\ \frac{\partial \lambda_3}{\partial x} = -\mu_1, \\ \lambda_3 + \mu_2 = 0. \end{cases} \quad (32)$$

Since $\lambda_3 = -\mu_2$, the second and third equations of this system are similar to the first two equations of the system (22). Thus, we might make the following assumptions [8, 20]:

$$\lambda_1 = \frac{z_3}{\alpha}, \quad \mu_1 = -\frac{z_2}{\alpha}, \quad \lambda_3 = -\mu_2 = \frac{z_1}{\alpha}, \quad (33)$$

where α is an arbitrary constant. These assumptions are compatible with the boundary conditions (31). If we substitute (33) into the first equation of the system (30) then we get:

$$\left(\frac{\partial^2 z_1}{\partial x \partial y} - \frac{\partial^2 z_1}{\partial x \partial y} \right) = -\alpha (z_1 - z_0),$$

from which $\alpha (z_1 - z_0) = 0$ in the whole domain Ω. Thus, the cost functional (21) can achieve its absolute minimum under the constraint (20). To find the optimal control U that minimizes the functional $\mathcal{J}_A$, we need to solve the equation

$$U = -\left(\nabla^2 - \frac{1}{L_0^2} \right) z_0 + G_0$$

in the domain Ω with the boundary condition

$$z_0|_\Gamma = 0 .$$

The initial conditions for streamfunction ψ_0 are used to calculate the function G_0.

It is known [20] that planetary-scale waves in the atmosphere propagate westward relative to the mean quasi-zonal flow, so that it is possible for them to be stationary (with respect to the surface) in a westerly atmospheric flow. If the optimal control objective is to achieve the stationarity of these waves by manipulating the lower boundary vertical velocity ω_0, i.e.

$$z_0 \equiv (\partial \psi / \partial t)\big|_{t=t_0} = 0 ,$$

then the control U is calculated as

$$U = G_0$$

in the domain Ω .

At first glance the results obtained seem almost obvious. However, in this paper we rigorously prove the existence of absolute extremum of the cost functional (21) under the constraint (22). Note that if various constraints are imposed on the control variable U, then the cost functional (21) may not achieve its absolute extremum, and then the expression for calculating the control variable U is not so obvious.

7 Conclusion

Global warming and its potential negative impacts on human society and natural ecosystems necessitate the development of methods for manipulation and control of the state and dynamics of the earth's climate system and its components. However, the problem of control of geophysical processes and earth's climate system as a whole is extremely difficult and multidisciplinary, which requires the consideration of physical, technical, ethical and legal aspects and limitations.

This paper provides a basis for further research in the field of geophysical cybernetics that is a new research area of a self-regulating cybernetic system in which the geophysical system represents the control object and the human society plays the role of controller. Interest in climate manipulation and weather modification will likely continue to grow. In this context, the development of a theoretical framework for the optimal control of the earth's climate system seems as a problem of current interest. This paper lays the groundwork for further research in this area. In particular, the general formulation of the problem of optimal control of the geophysical system was considered, and the necessary optimality conditions were derived for the problem of optimal control of large-scale dynamics of the atmosphere.

References:

[1] IPCC 2913: *Climate Change 2013: The Physical Science Basis*, Cambridge University Press, Cambridge, 2013.

[2] R.M. Yusupov, *Theoretical bases of control of geophysical processes*, MHE, Moscow, 1979.

[3] E.N. Lorenz, Deterministic non-periodic flow, *Journal of the Atmospheric Sciences*, Vol. 20, 1963, pp. 130-141.

[4] R.E. Bellman, *Dynamic programming*, Princeton University Press, Princeton, N.J. 1957.

[5] W.H. Fleming, and R. Rishel, *Deterministic and stochastic optimal control*, Springer, Berlin, 1975.

[6] L.S. Pontryagin, *The mathematical theory of optimal processes*, CRC, Boca Raton, 1987.

[7] J.L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Pitman, London, 1982.

[8] K.A. Lurie, *Applied optimal control theory of distributed systems*, Springer-Verlag, New York, 1993.

[9] D.E. Kirk, *Optimal control theory: An introduction*, Dover, New York, 2004.

[10] R.M. Yusupov (Ed.), *An introduction to geophysical cybernetics and environmental monitoring*, St. Petersburg State University Press, St. Petersburg, 1998.

[11] S.A. Soldatenko, Some applications of the theory of optimal control of distributed parameter systems to weather numerical modeling. In: *Research Activities in Atmospheric and Oceanic Modelling*, Geneva, Switzerland: WMO, Vol. 28, 1999, pp. 7.29-7.30.

[12] M. Jacobson, *Fundamentals of atmospheric modeling*, Cambridge University Press, Cambridge, 1999.

[13] S. Griffies, *Fundamentals of ocean climate models*, Princeton University Press, Princeton N.J., 2004.

[14] W. Robinson, *Modeling dynamic climate systems*, Springer-Verlag, New York, 2001.

[15] K.E. Trenberth, *Climate system modeling*, Cambridge University Press, Cambridge, 2010.

[16] H.A. Dijkstra, Nonlinear climate dynamics, Cambridge University Press, Cambridge, 2013.

[17] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGrow-Hill, New York, 1955.

[18] R. Daley, *Atmospheric data analysis*, Cambridge University Press, Cambridge, 1991.

[19] G.J. Haltiner, *Numerical weather prediction*, Wiley & Sons, New York, 1971.

[20] J.R. Holton, *An introduction to dynamic meteorology*, 4th edition, Elsevier, London, 2004.

[21] J.-L. Armand, *Applications of the theory of optimal control of distributed parameter systems to structural optimization*, NASA CR-2044, Washington, D.C., 1972.