MathCAD solutions for problem of
laminar boundary-layer flow

DANIELA CÂRSTEA
Industrial High-School Group of Railways Transport, Craiova, ROMANIA
danacrst@yahoo.com

Abstract: -The problem of laminar boundary-layer flow past a flat plate is presented from the viewpoint of an engineer where the numerical results are of great interest. We compute the velocity profile in the boundary layer and other relevant physical parameters. We also compute the temperature distribution in the thermal boundary layer associated with the forced flow. We consider some algorithmic skeletons both for uncoupled fields and coupled fields. The natural coupling between the velocity field and thermal field can lead to sophisticated algorithms and these aspects are considered in our target example.

The non-linear partial differential equations of the laminar boundary-layer flow past a flat plate are transformed into a system of ordinary differential equations by using usual similarity transformations. This system is solved numerically using a software product as MathCAD. Blasius problem is presented from the computational viewpoint and direct transformation and inverse transformations are presented. In the study of the simplified model one encounters what is called a two-point boundary-value problem. Shooting method is effective for this case using functions in MathCAD software.

Keywords: - Boundary layer; Similarity solutions; Blasius equation; Numerical simulation; MathCAD.

1   Introduction

The physical systems can be described   by mathematical models defined by ordinary differential or partial derivative equations with initial and boundary conditions. The mathematical models are non-linear so the analytical solutions are not possible. The numerical solutions are single approaches for these systems.

Our work presents the laminar boundary-layer flow on a flat plate. The *x-y* coordinate system is chosen so that *x* is along the plate, and *y* is perpendicular to the plate. The leading edge of the plate is at *x* = 0.This example is of great importance in a wide range of areas including industrial and environmental applications so that the motivation for this example remains an open problem.

We consider the laminar fluid flow over a semi-infinite vertical flat plate, which is embedded in a fluid of constant ambient temperature. It is assumed that the wall (plate) is aligned parallel to a free stream velocity oriented in the upward or downward direction. Let us consider a rectangular Cartesian co-ordinate system with the origin fixed at the leading edge of the vertical surface with the x-axis directed upwards along the wall, and y-axis directed to normal to the surface.

We consider the following assumptions:
- Flow field unbounded in the direction of axis Oy
- No pressure variation in flow direction
- Fluid is incompressible and gravity effects are negligible in the thin boundary layer
- No-slip condition of the fluid particles in contact with plate surface

There is a large professional literature in the area of this subject so that we shall not present the physical considerations of the mathematical models. By using either analytical or numerical approaches, many authors reported remarkable results. Our goal is to present some numerical results using the similarity method in a coupled model.

The non-uniform distribution of temperature in medium can generate spatial variations in density and a natural convection occurs. The purpose of this work is to present numerical models for the laminar flow with different boundary conditions. The interest is to present some computational aspects generated by the problem complexity. The computation complexity is generated by the fact a two-point boundary-value problem appears. A numerical method for Caughy problem can be combined with a shooting method to obtain an approximate solutions.

It is proved by physical considerations that the fluid velocity and temperature field are coupled fields, that is, the fields mutually interact. This fact is a consequence of the dependence of the temperature distribution by the velocity distribution and dependence of the velocity components by the

temperature. An uncoupled analysis is a simplified and unreal approach so that our interest is for coupled problems.

The underlying principle of the flow solution by the similarity technique involves the transformation of the boundary value problem in an ordinary differential equation. This technique consists in the following steps:

- define a similarity variable, which is a function of spatial coordinates
- express the dependant variables as functions of the similarity variable
- reduce the boundary layer equations, which are partial differential equations, to ordinary differential equations in terms of functions of the similarity variable. The boundary conditions are in terms of the similarity variables.
- solve the ordinary differential equations
- post process the numerical solution by inverse transformations and scale factors
- display the graphical solutions as a fast interpretation of the solution from the engineer viewpoint

The mathematical model is the famous Blasius equation that describes the properties of steady-state two dimensional boundary layer. This is the target example of this work with emphasis on the computational aspects.

2 Mathematical modelling

The mathematical model is based on the boundary-layer theory described in extenso in the reference [3]. Boundary layer equations are parabolic in nature and the solution techniques are based on the laws of similarity for boundary layer flows. We consider the steady-state case. The assumptions in this case are:

- Away from the surface of the plate, viscous effects can be considered negligible and we have a potential flow (potential flow pressure gradient is zero)
- In the boundary layer, viscous effects cannot be neglected, and are as important as inertia
- The pressure variation can be calculated from the potential flow solution along the surface of the plate
- A temperature field exists only if there is a difference in temperature between the plate and external flow
- The fluid properties are constant (i.e. independent of temperature)

- The buoyancy forces are equal to zero

Under the mentioned assumptions, the mathematical model of the problem in the steady case is defined by a system of partial derivative equations. The governing equations of motion and temperature for the classical Blasius flat-plate flow problem can be summarized by the equations of continuity, momentum and energy are ([3]-[4]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (3)$$

These equations describe continuity, momentum transport and energy transport through the steady state boundary layer.

The significances of the variables from Eqs. (1) – (3) are:

- u, v – Darcy speed components in the directions Ox and Oy
- ν – is the kinematic viscosity
- λ –the equivalent thermal diffusivity calculated as the ratio of the thermal conductivity of the fluid, divided by the volumetric heat capacity ρc_p of the fluid alone
- $T(x,y)$- fluid temperature inside the thermal boundary in the point (x,y)

In practical applications the system (1)-(3) is solved subjected to boundary conditions. At the wall surface we can have one of the following boundary conditions:

- The wall temperature is uniform
- The wall temperature is nonuniform with a prescribed law
- The heat flux is uniform
- The heat flux is nonuniform with a prescribed law

We consider the case of an uniform temperature of the wall and for the flow there is no-slip at the wall.

The boundary conditions for velocity are as follows [5]:

$$u(x,0) = 0 \quad x \geq 0$$
$$v(x,0) = 0 \quad x \geq 0 \qquad (4)$$
$$u(x,y) \rightarrow U \quad as \quad y \rightarrow \infty$$

In the boundary layer the velocity of the fluid increases from zero to its full value that is U.

The boundary conditions for the temperature are ([7],[8]):

$$T(x,0) = T_w \quad x \geq 0$$
$$T(x,y) \to T_\infty \quad as \quad y \to \infty \quad (5)$$

The absence of a length scale suggests a similarity solution. Introducing a similarity variable η and a dimensionless stream function $f(\eta)$ we can reduce the system of PDEs (1)-(3) to a system of ordinary differential equations. We define the dimensionless similarity variables and direct transformations ([1],[3]):

$$\eta(x,y) = y\sqrt{\frac{U}{vx}} = \frac{y}{x}\text{Re}_x^{1/2} \quad (6)$$

$$\psi(x,y) = \sqrt{vxU}\,f(\eta) \quad (7)$$

$$\theta(\eta) = \frac{T - T_w}{T_\infty - T_w} \quad (8)$$

where:
- T_w - is the wall temperature;
- T_∞ - is the fluid temperature in the free stream;
- U - is the fluid velocity in the free stream;
- ψ – is the corresponding stream function for the flow;
- Re_x – is the Reynolds number
- f - is reduced stream function

The formula for Reynolds number is:

$$\text{Re}_x = \frac{U \cdot x}{v}$$

With the notation (7) we have:

$$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x} \quad (9)$$

The boundary conditions for the new function ψ are:

$$\frac{\partial \psi}{\partial y} = U \quad as \quad y \leftarrow \infty$$

$$\frac{\partial \psi}{\partial x} = 0; \quad \frac{\partial \psi}{\partial y} = 0 \quad at \quad y = 0$$

Now substituting equation equations (6)-(8) in equations (2)-(3) we obtain [1] :

$$f''' + \frac{1}{2}ff'' = 0 \quad (10)$$

$$\theta'' + \frac{1}{2}\text{Pr} \cdot f \cdot \theta' = 0 \quad (11)$$

where Pr is the Prandtl number and derivatives are with respect to η. The equation (10) is known as the Blasius equation and is a two-point boundary value problem. Its solutions offer the velocity profiles u and v and some relevant parameters of the flow.

The boundary conditions (4)-(5) turn into:

$$f(0) = 0; \quad f'(0) = 0; \quad \theta(0) = 0 \quad (12)$$

$$f'(\infty) = 1; \quad \theta(\infty) = 1 \quad (13)$$

In these mathematical models we identify some physical constraints that limit the space of the solutions. One of these physical constraints is:

$$0 \leq f'(\eta) \leq 1 \quad \forall \eta \in [0, \infty) \quad (14)$$

Consequently, we have two solution types:
- A mathematical solution for the problem with the boundary conditions (12)-(13)
- A physical solution for the problem with boundary condition (12)-(14)

It is obviously that the mathematical solution does not depend on physical solution, but from the engineer's viewpoint we must find the physical solution. The problem takes the form of a third-order two-point boundary value problem for f so that we must seek a numerical solution by an iterative technique.

In equation (10) the physical significances of the f and its derivatives are obviously: f has the significance of a reduced stream flux, f' is a profile of the velocity and f'' is the profile of the velocity gradient. We operate with these variables in the processing stage of a program but we must not forget that in post-processing stage we need physical quantities (engineering significances).

3   Physical variables

If we have the solutions for the equations (10)-(11), we can obtain the velocity components([1],[3]):

$$u = U \cdot f'; \quad v = \frac{1}{2}\sqrt{\frac{vU}{x}}(\eta \cdot f' - f) \quad (15)$$

In the boundary layer the velocity of the fluid increases from zero at the wall to its full value which corresponds to external frictionless flow.

As engineer, other physical variables are of interest as the shear stress at wall $\tau_w(0)$. We must compute the thickness of the boundary layer where the viscosity is very important. This thickness depends on the viscosity in the sense that the thickness of boundary layer increases with increasing viscosity.

In a boundary layer, the fluid asymptotically approaches the free-stream velocity. Conventionally, we define the edge of the boundary layer to be the point at which the fluid velocity equals 99% of the free-stream velocity U.

There is no general formula for the thickness. For a boundary layer the thickness δ can be approximated as :

$$\delta \approx \sqrt{\frac{\mu x}{\rho U}} = \sqrt{\frac{v x}{U}} \quad (16)$$

The unknown numerical factor remaining in this formula can be determined from the exact solution of Blasius.

Another interesting physical variable is the shear stress in the boundary layer, defined as [3]:

$$\tau = \mu \frac{\partial u}{\partial y} = Ct \cdot f''(\eta) \qquad (17)$$

where Ct is a constant that depends on the material properties because μ is a fluid property which is strongly dependent on the temperature and called the dynamic fluid viscosity. The value of constant Ct is:

$$Ct = -\rho U^2 \sqrt{\frac{\nu}{xU}}$$

A large velocity gradient across the flow creates important shear stress in the boundary layer. In all flows where friction forces act together with inertial forces, the ratio formed by the viscosity μ and the density ρ denoted kinematic viscosity ν in [m²/s].

The shear stress near the plate is [11]:

$$\tau_W = \mu \frac{\partial u}{\partial y}(x,0) = \frac{f''(0)\rho U^2 \sqrt{\nu}}{\sqrt{Ux}}$$

where f'(0) is from the numerical solution of Blasius equation. In our target example f'(0)≈0.332

Temperature in the boundary layer is computed with formula:

$$T = T_W + (T_\infty - T_W)\theta \qquad (18)$$

4   Numerical results

It is obviously that the mathematical model of the fluid flow is a set of partial derivatives differential equations which are difficult to be solved by analytical methods. In a similarity solution we have an ordinary differential equation with boundary conditions at the both terminals of the spatial domain. This is a bilocal problem that must be solved by transformation in a Caughy problem with initial conditions that are guessed. Boundary-value problems generally require an iterative procedure involving satisfaction of the boundary conditions when specified at different values of the independent variable.

A simple technique is the shooting method. In a shooting method, the missing (unspecified) initial condition at the initial point of the interval is assumed, and the differential equation is then integrated numerically as a Caughy problem. This is an iterative process. The accuracy of the solution depends on the accuracy of the assumed missing initial condition. The algorithm starts with a guess of a part of the initial conditions. If a difference exists between the value of the dependent variable at the terminal point with its given value there, another value of the missing initial condition must be assumed and the process is repeated. Hence, a method must be devised to estimate logically the new initial conditions for the next trial integration. Our target example there are two asymptotic boundary conditions and hence two unknown surface conditions f'(0) and θ'(0).

For this type of problems in cases in discussion, we identify two major difficulties in terms of numerical computing:

- One of the boundary conditions is specified to infinity
- There is no general method of estimating the unspecified conditions for associated Caughy problem

The solutions at these sub problems can influence the convergence of the solution and computer time. In the trial integration infinity is numerically approximated by some large value of the independent variable, or we consider a finite interval [0, η_{max}] where η_{max} can be estimated or computed. Selecting a large value may result in divergence of the trial integration or in slow convergence of the final boundary conditions. More, selecting too large value of the independent variable is expensive in terms of computer time. A small value may not allow the solution to asymptotically converge to the   exact solution. The problem is open.

4. 1. Coupled or uncoupled fields

For laminar free convection, the two equations (10) –(11) are coupled naturally via the temperature (θ) and must solved simultaneously. The procedure involves that two unspecified (missing) initial conditions must be found. An implicit coupling exists by the dependence of the material properties on the temperature.

In many works in this area the momentum equation (10) is solved independently of the energy equation, and then energy equation (11) is solved. Obviously there is a natural coupling between the momentum equation (2) and the energy equation (3) that can not be ignored. This is due to the dependence of the material properties on the temperature (as viscosity).

4.2.Computational algorithm

The idea of shooting method for our target example involves a guess for the unknown initial value f'(0),

that is, we choose a value s_0 for this condition f''(0)=s_0. Afterwards we must build subsequent corrections s_k, k≥1 so that the final boundary condition is fulfilled. The problem is to have a finite number of corrections and to have a good approximation of the final condition. The most important sub problem that appears in this approach is how do we use the error values of each iteration step to correct the guesses from the previous steps.

To develop an algorithm or to use the software from the professional literature in the area of differential equations, it is imperative to write the differential equation as an equivalent first-order system, with specified boundary conditions.

Momentum equation in uncoupled approach
In the case of incompressible flow (i.e. uniform density) and constant temperature, the kinematic viscosity ν does not depends on temperature (is a constant) and ρ is constant, also. The equation (10) can be solved independently.

The differential equation (10) must be reformulated as a system of differential equations of the first order. The reformulation as a first-order system is straightforward. For this we define the column vector y ={y0, y1, y2}={f, f', f''}. Equation (10) can easily be written as the equivalent first-order system

The equations system is as follows [5]:

$$y0' = y1$$
$$y1' = y2 \qquad (19)$$
$$y2' = -\frac{1}{2} y0 \cdot y2$$

with the boundary conditions:

$$y0(0) = 0 \qquad (20)$$
$$y1(0) = 0 \qquad (21)$$
$$y1(\infty) = 1 \qquad (22)$$

It is obviously that the system (19) with conditions (20)-(22) can not be solved by classic methods as Euler, Runge-Kutta etc. This two-point boundary-value problem must be re-written as a Caughy problem which requires to replace the final condition with an initial condition f''(0)=y2(0)=s. The value of s is guessed by the user and then the equation is approximated using variations of Euler's method, Runge-Kutta schemes etc. The real parameter *s* must be determined so that the conditions (22) are fulfilled.

The algorithm for the solution of the system (19) involves the following steps [13]:

1. *Select an initial value for f''(0)*

2. *Solve the system (19) as a Caughy system (an initial value problem)*
3. *From the solution of the system (19) the value of f''(η_{max}) is found*
4. *If the final value for f'' is not equal to 1, another guess for f''(0) is made and go to the step 2 (the entire procedure is repeated)*
5. *Continue iterative process with successively better guesses for f''(0) till value of f''(η_{max}) is close enough to 1*

The guess of η_{max} is a subsidiary important problem because the infinity can not be included in a computer program. Essentially in this algorithm is the step 4 because a trial and error approach must be implemented ("shooting method").

The practical algorithm for solving the system (19) is classical. It can be described in a pseudo-code as follows [2]:

1. *take initial guesses s_0 and s_1 arbitrarily in R*
2. *set the iteration counter k← 1*
3. *repeat*
   - *Compute $y2_{k-1} = y2(s_{k-1})$ and $y2_k = y2(s_k)$*
   - *Compute s_{k+1} with*
   - $$s_{k+1} = s_k + (s_k - s_{k-1}) \frac{t - y2_k}{y2_k - y2_{k-1}}$$
   - *k←k+1*
4. *until* **convergence**

The parameter **t** is the target, that is y1(∞)=1. The test of convergence is the accuracy of the computation and is specified initial data of the program. In this approach we use a linear interpolation for the control parameter *s*. Obviously, the initial guess of the control parameter has a great importance on the run-time of the program or the number of iterations.
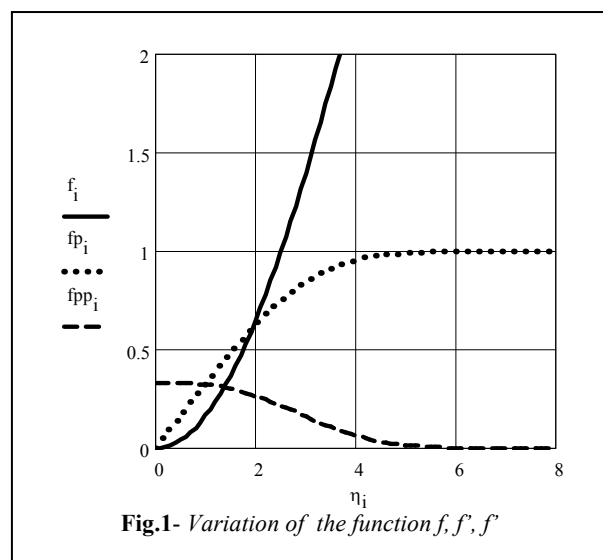

Fig.1- *Variation of the function f, f', f''*

A program in MathCAD for implementation of the above algorithm is presented in reference [2]. The number of iterations was 4 for a tolerance (accuracy) of 10^{-6}.

In Fig. 1 the variation of f, fp=f' and fpp=f'' are plotted for an interval [0,8]. In Fig. 2 some relevant physical quantities as u, v and τ are plotted. From this solution we see that there is a constant vertical velocity at the outer limit of the boundary layer. This is due to the retardation in the boundary layer, which in turn deflects some of the oncoming fluid upward. From formula (15) we see that the velocity component at the infinity is defined by the asymptotic value:

$$c = \eta \cdot f'(8) - f(8)$$

We find that c=1.721 and for a large Reynolds number we conclude that [11]

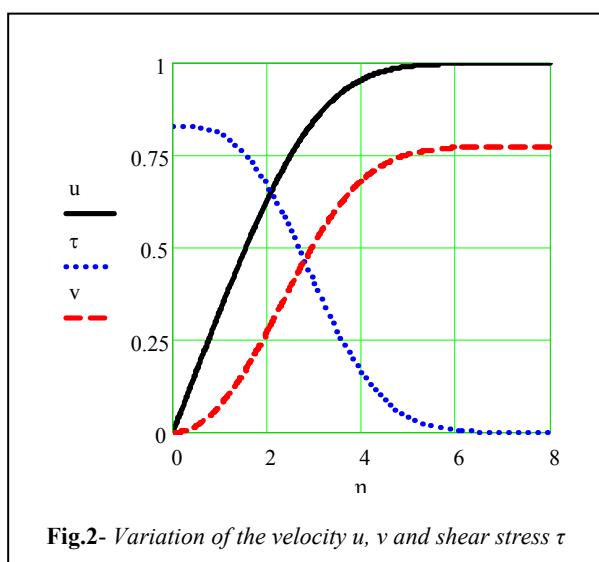$$v = \frac{1}{2}\sqrt{\frac{vU}{x}} . c = \frac{c}{2 \cdot \mathrm{Re}_x} \to 0$$

Fig.2- *Variation of the velocity u, v and shear stress τ*

Certainly, other algorithms for boundary-value problem (10) - (11) are possible and are described in a rich professional literature. For example, the sensitivity analysis of the target with respect of the control parameter *s* is one of them, but some of them can be successful for specific kinds of problems.

MathCAD functions for Blasius equation
The solution of the mathematical model (10)-(11) can be obtained using some standard MathCAD tools for manipulating the numerical solution of a differential equation [10]. The MathCAD solution to this problem involves using function *sbval* to find the value of f''(0) and then we use the high-order Runge-Kutta function *rkfixed* to integrate the equation as an initial-value problem. The function *rkfixed* offers the numerical results in a matrix z

with the first column with values of η and the following columns with values of the vector y.

We limit our discussion to the equation (10) in the formulation (19). This is solved independently. Thus, the MathCAD function *rkfixed* returns a matrix with the following entries: the first column contains the points at which the solution is evaluated, and the remaining columns contain the corresponding values of the solution and its first n-1 derivatives. In other words the matrix z is computed by [1]:

$$z = rkfixed(y,0,8,800,d)$$

where *d* is a n-element vector-valued function containing first derivatives of unknown functions, that is the right-hands from the system (19). The physical variables are obtained by the inverse transformations (15) and (17). In Fig. 2 the velocity components and shear stress are plotted for a particular fluid [2].
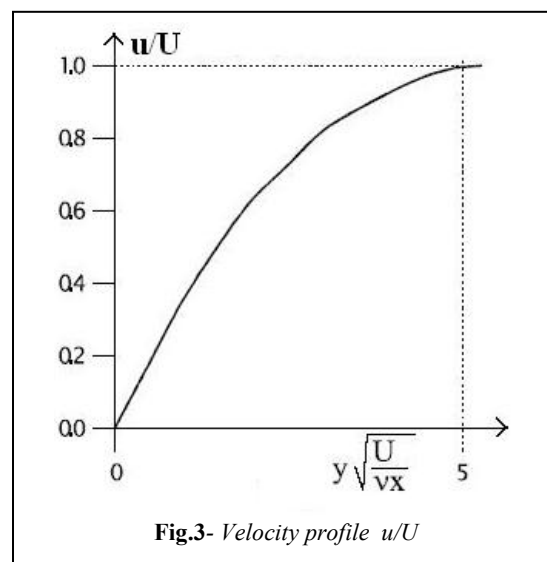

Fig.3- *Velocity profile u/U*

In the boundary layer the velocity of the fluid increases from zero at the wall to its full value equal to 1. The velocity reaches 99% of its bulk value when f'(η)=0.99. We can use a MathCAD function to find the value of η corresponding to this value of f' or the curve from figure 3. Then, the corresponding boundary layer thickness can be computed. By simulation program we can analyze the dependence of thickness of the boundary layer on the viscosity and finally on the fluid temperature. The thickness decreases with decreasing viscosity.

The analysis can be continued to obtain relevant parameters such as the displacement thickness, drag coefficient etc. These engineering quantities are presented in any textbook of the boundary layer flow. Some relevant physical quantities are presented as follows but the reader is invited to the references [3] and [4].

Boundary layer thickness

There is no accurate edge of the boundary layer. Rather it merges smoothly and gradually with the outer free stream. Many authors reported different definitions of the boundary layer thickness.

However, there are several ways of defining a boundary layer thickness, so that we select one of them. The idea is to determine where the horizontal velocity is "nearly" equal to the free stream velocity. The first thickness, which we call δ is the *y* value at which the velocity is 0.99 of the free stream velocity. The quantity δ is computed from the Blasius solution starting from the condition:

$$u = 0.99 \cdot U \quad or \quad f'(\eta) = 0.99$$

In other words we obtain η = 5 from the Blasius curve plotted in Fig. 3. Hence

$$\delta = \frac{5}{\sqrt{Re_x}} \cdot x$$

Other boundary layer thickness is the momentum thickness.

Displacement thickness

Instead of the geometric boundary layer thickness, the quantity δ_d is preferred which is a measure of displacing action of the boundary layer. This quantity is defined as

$$\delta_d = \int_0^\infty (1 - \frac{u}{U})dy = \frac{1.73}{\sqrt{Re_x}}x$$

Momentum thickness

A measure of the loss of momentum is the momentum thickness defined as:

$$\delta_m = \int_0^\infty (1 - \frac{u}{U}) \cdot \frac{u}{U} dy = \frac{1.33}{\sqrt{Re_x}}x$$

Local skin friction coefficient

This parameter is defined by formula:

$$c_{fx} = \frac{\tau_w(x)}{0.5\rho U^2} = \frac{0.66}{\sqrt{Re_x}}$$

Frictional force

The total frictional force per unit width on a plate of length L is a global parameter defined as

$$F = \int_0^L \tau_w dx$$

Energy equation in an uncoupled model

The two equations (10) –(11) can be solved in two approaches. The first approach is to consider the two equations as independent equations. In this approach we consider two stages:

- Firstly, we solve the equation (10) as a non-linear bi-local problem
- Secondly, we consider the values of f are known and we can integrate directly the equation (11)

The second approach involves a coupled problem where the coupling is implicitly and explicitly. An implicit coupling is due to the dependence of the material properties (as kinematic viscosity, density) on the temperature

The explicit coupling is due to the function f in the equation (11).

In an uncoupled approach, the algorithmic skeleton is as follows:

1. *Find the missing value of the f"(0) using the MathCAD function* **sbval**
2. *Solve the initial – value problem for Blasius equation and find the unknowns f, f' and f" using the function* **rkfixed**
3. *Solve equation (11) as a linear equation in θ because where f is Blasius function known from the step 2.*

Thermal boundary layer

The similarity solution of Blasius for the velocity in the boundary layer may be extended to include a calculation of the temperature distribution in the thermal boundary layer. We start by introducing a dimensionless variable θ of the temperature T(x,y) defined by the formula (8). The energy equation (3) becomes:

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \lambda\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right)$$

Using the assumptions from the boundary layer theory, we may drop the second *x*-derivative of θ compared with the second *y*-derivative. In this case, the analogy of the equation (2) with (equation (3) is obviously by

$$u \leftrightarrow T \quad v = \lambda$$

That is, the two equations become identical if T is replaced by u in equation (3) and the Prandtl number Pr=1. In other words, at small velocities the temperature and velocity distributions are identical :

$$\frac{T - T_w}{T_w - T_\infty} = \frac{u}{U}$$

Generally, by similarity transformations the equation (11) is obtained and must be solved for different Prandtl numbers. The solution of the equation (11) is obtained sequentially, that is: we

solve equation (10) and then equation (11) where f is known.

The similarity solution of equation (10) for the velocity in the boundary layer may be extended to include a calculation of the temperature distribution in the thermal boundary layer. We defined a dimensionless variable θ of the temperature T(x,y) as (8) where T_w is the wall temperature and T_∞ is the upstream temperature in fluid. By similarity transformation the equation (8) was obtained.

By considering the equation (8) as a first-order linear equation, we get

$$\frac{d\theta}{d\eta} = c \cdot e^{-\frac{\text{Pr}}{2}g(\eta)}; \quad g(\eta) = \int_0^\eta f(t)dt \quad (23)$$

with c is a constant that must be determined from the boundary conditions for θ.

Finally, from (23) we obtain:

$$\theta(\eta) = \frac{\int_0^\eta e^{-\frac{\text{Pr}}{2}g(t)}dt}{\int_0^\infty e^{-\frac{\text{Pr}}{2}g(t)}dt} \quad (24)$$

With the value of θ we can compute the local heat transfer coefficient h defined as:

$$h = k\sqrt{\frac{v}{Ux}}\frac{d\theta}{d\eta}(0) \quad (25)$$

We can compute the value of *h* at which the temperature has a value of 0.99 of the free stream temperature. With this value of *h* we can obtain the thickness of the thermal boundary layer relative to the thickness of the velocity boundary layer by dividing the value of *h* to 4.909. For Pr=1 we have θ= f'(η).

It can be proved that the thickness of the velocity boundary layer corresponds to a value of η=4.909. We can compute h using the equation (25).

A simplified approach can be developed by using the Blasius equation (10) for f(h). From equation (10) we see that

$$f = -\frac{2f'''}{f''}$$

or equivalently

$$f = \frac{d}{d\eta}(-2\ln f'')$$

In other words

$$g(\eta) = -2\ln(\frac{f''(\eta)}{f''(0)})$$

Thus, equation (24) becomes

$$\theta(\eta) = \frac{\int_0^\eta f''(s)^{\text{Pr}}ds}{\int_0^\infty f''(s)^{\text{Pr}}ds}$$

We can compute the thermal boundary layer by using this formula for θ. For practical interest we must compute the thermal boundary layer thickness as a function of Pr.

Energy equation in a coupled approach

In the same manner as for momentum equation, the similarity equation (11) for the temperature in the boundary layer can be solved. But we consider this case as a coupled model defined by the two equations (10)-(11).
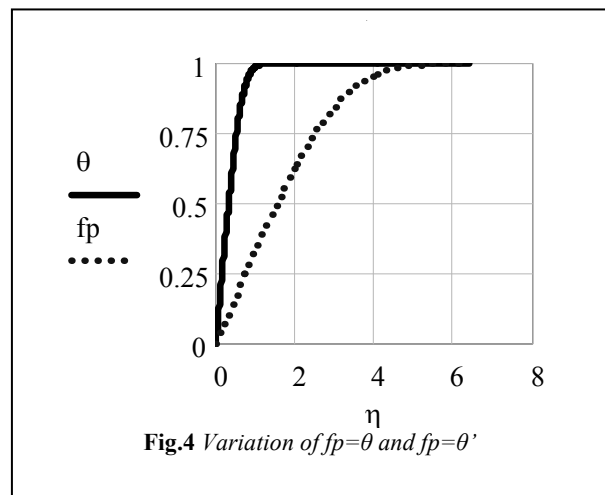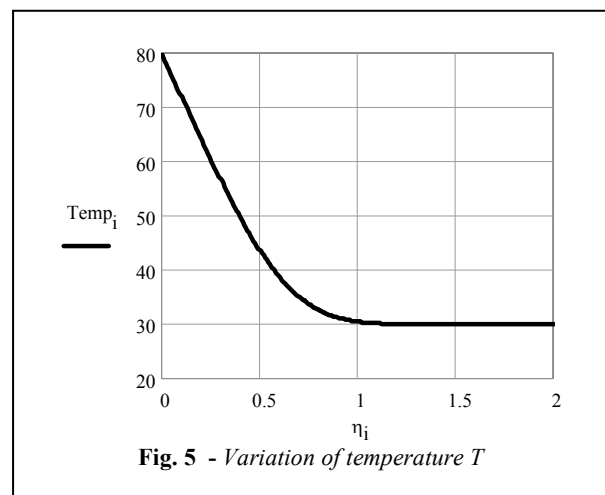

Fig.4 *Variation of fp=θ and fp=θ'*

Fig. 5 - *Variation of temperature T*

We define the column vector of the unknowns w={y0, y1, y, y3, y4}={f, f', f', θ, θ'}. In this case the vector d(x,y)={y1, y2, -0.5 y0.y2, y4, -0.5 Pr.y0.y4}. The Prandtl number Pr is specified. Using MathCAD built-in functions **sbval** and **rkfixed,** we obtain the values of the vector w. In this way we obtain the profiles of the velocity and temperature.

In Fig. 4 we have the graphical representations of θ and θ' for Pr=100 and the spatial interval [0,8].

For a wall temperature Tw=80 ^{0}C and T$_∞$=30 ^{0}C, the variation of the temperature in the boundary layer is plotted in Fig. 5 for Prandtl number Pr=100.

A version of the algorithm for coupled fields is starting with a guess for both missing initial conditions f'(0) and θ'(0). In other words we do not consider the momentum equation (10) independently. The procedure is the same as for uncoupled case except that two initial conditions must be found. A subsidiary problem appears: we must find the maximum value of η to capture the evolution of the two dependent variables f and θ. The solution at this sub problem is following: we guess an initial value for $η_{max}$ so that the system (10)-(11) has a solution. Then this guessed value is increased until the solution is asymptotic with $η_{max}$.

The first step is the transformation of the two equations (10) and (11) in a system of five first-order differential equations:

$$\frac{dy}{d\eta} = D(\eta, y) \qquad (26)$$

where the column vector D(η, y) has the following entries:

$D(\eta, y) = \{y1, y2, -0.5y0 \cdot y2, y4, -0.5 \cdot Pr \cdot y3 \cdot y4\}$

A coupling algorithm involves the following steps:

1. *Guess the missing value of the f''(0) and θ'(0) using the MathCAD function* **sbval**
2. *Solve the system (26) and find the unknowns f, f', f'', θ and θ' using the function* **rkfixed**
3. *Compute the physical variables by inverse transformations*

Unfortunately the algorithm has a drawback: the estimation of $η_{max}$ is a difficult task.

An algorithm based on the sensitivity analysis
The shooting method has a serious drawback: the "guess" of the initial conditions for the missing values of the dependent variables. This is art or science? I think that the experience, the knowledge database in the area of fluids mechanics can be a good start.

In the professional area of differential equations the sensitivity analysis of the objective function can be used in a shooting technique [9]. It is not the goal of this paper to present the general theory so that we limit our discussion to the case of the laminar boundary-layer flow with emphasis on the target example from this paper.

Let us define the objective function for the momentum equation (19):

$$F(s) = f'(\eta_{max}) - 1 = y1(\eta_{max}) - 1 \quad (27)$$

where s is the control parameter defined by y2(0)=s. The initial condition for f'(0) is unknown, and the boundary condition at $η_{max}$ is the control for this problem. We try to find the minimum of F(s) for s a real parameter; practically the minimum of F must be zero. It is obviously that this process involves an iterative procedure to construct a sequences s_k that converges to a zero of F. The Newton-Raphson method is most commonly used. The sequence $\{s_k\}$ can be constructed using shooting technique with Newton-Raphson method [12]:

$$s_{k+1} = s_k - F'(s_k)^{-1} F(s_k) \qquad (28)$$

where F(s$_k$) can be computed directly from the solution of the system (19) but F' requires the values of $\partial y2(\eta_{max}; s_k)/\partial s_k$. These values can be obtained by solving the system (19) together with the sensitivity equations:

$$y0'_s = y1_s$$
$$y1'_s = y2_s \qquad (29)$$
$$y2'_s = -\frac{1}{2}(y0_s \cdot y2 + y0 \cdot y2_s)$$

In (29) , for example

$$y1_s(\eta, s) = \frac{\partial y1(\eta, s)}{\partial s}$$

The initial conditions for the system (25) are:
$$y0_s(0) = 0; \quad y1_s(0) = 0; \quad y2_s(0) = 1$$

If the objective function is not sufficiently small, a new value for s is computed as

$$s_{k+1} = s_k - \frac{y1(\eta_{max}, s_k) - 1}{y1_s(\eta_{max}, s_k)}$$

5 Conclusions

In summary, we have used in this paper a numerical approach to give suitable solutions to classical boundary-layer problem in fluid mechanics. We presented some computational aspects in non-linear analysis of laminar-layer flow in a horizontal medium of infinite extent. The mathematical model were obtained from Navier-Stokes equations considering the boundary-layer approximation . The simplified model was obtained by some transformations so that a system of ordinary differential equations were obtained.

The nonlinear mathematical model of the problem prohibits the use of the analytical methods. A numerical solution is the single approach for these problems. The two-point boundary problem was solved by a Runge-Kutta method and shooting method. MathCAD functions make numerical

solution of the mathematical models of the fluid flow relatively simple and quick. MathCAD solutions are presented for Blasius equations with additional computations based on the numerical results obtained by the MathCAD function *rkfixed.* The MathCAD solution to this problem involves using function **SBVAL** to find the values of f'(0) and then θ'(0) using a conventional higher-order Runge-Kutta function, such as **RKFIXED** in MathCAD, to integrate the equations as a Caughy problem.

We limited our presentation to an iterative procedure for Blasius equation. Fortunately, it is possible to avoid the iteration by introducing a new dependent variable.

References:

[1]. Alam, M.S., Rahman, M.M., Samad, M.A. Numerical Study of the Combined Free-Forced Convection and Mass Transfer Flow Past a Vertical Porous Plate in a Porous Medium with Heat Generation and Thermal Diffusion. In: *Nonlinear Analysis: Modelling and Control, 2006, Vol. 11*, No. 4, 331–343

[2]. Kostic, M., "The famous Blasius Boundary-layer flow over the flat plate". http://www.ceet.niu.edu/faculty/kostic

[3]. Schlichting, H., Gersten, K., *Boundary layer theory.* Springer- Verlag, Berlin , 2000.

[4]. Schlichting., H. , Boundary layer theory. McGraw Hill, Inc., 1979

[5]. Aurel Chiriță, Horia Ene, Bogdan Niculescu, Ion Cârstea. **"**Numerical simulation of the free convection flow in porous medium". In Volume: *SYSTEM SCIENCE and SIMULATION in ENGINEERING.* Published by WSEAS Press, 2007. ISSN: 1790-5117; ISBN: 978-960-6766-14-5. Pg. 334-338.

[6]. Belchami, Z., Brighi,B., Taous,K. On a family of differential equations for boundary layer approximations in porous medium. In: *Prepublications Mathematiques. Departement de Mathematiques, no. 9/99.* Universite de Metz.

[7]. Belchami, Z., Brighi, B., Taous, K. On the concave solutions of the Blasius equation. In: *Acta Math. Univ. Comenianae, vol. LXIX, 2(2000).* Pp.199-214

[8]. Guedda, M., Hammouch, Z., On similarity and pseudo-similarity solutions of Falkner-Skan boundary layers. In: *Fluid Dynamics Research 38 (2006).* The Japan Society of Fluid Mechanics and Elsevier B.V.

[9]. Holsapple, R., et all. A modified simple shooting method for solving two-point boundary-value problems. In: *www. ScienceDirect.com*

[10].*** *MathCAD Professional 7. User's Manual.* MathSoft Inc. USA

[11].Gautam Biswas. *Course.* Department of Mechanical Engineering, Indian Institute of Technology Kanpur

[12].Meade, D., Haran, B., White,R. The shooting technique for the solution of two-point boundary value problems. University of South Carolina, Columbia, USA. In the web page: *http://www.math.sc.edu/~meade/papers/mtn-shoot.pdf*

[13].Hodge, B.K., The Use of MathCAD in a Convective Heat Transfer Course. In: *ASEE Southeast Section Conference 2004*