A new solution of Euler's equation of motion with explicit expression of helicity

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Abstract: - Improving the variational formulation for an ideal compressible fluid, a new expression of velocity field is presented by using vector potentials of frozen field, *i.e.* the potentials convected by the fluid flow under effect of stretching. This has been deduced from the action principle. It is verified that the system of new expressions in fact satisfies the Euler's equation of motion. The Lagrangian consists of main terms of total kinetic energy and internal energy, together with two terms yielding the equations of continuity and entropy and the third term which provides rotational component of velocity field. The last term leads to an explicit expression of non-vanishing helicity.

Keywords: Ideal fluid, New solution, Variational formulation, Helicity.

1 Introduction

A symmetry of a physical system means invariance with respect to a certain group of transformations and plays an essential role in the gauge theory of theoretical physics. *Fluid mechanics* is a field theory of Newtonian mechanics of Galilean symmetry. Two symmetries are known as subgroups of the Galilean group: translation (space and time) and space-rotation.

Guided by the gauge theory, Kambe [1, 2] studied flow fields of an ideal compressible fluid and investigated consequence of both global and local invariances of the fields in the space-time $(\boldsymbol{x}, t)$, where $\boldsymbol{x} = (x^1, x^2, x^3)$ is the three-dimensional space coordinates. Among the results obtained in the previous studies, it is particularly remarkable that the convective derivative D_t defined by

$$\mathrm{D}_t \equiv \partial/\partial t + \boldsymbol{v}\cdot\nabla\,, \qquad \nabla = (\partial/\partial x^i), \qquad (1)$$

(the Lagrange derivative in the fluid mechanics) is in fact a *covariant derivative* which is a building block in the gauge theory, where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity field. On the basis of the gauge-theoretic point of view, appropriate Lagrangian functionals are defined for motion of an ideal fluid. Total Lagrangian consists of space integrals of kinetic energy and internal energy ϵ (with negative sign). Traditionally, this is supplemented by additional two terms associated with conservations of mass and entropy. Thus the total Lagrangian L_* is defined by

$$L_* = \int_V \Lambda_*(\boldsymbol{v}, \rho, s, \phi, \psi)\, \mathrm{d}^3\boldsymbol{x}\,, \qquad (2)$$

$$\Lambda_* = \tfrac{1}{2}\,\rho\langle \boldsymbol{v}, \boldsymbol{v}\rangle - \rho\epsilon(\rho, s) - \rho\mathrm{D}_t\phi - \rho s\mathrm{D}_t\psi, \qquad (3)$$

where V is a volume in the $\boldsymbol{x}$-space (chosen arbitrarily), Λ_* is the Lagrangian density, $\rho(\boldsymbol{x}, t)$ and $s(\boldsymbol{x}, t)$ are the fluid density and specific entropy (*per unit mass*), and $\phi(\boldsymbol{x}, t)$ and $\psi(\boldsymbol{x}, t)$ are potentials associated with mass and entropy respectively.[1] An additional term will be added to L_* in a later section (§3).

An action integral is defined by

$$J = \int_{t_1}^{t_2} L_*\mathrm{d}t = \int \Lambda_*(\boldsymbol{v}, \rho, s, \phi, \psi)\mathrm{d}^4x, \qquad (4)$$
$$\mathrm{d}^4x = \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x},$$

where I_t is a time interval $[t_1, t_2]$ (chosen arbitrar-

[1] *Scalar product* between two tangent vectors $\boldsymbol{a} = (a^i)$ and $\boldsymbol{b} = (b^i)$ is denoted in this paper by four ways, $\langle \boldsymbol{a}, \boldsymbol{b}\rangle$, $\boldsymbol{a}\cdot\boldsymbol{b}$, $\delta_{ij}a^ib^j$, or a_ib^i, where $a_i(= \delta_{ij}a^j)$ is a cotangent vector (derived from the tangent vector a^i).

ily). The action principle is

$$\delta J = \int_{V\otimes I_t} \delta \Lambda_*(\boldsymbol{v}, \rho, s, \phi, \psi)\, \mathrm{d}^4x = 0, \qquad (5)$$

for its variation δJ with respect to arbitrary variations of the variables $\boldsymbol{v}, \rho, s, \phi$ and ψ.

Total energy and momentum are global integrals of $\frac{1}{2}\rho\langle \boldsymbol{v}, \boldsymbol{v}\rangle$ and $\rho\boldsymbol{v}$ respectively, that characterize the flow field globally. There is another important global integral (integral over the whole space), which is the *helicity* $\mathcal{H}$ defined by

$$\mathcal{H} \equiv \int \langle\, \boldsymbol{v}, \boldsymbol{\omega}\,\rangle\, \mathrm{d}^3\boldsymbol{x}\,, \qquad (6)$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the vorticity.

The section 2 is a review of the action principle applied to the above Lagrangian under the Eulerian variation in which variations are taken for all the field variables independently. This yields a general solution equivalent to the classical Clebsch solution (Clebsch [3], [1]). In this solution the vorticity has a special form such that the helicity vanishes. In a particular case of isentropic fluid in which the entropy s is a constant, the flow field thus obtained becomes *irrotational*. It is well-known that, even in such an isentropic fluid, the fluid flow can support rotational velocity fields. In fact, Euler [4] showed already in 1755 that his equation of motion can describe rotational flows. In addition, most traditional formulations of the action principle take into account both the continuity equation and isentropic condition as constraint conditions for variations. To do it, Lagrange multipliers are used. This is a mathematical artifact, while physical meaning of the multipliers is not clear.

In order to resolve the issue just mentioned above, a new term L_A of the form,

$$L_A = -\int (\mathcal{L}_t^*[\boldsymbol{A}])_i\, \Omega^i\, \mathrm{d}^3\boldsymbol{x}, \qquad (7)$$

is added to the total Lagrangian for flows of an inviscid fluid, where the vector $\boldsymbol{\Omega} = (\Omega^i)$ satisfies the frozen field equation and $\mathcal{L}_t^*[\boldsymbol{A}]$ is the Lie derivative of a cotangent vector $\boldsymbol{A}$ (details are given in §3 and Appendix B). This new term was introduced by [1, 2] to account for the rotation symmetry of the flow field, expecting that this yields non-vanishing rotational component in the velocity field and non-vanishing helicity. It is confirmed here that the expectation is valid..

In the present formulation, the term L_A is added to the total Lagrangian, as well as those associated with the last two terms of (3). Forms of the three terms are determined so that the Euler-Lagrange equation is not influenced by addition of the new terms (see Appendix A and B). In the next section 2, in addition to showing the Clebsch-type solution resulting in *irrotational* flow for the particular case of isentropy, the last two terms of (3) yield the conservation equations of mass and entropy.

The present paper investigates outcome of the new term L_A when it is added to the total Lagrangian L_*. It is verified newly and explicitly that the fields obtained by Eulerian field variation *in fact* satisfy the Euler's equation of motion. Details of the background of the present investigation are left in Appendices A $\sim$ C, so that the main part is described smoothly.

Simple examples are presented to show how rotational flow fields are expressed by a new representation. In addition, advantage of the present formulation is emphasized by showing that the vector potentials $\boldsymbol{A}$ and $\boldsymbol{\Omega}$ generate the helicity and that the potential fields take account of the effect of frozen vector fields explicitly. The Clebsch-type solution obtained from L_* lacks the last effect, as shown in §5.

2 Review of previous Eulerian variations

In general field theory, variations are taken for all the field variables independently. This is called *Eulerian* variation of flow field (Serrin [5]). In regard to the Lagrangian density (3), we take variations of the variables $\boldsymbol{v}$, ρ, s and potentials ϕ and ψ, by assuming all the variations being independent. In this section, we disregard the Lagrangian L_A in order to consider what is the result of neglecting it.

2.1 Solution by the variation

Substituting the varied variables $\boldsymbol{v}+\delta\boldsymbol{v}$, $\rho+\delta\rho$ $s+\delta s$, $\phi+\delta\phi$ and $\psi+\delta\psi$ into $\Lambda_*(\boldsymbol{v}, \rho, s, \phi, \psi)$

and writing its variation as $\delta\Lambda$, we obtain

$$\begin{aligned}\delta\Lambda_* &= \delta\boldsymbol{v}\cdot\rho\,(\,\boldsymbol{v}-\nabla\phi - s\,\nabla\psi\,) - \delta s\ \rho\ \mathrm{D}_t\psi \\ &\quad + \delta\rho\ (\tfrac{1}{2}\,v^2 - h - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi) \\ &+ \delta\phi\,\big(\,\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\,\big) - \partial_t(\rho\,\delta\phi) - \nabla\cdot(\rho\boldsymbol{v}\,\delta\phi) \\ &+ \delta\psi\,\big(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\big) - \partial_t(\rho s\delta\psi) \\ &\quad -\nabla\cdot(\rho s\boldsymbol{v}\delta\psi), \end{aligned} \tag{8}$$

where $h = \epsilon + p/\rho$ is the specific enthalpy, and standard relations of thermodynamics are used.[2] As usual, the variation fields are assumed to vanish on the boundary surface S enclosing the domain $V\otimes I_t$ when integration by parts is carried out in (5). By substituting (8), the action principle (5) for independent variations $\delta\boldsymbol{v}$, $\delta\rho$ and δs gives

$$\delta\boldsymbol{v}\ :\quad \boldsymbol{v} = \nabla\,\phi + s\,\nabla\psi\,, \tag{9}$$

$$\delta\rho\ :\quad \tfrac{1}{2}\,v^2 - h - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi = 0\,, \tag{10}$$

$$\delta s\ :\quad \mathrm{D}_t\psi \equiv \partial_t\psi + \boldsymbol{v}\cdot\nabla\psi = 0\,. \tag{11}$$

Using (1), (9) and (11), we have

$$\begin{aligned}\mathrm{D}_t\phi = \partial_t\phi + \boldsymbol{v}\cdot\nabla\phi &= \partial_t\phi + \boldsymbol{v}\cdot(\boldsymbol{v} - s\nabla\psi) \\ &= v^2 + \partial_t\phi + s\,\partial_t\psi.\end{aligned}$$

Equation (10) can be rewritten, by using this and (11), as

$$\tfrac{1}{2}\,v^2 + h + \partial_t\phi + s\,\partial_t\psi = 0\,. \tag{12}$$

From the variations of $\delta\phi$ and $\delta\psi$, we obtain

$$\delta\phi\ :\quad \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \tag{13}$$

$$\delta\psi\ :\quad \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) = 0\,.$$

Using (13), the second reduces to the *adiabatic* equation:

$$\partial_t s + \boldsymbol{v}\cdot\nabla s = \mathrm{D}_t s = 0\,. \tag{14}$$

Thus, we obtain the continuity equation (13) and entropy equation (14) from the action principle. With the velocity (9), the vorticity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{v} = \nabla s\times\nabla\psi. \tag{15}$$

The scalar product $\boldsymbol{\omega}\cdot\boldsymbol{v}$ is given by

$$\begin{aligned}\boldsymbol{\omega}\cdot\boldsymbol{v} &= (\nabla s\times\nabla\psi)\cdot(\nabla\,\phi + s\,\nabla\psi) \\ &= \boldsymbol{\omega}\cdot\nabla\phi = \nabla\cdot[\phi\,\boldsymbol{\omega}\,]\,.\end{aligned}$$

[2] $(\partial\epsilon/\partial\rho)_s = p/\rho^2$, and $(\partial/\partial\rho)_s(\rho\,\epsilon) = \epsilon + \rho\,(\partial\epsilon/\partial\rho)_s = \epsilon + p/\rho = h$.

The helicity $\mathcal{H}$ is defined by the integral (6). Assuming that $\boldsymbol{\omega} = 0$ out of V, $\mathcal{H}$ vanishes in this case:

$$\mathcal{H} \equiv \int_V \boldsymbol{\omega}\cdot\boldsymbol{v}\mathrm{d}^3\boldsymbol{x} = \int_V \nabla\cdot[\phi\,\boldsymbol{\omega}]\mathrm{d}^3\boldsymbol{x} = 0. \tag{16}$$

However, for general velocity fields, the helicity $\mathcal{H}$ is a measure of linkage and knottedness of vortex lines, and does not vanish in general.

2.2 Clebsch solution

Above results are summarized as follows:

$$\boldsymbol{v} = \nabla\phi + s\,\nabla\psi, \tag{17}$$

$$\tfrac{1}{2}\,v^2 + h + \partial_t\phi + s\,\partial_t\psi = 0, \tag{18}$$

$$\mathrm{D}_t s = 0, \qquad \mathrm{D}_t\psi = 0. \tag{19}$$

The velocity field (17) is equivalent to the classical Clebsch solution [3]. In fact, using (17) and (15), and using a vector identity of the footnote[3], we have

$$\begin{aligned}\boldsymbol{\omega}\times\boldsymbol{v} &= (\boldsymbol{v}\cdot\nabla s)\nabla\psi - (\boldsymbol{v}\cdot\nabla\psi)\nabla s, \\ \partial_t\boldsymbol{v} &= \nabla\partial_t\phi + \partial_t s\,\nabla\psi + s\,\nabla\partial_t\psi.\end{aligned}$$

Adding the last two equations, we obtain

$$\begin{aligned}\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} &= \nabla(\partial_t\phi + s\,\partial_t\psi) \\ &\quad +(\mathrm{D}_t s)\,\nabla\psi - (\mathrm{D}_t\psi)\,\nabla s,\end{aligned}$$

Last two terms vanish due to (19). Thus, by the help of (18), it is found that the following Euler's equation is satisfied:

$$\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla\big(\tfrac{1}{2}\,v^2 + h\,\big). \tag{20}$$

2.3 Isentropic fluid

For an isentropic fluid where s takes a constant value s_0 at all points (but the density ρ is not necessarily a constant), the equation of motion (20) can be written as

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\,\nabla p, \tag{21}$$

or

$$\mathrm{D}_t\boldsymbol{v} = -\nabla h. \tag{22}$$

[3] $(\nabla s\times\nabla\psi)\times\boldsymbol{v} = -(\boldsymbol{v}\cdot\nabla\psi)\,\nabla s + (\boldsymbol{v}\cdot\nabla s)\,\nabla\psi$.

where the two relations (a) and (b) of the footnote[4] are used.

Moreover, the motion is *irrotational.* In fact, from (9), we have $\boldsymbol{v} = \nabla\,\Phi$ where $\Phi = \phi + s_0\,\psi$, and we obtain $\boldsymbol{\omega} = 0$ from (15). Thus, the motion reduces to *potential* flows for an isentropic fluid.

Traditionally, this property is considered to be a defect of the formulation of the Eulerian variation carried out just above, because the action principle should yield equations of rotational flows (*e.g.* [4]). This is a long-standing problem ([5]∼[8]). In particular, Lin [6] tried to resolve this difficulty by introducing a *constraint* as a side condition, which imposes invariance of Lagrangian particle labels (a_1, a_2, a_3) along particle trajectories. The additional Lagrangian introduced by him was expressed by the following form:

$$L_{\rm L} = \int B_k \cdot {\rm D}_t a_k \; {\rm d}^3\boldsymbol{x}\,, \qquad (23)$$

This introduces three potentials $(B_1,\, B_2,\, B_3)$ as a set of Lagrange multipliers of conditional variation, but physical significance of B_k is not clear. In order to resolve this issue, Kambe [1] proposed a new term L_A instead of the above $L_{\rm L}$. This is the subject to be considered in the next section.

In the expression ${\rm D}_t a_k$ of (23), the Lagrangian particle labels are regarded as functions of the Eulerian space coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$, *i.e.* $\boldsymbol{a} = (a_1, a_2, a_3) = \boldsymbol{a}(\boldsymbol{x}, t)$. On the other hand, the position of a fluid particle of the name tag $\boldsymbol{a} = (a_1, a_2, a_3)$ at a time t may be denoted by $X^k(t, \boldsymbol{a})$. Usually, the parameters a_k $(k = 1, 2, 3)$ are defined by $a_k = X^k(0, \boldsymbol{a})$. Then, the conservation of mass in a volume element ${\rm d}^3\boldsymbol{X}(t)$ at a time t imposes the following:

$$\rho(t, \boldsymbol{X})\, {\rm d}^3\boldsymbol{X}(t) = \rho(0, \boldsymbol{X})\, {\rm d}^3\boldsymbol{a}, \qquad (24)$$

where $\rho(t, \boldsymbol{x})$ is the fluid density. Using the Jacobian J of the transformation, we have ${\rm d}^3\boldsymbol{X}(t) = J\,{\rm d}^3\boldsymbol{a}$, where $J = \partial(\boldsymbol{X})/\partial(\boldsymbol{a})$. Thus the following relation must be satisfied at all points at any time:

$$J = \frac{\partial(X^1, X^2, X^3)}{\partial(a_1, a_2, a_3)} = \frac{\rho(0, \boldsymbol{X})}{\rho(t, \boldsymbol{X})}\,. \qquad (25)$$

[4] (a) $(\nabla\times\boldsymbol{v})\times\boldsymbol{v} + \nabla(\frac{1}{2}\,v^2) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$; and (b) ${\rm d}h = (1/\rho){\rm d}p$ by the thermodynamics, since we have $\epsilon = \epsilon(\rho)$, and ${\rm d}\epsilon = (p/\rho^2)\,{\rm d}\rho$.

3 Improvement of the Eulerian variation

In order to derive *rotational* component of velocity field by the Eulerian variation, an additional Lagrangian L_A was proposed with using the invariance property of the vorticity $\boldsymbol{\omega}_a$ by [1, 2], where $\boldsymbol{\omega}_a$ is the vorticity transformed to the space of Lagrangian coordinates $\boldsymbol{a}$. The Lagrangian L_A is represented in the Eulerian $\boldsymbol{x}$-space as follows:

$$\begin{aligned} L_A &= -\int_V \langle \mathcal{L}_t^*[\boldsymbol{A}],\, \boldsymbol{\Omega}\rangle\, {\rm d}^3\boldsymbol{x} && (26)\\ &= -\int_V (\mathcal{L}_t^*[\boldsymbol{A}])_i\, \Omega^i\, {\rm d}^3\boldsymbol{x} \\ &= \int_V A_i\, (\mathcal{L}_t[\boldsymbol{\Omega}\; {\rm d}^3\boldsymbol{x}])^i + {\rm Int}_S, && (27)\\ &\qquad {\rm div}\,\boldsymbol{A} = 0, \quad {\rm div}\,\boldsymbol{\Omega} = 0, && (28)\end{aligned}$$

(see Appendix B for its derivation), where ${\rm Int}_S$ denotes surface integrals over the surface S bounding the integration volume V. Lie-derivative takes different forms depending on the objects of operation [2, 9]:

$$\begin{aligned} \mathcal{L}_t[\phi] &\equiv \partial_t\phi + v^k\partial_k\phi = {\rm D}_t\,\phi, && (29)\\ (\mathcal{L}_t[\boldsymbol{\Omega}])^i &\equiv \partial_t\Omega^i + v^k\partial_k\Omega^i - \Omega^k\partial_k v^i, && (30)\\ (\mathcal{L}_t^*[\boldsymbol{A}])_i &\equiv \partial_t A_i + v^k\partial_k A_i + A_k\partial_i v^k, && (31)\\ \mathcal{L}_t[{\rm d}^3\boldsymbol{x}] &\equiv (\partial_k v^k)\; {\rm d}^3\boldsymbol{x}, && (32)\end{aligned}$$

where ϕ is a scalar field (a *zero-form*), $\boldsymbol{\Omega} = (\Omega^i)$ is a tangent vector, $\boldsymbol{A} = (A_i)$ a cotangent vector (a *one-form*), and ${\rm d}^3\boldsymbol{x}$ a volume *three-form.* By using these definitions, we have the following:

$$\begin{aligned} \mathcal{L}_t\Big[\langle \boldsymbol{A},\, \boldsymbol{\Omega}\rangle\, {\rm d}^3\boldsymbol{x}\Big] &= \langle \mathcal{L}_t^*[\boldsymbol{A}], \boldsymbol{\Omega}\rangle\, {\rm d}^3\boldsymbol{x} \\ &\quad + \langle \boldsymbol{A},\, \mathcal{L}_t[\boldsymbol{\Omega}]\rangle\, {\rm d}^3\boldsymbol{x} + \langle \boldsymbol{A},\, \boldsymbol{\Omega}\rangle\, \mathcal{L}_t[{\rm d}^3\boldsymbol{x}] \\ &= \Big[\partial_t\langle \boldsymbol{A},\, \boldsymbol{\Omega}\rangle + \partial_k(v^k\langle \boldsymbol{A},\, \boldsymbol{\Omega}\rangle)\Big]{\rm d}^3\boldsymbol{x}.\end{aligned}$$

This is required to vanish by the equations (41) and (42) obtained from the variational principle of the next subsection §3.1. Therefore, the scalar product $\langle \boldsymbol{A},\, \boldsymbol{\Omega}\,\rangle$ is a density satisfying a conservation equation.

Invariance of mass during the motion requires $\mathcal{L}_t[\rho\,{\rm d}^3\boldsymbol{x}] = \rho\,\mathcal{L}_t[{\rm d}^3\boldsymbol{x}] + \mathcal{L}_t[\rho]\,{\rm d}^3\boldsymbol{x} = 0$. Using (29) and (32), we obtain

$$\begin{aligned} \mathcal{L}_t[\rho] = {\rm D}_t\rho &= -\rho\,(\partial_k v^k), \\ \therefore \quad \partial_t\rho + \partial_k(\rho v^k) &= 0. && (33)\end{aligned}$$

By including L_A of (27), the Lagrangian density of (3) is modified to

$$\begin{aligned}\Lambda[v_i,\rho,s,\phi,\psi,\boldsymbol{A}] &= \tfrac{1}{2}\,\rho\, v_k v^k - \rho\,\epsilon(\rho,s)\\ &\quad -\rho\,(\partial_t + v^k\partial_k)\phi - \rho s\,(\partial_t + v^k\partial_k)\psi\\ &\quad -\big(\partial_t A_i + v^k\partial_k A_i + A_k\partial_i v^k\big)\,\Omega^i, \qquad (34)\end{aligned}$$

where D_t is replaced by $\partial_t + v^k\partial_k$. Note that v^k(tangent vector) $= v_k$(cotangent vector) in the cartesian flat space (see the footnote below Eq.(3)). The same is said to $\boldsymbol{\Omega}$, *i.e.* $\Omega^k = \Omega_k$.

3.1 New variational solution

Now, we take variations of the field variables v_i, ρ, s and potentials ϕ, ψ and $\boldsymbol{A}$. Substituting the varied variables $v_i+\delta v_i$, $\rho+\delta\rho$, $s+\delta s$, $\phi+\delta\phi$, $\psi+\delta\psi$ and $\boldsymbol{A}+\delta\boldsymbol{A}$ into $\Lambda[v_i,\rho,s,\phi,\psi,\boldsymbol{A}]$ and writing its variation as $\delta\Lambda$, we obtain

$$\begin{aligned}\delta\Lambda &= \delta v_i\Big[\rho\,(v_i - \partial_i\phi - s\,\partial_i\psi) - \Omega_k\,\partial_i A_k\\ &\qquad +\Omega_k\,\partial_k A_i\Big] - \partial_k\big(\Omega_k A_i\,\delta v_i\big) \qquad (35)\\ &\quad + \delta\rho\,\big(\tfrac{1}{2}u^2 - h - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi\big) - \delta s\;\rho\,\mathrm{D}_t\psi\\ &\quad + \delta\phi\,\big(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\big) - \partial_t(\rho\,\delta\phi) - \nabla\cdot(\rho\boldsymbol{v}\,\delta\phi)\\ &\quad + \delta\psi\,\big(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\big) - \partial_t(\rho s\,\delta\psi)\\ &\qquad -\nabla\cdot(\rho s\boldsymbol{v}\,\delta\psi).\\ &\quad -\langle\mathcal{L}_t^*[\boldsymbol{A}],\,\delta\boldsymbol{\Omega}\,\rangle + \Big\langle\delta\boldsymbol{A},\,\Big(\mathcal{L}_t[\boldsymbol{\Omega}] + \boldsymbol{\Omega}\,\partial_k v^k\Big)\Big\rangle\end{aligned}$$

Thus, the variational principle $\delta J = \int\delta\,\Lambda\;\mathrm{d}^4x = 0$ for independent variations of δv_i, $\delta\rho$, δs, *etc.* results in the followings:

$$\begin{aligned}\delta v_i &: \quad \rho\,(v_i - \partial_i\phi - s\,\partial_i\psi) - \Omega_k\,\partial_i A_k\\ &\qquad\qquad + \Omega_k\,\partial_k A_i = 0\,, \qquad (36)\\ \delta\rho &: \quad \tfrac{1}{2}v^2 - h - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi = 0\,, \qquad (37)\\ \delta s &: \quad \mathrm{D}_t\psi \equiv \partial_t\psi + \boldsymbol{v}\cdot\nabla\psi = 0\,, \qquad (38)\\ \delta\phi &: \quad \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \qquad (39)\\ \delta\psi &: \quad \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) = 0\,, \qquad (40)\\ \delta\boldsymbol{\Omega} &: \quad \mathcal{L}_t^*[\boldsymbol{A}] = 0\,. \qquad (41)\\ \delta\boldsymbol{A} &: \quad \mathcal{L}_t[\boldsymbol{\Omega}] + \boldsymbol{\Omega}\,\partial_k v^k\\ &\qquad = \partial_t\boldsymbol{\Omega} + \nabla\times(\boldsymbol{\Omega}\times\boldsymbol{v}) = 0\,, \qquad (42)\\ &\qquad \partial_k\Omega^k = 0. \qquad (43)\end{aligned}$$

Using the continuity equation (39), we obtain the same adiabatic equation as (14),

$$\mathrm{D}_t s = \partial_t s + \boldsymbol{v}\cdot\nabla s = 0\,. \qquad (44)$$

from (40). The equation (36) gives a *new expression* for the velocity $\boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{v} &= \nabla\phi + s\,\nabla\psi + \frac{1}{\rho}\,\boldsymbol{w}\,, \qquad (45)\\ \text{where}\quad \boldsymbol{w} &= (w_i) = \Omega^k\,\nabla A_k - (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{A}\\ &= \boldsymbol{\Omega}\times(\nabla\times\boldsymbol{A}), \qquad (46)\\ w_i = \Omega^k\,C_{ik}, &\quad C_{ik} = \partial_i A_k - \partial_k A_i, \qquad (47)\end{aligned}$$

where we have the equality,

$$\Omega^k\,C_{ik} = [\boldsymbol{\Omega}\times(\nabla\times\boldsymbol{A})]_i,$$

since the right hand side is rewritten as

$$\begin{aligned}&\varepsilon_{ijk}\,\Omega_j(\nabla\times\boldsymbol{A})_k = \varepsilon_{ijk}\varepsilon_{klm}\,\Omega_j\;\partial_l A_m\\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_j\partial_l A_m = \Omega_j\partial_i A_j - \Omega_j\partial_j A_i.\end{aligned}$$

The vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ is given by

$$\boldsymbol{\omega} = \nabla s\times\nabla\psi + \frac{1}{\rho}\,\nabla\times\boldsymbol{w} - \frac{1}{\rho^2}\,\nabla\rho\times\boldsymbol{w}. \qquad (48)$$

The second and third terms express non-vanishing vorticity *even* in an isentropic fluid of uniform s. Defining $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, we hvae

$$\begin{aligned}\nabla\times\boldsymbol{w} &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{B} \qquad (49)\\ &= \nabla\,\Omega_k\times\nabla A_k - \nabla\Omega_k\times\partial_k\boldsymbol{A} - (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{B}. \qquad (50)\end{aligned}$$

Thus, the present formulation yields the rotational component naturally by the variational principle.

3.2 Euler's equation of motion is satisfied

Next step is to verify that the set of equations derived in the previous subsection *in fact* satisfy the Euler's equation of motion. This is carried out as follows for the flow field described by (36), (37) and (38). Applying the covariant derivative $\mathrm{D}_t = \partial_t + \boldsymbol{v}\cdot\nabla$ to $\boldsymbol{v}$ of (45), we have

$$\mathrm{D}_t[\boldsymbol{v}] = \mathrm{D}_t\nabla\,\phi + \mathrm{D}_t(s\,\nabla\psi) + \frac{1}{\rho}\,\mathrm{D}_t\boldsymbol{w} - \frac{1}{\rho^2}\,(\mathrm{D}_t\rho)\,\boldsymbol{w}, \qquad (51)$$

where, from (33),

$$\mathrm{D}_t\rho = -\rho\,(\partial_k v^k). \qquad (52)$$

The first term can be rewritten as

$$\mathrm{D}_t(\nabla\,\phi) = \nabla(\mathrm{D}_t\phi) - \partial_k\phi\,\nabla v^k. \qquad (53)$$

Using the isentropic property (44), the second term is

$$\mathrm{D}_t(s\,\nabla\psi) = s\,\mathrm{D}_t(\nabla\psi) = s\,\nabla(\mathrm{D}_t\psi) - s\,\partial_k\psi\,\nabla v^k = -s\,\partial_k\psi\,\nabla v^k, \qquad (54)$$

where (38) is used. By using the expression (47), the third term is

$$\begin{aligned}\mathrm{D}_t w^i &= \mathrm{D}_t(\Omega^k\,C_{ik})\\ &= \mathrm{D}_t(\Omega^k)\,C_{ik} + \Omega^k\,\mathrm{D}_t(C_{ik}). \qquad (55)\end{aligned}$$

By (42) and (30), we have

$$\mathrm{D}_t(\Omega^k) = \Omega^l\,\partial_l v^k - \Omega^k\,\partial_l v^l\,,$$

while for $\mathrm{D}_t(C_{ik})$, by using (47), we have

$$\begin{aligned}\mathrm{D}_t C_{ik} &= \mathrm{D}_t(\partial_i A_k) - \mathrm{D}_t(\partial_k A_i)\\ &= \partial_i(\mathrm{D}_t A_k) - \partial_k(\mathrm{D}_t A_i)\\ &\quad -(\partial_i v^l)\,\partial_l A_k + (\partial_k v^l)\,\partial_l A_i\,.\end{aligned}$$

Substituting these two into (55),

$$\begin{aligned}\mathrm{D}_t w_i &= \Omega^l\,\partial_l v^k\,(\partial_i A_k - \partial_k A_i)\\ &\quad -\Omega^k\,\partial_l v^l\,(\partial_i A_k - \partial_k A_i) + \Omega^k\Big(\partial_i(\mathrm{D}_t A_k)\\ &\quad -\partial_k(\mathrm{D}_t A_i) - (\partial_i v^l)\,\partial_l A_k + (\partial_k v^l)\,\partial_l A_i\Big).\end{aligned}$$

The right hand side simplifies greatly by cancellation, and we finally obtain

$$\begin{aligned}\mathrm{D}_t \boldsymbol{w} &= -w_k\,\nabla v^k - (\partial_l v^l)\,w_i\,, \qquad (56)\\ w_k &= \Omega^l\,C_{kl}\,.\end{aligned}$$

Substituting (52), (53), (54) and (56) into (51), we obtain

$$\begin{aligned}\mathrm{D}_t\boldsymbol{v} &= \nabla(\mathrm{D}_t\phi) - (\partial_k\phi + s\,\partial_k\psi + \frac{1}{\rho}\,w_k)\,\nabla v^k\\ &= \nabla(\mathrm{D}_t\phi) - v_k\,\nabla v^k = \nabla(\mathrm{D}_t\phi - \tfrac{1}{2}\,v^2)\end{aligned}$$

Using (37) and (38), this reduces to the Euler's equation of motion (22):

$$\mathrm{D}_t\boldsymbol{v} = -\nabla\,h\,. \qquad (57)$$

Thus, it is found that the present *improved* variation which takes account of the Lagrangian L_A leads to a *new* result, *i.e.* Euler's equation of motion is satisfied by the new set of (36), (37) and (38).

3.3 Examples

In order to show how the above representation of velocity is applied, we consider two simple examples of flow of an incompressible fluid of constant density ρ_0 and constant entropy s_0 by showing representation in terms of $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ in a cylindrical frame of reference (x, r, ϕ):

$$\begin{aligned}\boldsymbol{v} &= \nabla\Phi + \frac{1}{\rho_0}\,\boldsymbol{w},\\ &\Phi = \phi + s_0\psi, \qquad \boldsymbol{w} = (w_x,\, w_r,\, w_\phi)\,,\\ &\boldsymbol{w} = \boldsymbol{\Omega}\times\boldsymbol{B}, \qquad \boldsymbol{B} = (\nabla\times\boldsymbol{A}),\\ &\boldsymbol{\Omega} = (\Omega_x,\,\Omega_r,\,\Omega_\phi), \qquad \boldsymbol{A} = (A_x,\,A_r,\,A_\phi)\end{aligned}$$

(*a*) *Rectilinear vortex*

First example is a rectilinear vortex L of strength γ which coincides with the x-axis, *i.e.* the axis of cylindrical symmetry. With respect to this coordinate frame, we can define

$$\boldsymbol{\Omega} = (\gamma,\,0,\,0), \qquad \boldsymbol{A} = (0,\,0,\,A_\phi), \qquad A_\phi = -\frac{\rho_0 x}{2\pi r}, \qquad (58)$$

$(\Phi = 0)$. This gives

$$\boldsymbol{B} = (B_x, B_r, B_\phi) = \nabla\times\boldsymbol{A} = (0,\,\frac{\rho_0}{2\pi r},\,0).$$

Then we have the velocity field of vortex L:

$$\boldsymbol{w} = \boldsymbol{\Omega}\times\boldsymbol{B} = (0,\,-\gamma B_\phi,\,\gamma B_r) = (0,\,0,\,\frac{\gamma\rho_0}{2\pi r}), \qquad (59)$$

a well-known line-vortex of circulating velocity $v_\phi = w_\phi/\rho_0 = \gamma/(2\pi r)$ of strength γ.

(*b*) *Hill's spherical vortex*

Next example is the Hill's spherical vortex of radius a moving with a constant velocity U in the x-direction. The frame of reference is taken to be fixed to the moving vortex with its symmetry axis coinciding with the x-axis of the cylindrical frame (x, r, ϕ). The flow field is steady in this frame. The flow outside the sphere of radius a is assumed to be irrotational with uniform velocity $-U$ at infinity. Its (Stokes's) stream function Ψ_o is given by

$$\begin{aligned}\Psi_o(x, r) &= -\tfrac{1}{2}\,Ur^2\Big(1 - \frac{a^3}{R^3}\Big), \qquad (60)\\ R^2 &= x^2 + r^2.\end{aligned}$$

Then the velocity components $\boldsymbol{v} = (v_x, v_r, 0)$ are given by

$$v_x = \frac{1}{r}\frac{\partial \Psi_o}{\partial r} = -U + U\Big(1 - \frac{3}{2}\frac{r^2}{R^2}\Big)\frac{a^3}{R^3}, \quad (61)$$

$$v_r = -\frac{1}{r}\frac{\partial \Psi_o}{\partial x} = \frac{3}{2}Ua^3\frac{xr}{R^5}. \quad (62)$$

Obviously, the velocity field $(v_x, v_r, 0)$ satisfies the continuity equation, $\nabla \cdot \boldsymbol{v} = 0$.

The Hill's vortex is characterized by the vorticity proportional to r within the spherical surface of radius a (accounting for stretching of circular vortex lines by convection). The vorticity has the ϕ-component ω_ϕ only. Denoting it as $\omega_\phi = cr$ (c a constant), the stream function is found to be

$$\Psi_i(x, r) = \frac{1}{10}c\,r^2\,(a^2 - R^2),$$

$$= \frac{3}{4}U\,r^2\Big(1 - \frac{R^2}{a^2}\Big). \quad (63)$$

where c is determined as $c = (15/2)(U/a^2)$, so that the tangential velocity matches with that of the outer flow given by (61) and (62). The velocity components $(v_x, v_r, 0)$ within the sphere are given by

$$v_x^{(i)} = \frac{1}{r}\frac{\partial \Psi_i}{\partial r} = \frac{3}{2}U\frac{a^2 - x^2 - 2r^2}{a^2}, \quad (64)$$

$$v_r^{(i)} = -\frac{1}{r}\frac{\partial \Psi_i}{\partial x} = \frac{3}{2}U\frac{xr}{a^2}. \quad (65)$$

On the surface of sphere $R = \sqrt{x^2 + r^2} = a$, it is seen that both of Ψ_i and Ψ_o give the same expressions of $v_x = -(3/2)U(r^2/a^2)$ and $v_r = (3/2)U(xr/a^2)$.

According to the present formulation, we define the fields $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ within the sphere $R < a$ as

$$\boldsymbol{\Omega} = (0,\, 0,\, k\,r), \qquad \boldsymbol{A} = (\phi F_x,\, \phi F_r,\, 0), \quad (66)$$

for $\sqrt{x^2 + r^2} < a$, in the cylindrical frame (x, r, ϕ), where F_x and F_r are functions of x and r, and k a constant. This gives

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (-r^{-1}F_r,\, r^{-1}F_x,\, \phi\, b_\phi),$$

where $b_\phi = \partial_x F_r - \partial_r F_x$. Then we have the velocity $\boldsymbol{w}$ within the vortex ($R < a$ and $\Omega_\phi = kr$):

$$\boldsymbol{w} = \boldsymbol{\Omega} \times \boldsymbol{B} = (-\Omega_\phi B_r,\, \Omega_\phi B_x,\, 0)$$

$$= k\,(-F_x,\, -F_r,\, 0). \quad (67)$$

Let us introduce a stream function Ψ_A to represent the field $(-F_x,\, -F_r,\, 0)$ by

$$-F_x = \frac{1}{r}\frac{\partial \Psi_A}{\partial r}, \qquad -F_r = -\frac{1}{r}\frac{\partial \Psi_A}{\partial x}. \quad (68)$$

Corresponding vorticity is given by

$$\partial_x(-F_r) - \partial_r(-F_x) = -\frac{1}{r}\partial_x^2\Psi_A - \partial_r\Big(\frac{1}{r}\partial_r\Psi_A\Big) = r.$$

This is solved by

$$\Psi_A = \frac{1}{10}r^2\,(a^2 - x^2 - r^2).$$

Using this, (68) gives

$$F_x = -\frac{1}{5}(a^2 - x^2 - 2r^2), \qquad F_r = -\frac{1}{5}\,rx. \quad (69)$$

If k is replaced by $(15/2)(\rho_0 U/a^2)$, the equation (67) reduces to the velocity field of (64) and (65):

$$w_x = -kF_x = \rho_0\, v_x^{(i)}, \qquad w_r = -kF_r = \rho_0\, v_r^{(i)}.$$

Thus, the fields $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ have been determined by the functions of (69) with $k = (15/2)(\rho_0 U/a^2)$.

Out of the sphere is the irrotational flow described by Ψ_o of (60). Alternative expression of the outer velocity can be given by the following velocity potential Φ_o:

$$\Phi_o = -Ux - \tfrac{1}{2}\,Ua^3\frac{x}{R^3},$$

with $v_x = \partial_x\Phi_o$ and $v_r = \partial_r\Phi_o$. It is easily seen that these reduce to the same expressions as (61) and (62). The inner velocity field can be represented without using any velocity potential.

It remains to show that the fields $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ satisfy (42) and (41) respectively:

$$\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{v}) = 0, \quad (70)$$

$$\mathcal{L}_t^*[\boldsymbol{A}] = v^k\partial_k\boldsymbol{A} + A_k\nabla v^k = 0. \quad (71)$$

which are required by the variational principle, where the time derivative terms are omitted in the present problem.

By using $\nabla \cdot \boldsymbol{v} = 0$ and $\nabla \cdot \boldsymbol{\Omega} = 0$ (from (43)), the first equation (70) reduces to $(\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} = 0$. Only non vanishing component is its ϕ-component, which is given by

$$[(\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega}]_\phi - [(\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v}]_\phi = v_r\partial_r\Omega_\phi - \frac{\Omega_\phi v_r}{r} = 0,$$

since $\Omega_\phi = kr$. Thus, the first (70) is satisfied.

In regard to the second (71), it is useful to note that we have $\boldsymbol{A} = -(\phi/k)\,\boldsymbol{w}$ from (66) and (67). On the other hand, the steady form of the equation (56) can be written as $v^k\partial_k\boldsymbol{w} = -w_k\,\nabla v^k$, since $\partial_l v^l = 0$ for the present Hill's vortex. This is equivalent to the second (71) by using the relation $\boldsymbol{A} = -(\phi/k)\,\boldsymbol{w}$ and noting that $v^k\partial_k$ does not include ∂_ϕ since $v_\phi = 0$ and that the ϕ-components of both equations vanish identically. Thus, the fields $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ satisfy the conditions required.

The case of the rectilinear vortex can be verified similarly by using x-independence of the veloctiy v_ϕ and ϕ-independence of $\boldsymbol{A}$, $\boldsymbol{\Omega}$ and v_ϕ.

4 Helicity and the Lagrangian L_A

Now we can show that there is a close relation between the Lagrangian L_A of (27) and the helicity defined by (6) in which helicity density is given by $\langle \boldsymbol{v}, \boldsymbol{\omega}\rangle$.

In an *isentropic* fluid of $s = s_0$ (constant) and constant density ρ_0, we have

$$\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}/\rho_0, \qquad \boldsymbol{\omega} = \nabla\times\boldsymbol{w}/\rho_0,$$

from (45) and (48), where $\Phi = \phi + s_0\psi$. Thus, the helicity density is given by

$$\begin{aligned}\langle \boldsymbol{v},\,\boldsymbol{\omega}\rangle &= \langle \nabla\Phi + \frac{1}{\rho_0}\boldsymbol{w},\,\boldsymbol{\omega}\rangle \\ &= \nabla\cdot(\Phi\boldsymbol{\omega}) + \frac{1}{\rho_0^2}\,\langle \boldsymbol{w},\,\nabla\times\boldsymbol{w}\rangle,\end{aligned}$$

where we used the relation $\langle\nabla\Phi,\,\boldsymbol{\omega}\rangle = \nabla\cdot(\Phi\boldsymbol{\omega})$ for the first term, since $\nabla\cdot\boldsymbol{\omega} = 0$. By integration, this term is transformed to surface integrals over the surface S bounding the volume V which are assumed to vanish.[5] So that, main helicity density (multiplied by ρ_0) is given by

$$\rho_0\langle\boldsymbol{v},\boldsymbol{\omega}\rangle_{\boldsymbol{w}} \equiv \langle\boldsymbol{w},\boldsymbol{\omega}\rangle = \langle\boldsymbol{\Omega}\times(\nabla\times\boldsymbol{A}),\boldsymbol{\omega}\rangle \quad (72)$$

$$= -\langle\boldsymbol{\omega}\times(\nabla\times\boldsymbol{A}),\boldsymbol{\Omega}\rangle = \langle\boldsymbol{\omega}\times\boldsymbol{\Omega},\boldsymbol{B}\rangle. \quad (73)$$

by using (46), where $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. Thus, the helicity is given by

$$\begin{aligned}\mathcal{H} &= -\frac{1}{\rho_0}\int \Big\langle \boldsymbol{\omega}\times(\nabla\times\boldsymbol{A}),\,\boldsymbol{\Omega}\Big\rangle\,\mathrm{d}^3\boldsymbol{x} \\ &= \frac{1}{\rho_0}\int \Big\langle \boldsymbol{\omega}\times\boldsymbol{\Omega},\ \boldsymbol{B}\Big\rangle\,\mathrm{d}^3\boldsymbol{x}. \qquad (74)\end{aligned}$$

[5]Usually, decay of the term $|\Phi\boldsymbol{\omega}|$ is of higher-order than $O(r^{-2})$ as $r\to\infty$, so that the surface integral tends to vanish at large distances r from an origin in V.

The last expression of $\mathcal{H}$ claerly states that the helicity vanishes if the vector field $\boldsymbol{\Omega}$ is chosen such that it is parallel to the vorticity $\boldsymbol{\omega}$ at all points. Putting it differently, the helicity is non-vanishing if the vector field $\boldsymbol{\Omega}$ is not proportional to the vorticity $\boldsymbol{\omega}$ at every point. It is interesting to see that the density of Lagrangian L_A is given by

$$-\langle \mathcal{L}_t^*[\boldsymbol{A}],\,\boldsymbol{\Omega}\rangle\,\mathrm{d}^3\boldsymbol{x},$$

and that the helicity density is obtained by replacing $\mathcal{L}_t^*[\boldsymbol{A}]$ in this expression with the term $\boldsymbol{\omega}\times(\nabla\times\boldsymbol{A})/\rho_0$. Both of the factors are obtained once the two vector fields $\boldsymbol{v}$ and $\boldsymbol{A}$ are known. Thus, we have found a close relation between the Lagrangian and the helicity, *i.e.* if one is defined, then the other is obtained by replacement of corresponding terms.

5 Comparison of L_A with other forms

To help understanding of the present solution, it is useful to compare the form of the Lagrangian L_A with other forms. Associated with the present case,

$$L_A = -\int_V (\mathcal{L}_t^*[\boldsymbol{A}])_i\,\Omega^i\,\mathrm{d}^3\boldsymbol{x},$$

defined by (26), the principle of least action of §3 yielded the equations (41) and (42) for the potentials A_i and Ω^i, which are reproduced here:

$$\begin{aligned}\partial_t A_i + (\boldsymbol{v}\cdot\nabla)A_i &= -A_k\partial_i v^k, &(75)\\ \partial_t\Omega^i + (\boldsymbol{v}\cdot\nabla)\Omega^i &= \Omega^k\partial_k v^i - \Omega^i\,\partial_k v^k. &(76)\end{aligned}$$

where $\partial_k\Omega^k = 0$ and $\partial_k A_k = 0$. From (45) and (46), the velocity is given by

$$\begin{aligned}\boldsymbol{v} &= \nabla\phi + s\,\nabla\psi + \frac{1}{\rho}\,\boldsymbol{\Omega}\times(\nabla\times\boldsymbol{A}) \\ &= \nabla\phi + s\,\nabla\psi + \frac{1}{\rho}\,\boldsymbol{w}. \qquad (77)\end{aligned}$$

where $\boldsymbol{w} = \boldsymbol{\Omega}\times(\nabla\times\boldsymbol{A})$. The vorticity $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega} = \nabla s\times\nabla\psi + \nabla\times\Big(\frac{1}{\rho}\,\boldsymbol{w}\Big). \qquad (78)$$

On the other hand, the Lagrangian (23) introduced by Lin (1963) is

$$L_{\mathrm{L}} = \int B_k\cdot \mathrm{D}_t a_k\ \mathrm{d}^3\boldsymbol{x}\,.$$

where a_k $(k = 1, 2, 3)$ are Lagrange parameters satisfying

$$D_t a_k = \partial_t a_k + (\boldsymbol{v}\cdot\nabla) a_k = 0, \qquad k = 1, 2, 3. \quad (79)$$

The potential fields B_k $(k = 1, 2, 3)$ of the coefficient are so-called the *Lagrange multupliers* for which the variation principle leads to the followings:

$$D_t B_k = \partial_t B_k + (\boldsymbol{v}\cdot\nabla) B_k = 0, \qquad k = 1, 2, 3. \quad (80)$$

Appendix C gives the velocity and vorticity associated with the Lin's Lagrangian L_L as

$$\boldsymbol{v} = \nabla\phi + s\,\nabla\psi + \sum_{k=1}^{3} B_k\,\nabla a_k, \quad (81)$$

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{v} = \nabla s\times\nabla\psi + \sum_{k=1}^{3} \nabla B_k\times\nabla a_k. \quad (82)$$

Obviously, with both cases of (78) and (82), the vorticity does not vanish even for the case of isentropy of $\nabla s = 0$. Therefore both formulations can support *rotational* flows in isentropic flows. The number of potentials B_k can be reduced from three to two [10, 11]. This is possible because of the mass conservation condition of (25) connecting the density ρ with the derivatives $\partial X^i/\partial a_j$ (Appendix C).

However, there is an essential difference between the pair of equations [(75), (76)] and [(79), (80)]. In both pairs, the left hand sides are common, *i.e.* given by the convective derivative $\partial_t + (\boldsymbol{v}\cdot\nabla)$ applied to the potentials. In regard to the right hand side, those of the latter pair vanish, expressing the potentials convected by the fluid flow. In the former pair however, there are non-vanishing terms, which represent the effects of stretching or volume change. Namely, the potentials a_k and B_k in the L_L are simply convected by the flow without any influence of stretching or volume change. These effects are essential in the dynamics of the vorticity. Therefore it is expected that the former case of L_A describes the vorticity dynamics more faithfully.

6 Conclusion

An improvement of variational formulation is proposed for rotational flows of an ideal fluid by using an additional Lagrangian L_A. The system of new expressions derived from the principle of least action is verified to satisfy the Euler's equation of motion. Therefore we have obtained a new expression of solution to the Euler's equation of motion. The rotational part of velocity field is expressed by using two vector potentials $\boldsymbol{A}$ and $\boldsymbol{\Omega}$, governed by such equations that take account of the effects of stretching and volume change. As a result, the scalar product $\langle \boldsymbol{A}, \boldsymbol{\Omega} \rangle$ becomes the density of a scalar field that satisfies a conservation equation. Two simple examples are given to show how the velocity is represented by using $\boldsymbol{A}$ and $\boldsymbol{\Omega}$ for vortex flows of an incompressible fluid with a constant specific entropy. In this solution, the helicity density is given an explicit form.

Appendix

A Euler-Lagrange equation

— Variation with respect to particle coordinate —

We consider the following infinitesimal transformation of the position of particle $\boldsymbol{a}$, $\boldsymbol{x}\to\boldsymbol{x}'$, and associated volume change $d^3\boldsymbol{x}$:

$$\boldsymbol{x}(\boldsymbol{a}) \to \boldsymbol{x}'(\boldsymbol{a}) = \boldsymbol{x}(\boldsymbol{a}) + \boldsymbol{\xi}(\boldsymbol{a}, t), \quad (83)$$

$$d^3\boldsymbol{x} \to d^3\boldsymbol{x}' = (1 + \partial_k\xi^k)\,d^3\boldsymbol{x},$$

Then the variations of particle position and volume are $\delta\boldsymbol{x} = \boldsymbol{\xi}$ and $\delta(d^3\boldsymbol{x}) = \partial_k\xi^k\,d^3\boldsymbol{x}$. It is shown in [1, 2] that other variations of density ρ, velocity $\boldsymbol{v}$ and entropy s are given by

$$\delta\rho = -\rho\,\partial_k\xi^k, \qquad \delta\boldsymbol{v} = D_t\boldsymbol{\xi}, \qquad \delta s = 0.$$

The action J is defined by (4) in the main text, where the Lagrangian density Λ_* is given by (3), which is written simply by Λ here. Variation of J resulting from the above variations is expressed as

$$\delta J = \int d^4x \Big[\frac{\partial\Lambda}{\partial\boldsymbol{v}}\delta\boldsymbol{v} + \frac{\partial\Lambda}{\partial\rho}\delta\rho + \frac{\partial\Lambda}{\partial s}\delta s + \Lambda\,\partial_k\xi^k\Big].$$

Substituting the expressions for $\delta\rho$, $\delta\boldsymbol{v}$, δs and $\delta(d^3\boldsymbol{x})$, and requiring δJ vanishes for arbitrary variation ξ^k, we obtain the *Euler-Lagrange equation*:

$$\frac{\partial}{\partial t}\Big(\frac{\partial\Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^l}\Big(v^l\,\frac{\partial\Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^k}\Big(\Lambda - \rho\,\frac{\partial\Lambda}{\partial\rho}\Big) = 0,$$

(see [1, 2] for details). Substituting the expression of (3) into Λ, we obtain the following momentum conservation:

$$\partial_t(\rho \boldsymbol{v}) + \nabla \cdot \rho \boldsymbol{v}\boldsymbol{v} + \nabla p = 0.$$

This reduces to the Euler's equation of motion (21) by using the continuity equation (13).

B Lagrangian L_A

The Lagrangian L_A of (26) was defined by [1, 2] originally in the form of total time derivative as follows:

$$\begin{aligned} L_A &= -\frac{\mathrm{d}}{\mathrm{d}\tau}\int_M \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a} \\ &= -\int_M \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}, \qquad (84) \end{aligned}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ are the Lagrangian coordinates with $\mathrm{d}^3\boldsymbol{a}$ a volume element, and $\tau\,(= t)$ is the time used in combination with $\boldsymbol{a}$, and $\boldsymbol{A}_a$ is a vector potential. The vector $\boldsymbol{\Omega}_a$ is one defined in the Lagrangian $\boldsymbol{a}$-space which satisfies $\partial_\tau \boldsymbol{\Omega}_a = 0$, the same equation as the vorticity transformed to the $\boldsymbol{a}$-space. Hence, the time derivative ∂_τ is applied to the vector $\boldsymbol{A}_a$ only on the right side of (84), since $\partial_\tau \boldsymbol{\Omega}_a = 0$ and $\partial_\tau \mathrm{d}^3\boldsymbol{a} = 0$.

The action J_A associated with L_A is defined by the same integral as (4), which is written here:

$$J_A = \int \Big(\int L_A \, \mathrm{d}\tau \Big) \mathrm{d}^3\boldsymbol{a}.$$

Substituting the above L_A of (84), we can integrate it with respect to τ. Hence, even this new term is added to the total Lagrangian L_* of (2), it is eliminated when deriving the Euler-Lagrange equation by the variational principle, described in Appendix A.

Similarly, the last two terms of the Lagrangian density Λ_* of (3) were also derived from the Lagrangians originally defined in the form of total time derivative by

$$L_\phi = -\frac{\mathrm{d}}{\mathrm{d}\tau}\int \phi\, \mathrm{d}^3\boldsymbol{a}, \quad L_\psi = -\frac{\mathrm{d}}{\mathrm{d}\tau}\int s\,\psi\, \mathrm{d}^3\boldsymbol{a}.$$

Therefore, the action J defined by time integral is uninfluenced by these three additional terms because they are integrated with respect to τ.

However, these Lagrangians become non-trivial when transformed to the physical-space coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$ and when Eulerian variation is taken. This is because the Jacobian of the transformation $(a_1, a_2, a_3) \Leftrightarrow (x^1, x^2, x^3)$, *i.e.* $\partial(x^k)/\partial(a^l)$, is connected directly to the density ρ. Using the notations $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{a} = (a, b, c)$ instead of (x^1, x^2, x^3) and (a_1, a_2, a_3), the integration element $\mathrm{d}^3\boldsymbol{a} = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$ is replaced by

$$\mathrm{d}^3\boldsymbol{a} = \hat{\rho}\, \mathrm{d}^3\boldsymbol{x}, \quad \hat{\rho} \equiv \frac{\rho(t, \boldsymbol{x})}{\rho(0, \boldsymbol{x})} = \frac{\partial(a, b, c)}{\partial(x, y, z)}.$$

by (24) and by the definition of the Lagrangian coordinates $\boldsymbol{a} = (a, b, c)$. The vorticity $\boldsymbol{\omega}$ in the Eulerian $\boldsymbol{x}$-space is transformed to $\boldsymbol{\omega}_a$ in the Lagrangian $\boldsymbol{a}$-space, which can be shown to be invariant with respect to τ, *i.e.* $\partial_\tau \boldsymbol{\omega}_a = 0$.

Next, we consider transformation of the integrals in the $\boldsymbol{a}$-space (defined above) to those in the $\boldsymbol{x}$-space. For that purpose, it is useful to define a one-form V^1 by

$$\begin{aligned} V^1 &= V_a \mathrm{d}a + V_b \mathrm{d}b + V_c \mathrm{d}c \\ &= u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z. \end{aligned}$$

where $\boldsymbol{V}_a = (V_a, V_b, V_c)$, and $\mathrm{d}\boldsymbol{a} = (\mathrm{d}a, \mathrm{d}b, \mathrm{d}c)$. [In the above expressions, $V_a = ux_a + vy_a + wz_a$, where $x_a = \partial X/\partial a$, $u = X_\tau$, *etc.*] Its differential $\mathrm{d}V^1$ gives a two-form $\Omega^2 = \mathrm{d}V^1$:

$$\begin{aligned} \Omega^2 &= \Omega_a \mathrm{d}b \wedge \mathrm{d}c + \Omega_b \mathrm{d}c \wedge \mathrm{d}a + \Omega_c \mathrm{d}a \wedge \mathrm{d}b \qquad (85) \\ &= \omega_x \mathrm{d}y \wedge \mathrm{d}z + \omega_y \mathrm{d}z \wedge \mathrm{d}x + \omega_z \mathrm{d}x \wedge \mathrm{d}y, \qquad (86) \end{aligned}$$

where $\nabla_a \times \boldsymbol{V}_a = (\Omega_a, \Omega_b, \Omega_c) = \boldsymbol{\Omega}_a$, and $\nabla \times \boldsymbol{v} = (\omega_x, \omega_y, \omega_z) = \boldsymbol{\omega}$ is the vorticity.[6]

Furthermore, we introduce a vector potential $\boldsymbol{A}_a = (A_a, A_b, A_c)$ in the $\boldsymbol{a}$-space, and define its one-form A^1 by

$$\begin{aligned} A^1 &= A_a\, \mathrm{d}a + A_b\, \mathrm{d}b + A_c\, \mathrm{d}c \qquad (87) \\ &= A_x\, \mathrm{d}x + A_y\, \mathrm{d}y + A_z\, \mathrm{d}z. \qquad (88) \end{aligned}$$

The exterior product of A^1 of (87) and Ω^2 of (85) yields a three-form $\mathrm{d}^3\boldsymbol{a} = \mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c$ multiplied by a scalar product $\langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle$:

$$A^1 \wedge \Omega^2 = \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}\,, \qquad (89)$$

[6] Exterior product (represented by the symbol $\wedge$) of two one-forms $\mathrm{d}a$ and $\mathrm{d}b$ define a two-form $\mathrm{d}a \wedge \mathrm{d}b$, and other pairs of one-forms define corresponding two-forms. Three independent two forms make a vector-like composition such as defined by $(\mathrm{d}x^2 \wedge \mathrm{d}x^3,\ \mathrm{d}x^3 \wedge \mathrm{d}x^1,\ \mathrm{d}x^1 \wedge \mathrm{d}x^2)$, and $(\mathrm{d}a^2 \wedge \mathrm{d}a^3,\ \mathrm{d}a^3 \wedge \mathrm{d}a^1,\ \mathrm{d}a^1 \wedge \mathrm{d}a^2)$.

where $\langle \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle = A_a\Omega_a + A_b\Omega_b + A_c\Omega_c$ is a scalar product of $\boldsymbol{A}_a$ and $\boldsymbol{\Omega}_a$.

In the $\boldsymbol{x} = (x, y, z)$ space, the same exterior product of (88) and (86) gives the following form which is equivalent to (89):

$$A^1 \wedge \Omega^2 = \langle \boldsymbol{A}, \boldsymbol{\omega} \rangle \, \mathrm{d}^3\boldsymbol{x}\,, \qquad \mathrm{d}^3\boldsymbol{x} = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z. \tag{90}$$

It is obvious that the scalar product $\langle \boldsymbol{A}, \boldsymbol{\omega} \rangle$ is invariant under local rotational transformations in the $\boldsymbol{x}$ space.

From the equality of (89) and (90), we have $\langle \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle \mathrm{d}^3\boldsymbol{a} = \langle \boldsymbol{A}, \boldsymbol{\omega} \rangle \mathrm{d}^3\boldsymbol{x}$. Taking derivative with respect to τ, we have

$$\frac{\partial}{\partial \tau}\left[\langle \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle \, \mathrm{d}^3\boldsymbol{a} \right] = \langle \partial_\tau \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle \, \mathrm{d}^3\boldsymbol{a},$$

since $\partial_\tau \boldsymbol{\Omega}_a = 0$ and $\partial_\tau \mathrm{d}^3\boldsymbol{a} = 0$. In the $\boldsymbol{x}$-space, after integration with $-$ sign added, both sides are represented as

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int \left[\langle \boldsymbol{A}, \boldsymbol{\omega} \rangle \mathrm{d}^3\boldsymbol{x} \right] \;=\; -\int \langle \mathcal{L}_t^*[\boldsymbol{A}], \, \boldsymbol{\Omega} \rangle \, \mathrm{d}^3\boldsymbol{x}\,.$$

Thus, the expression (26) in the main text is deduced. See [2, Chap.7] for mathematical detail of the above derivation.

C Generalized form of Clebsch solution

Extending the Clebsch solution of §2.2, one can define a generalized form of Clebsch solution by

$$\boldsymbol{v} = \nabla\phi + s\,\nabla\psi + \sum_{k=1}^{3} B_k\,\nabla a_k, \tag{91}$$

$$\tfrac{1}{2}\,v^2 + h + \partial_t\phi + s\partial_t\psi + \sum_{k=1}^{3} B_k\,\partial_t a_k = 0, \tag{92}$$

$$\mathrm{D}_t s = 0, \qquad \mathrm{D}_t\psi = 0, \tag{93}$$

$$\mathrm{D}_t B_k = 0, \quad \mathrm{D}_t a_k = 0, \quad (k = 1, 2, 3). \tag{94}$$

The third term of the velocity (91) and the last equations (94) are new terms. With the velocity (91), we have $\boldsymbol{\omega}$, $\boldsymbol{\omega} \times \boldsymbol{v}$ and $\partial_t \boldsymbol{v}$ given by

$$\begin{aligned}
\boldsymbol{\omega} &= \nabla \times \boldsymbol{v} = \nabla s \times \nabla\psi + \nabla B_k \times \nabla a_k. \\
\boldsymbol{\omega} \times \boldsymbol{v} &= (\boldsymbol{v}\cdot\nabla s)\nabla\psi - (\boldsymbol{v}\cdot\nabla\psi)\nabla s \\
&\quad +(\boldsymbol{v}\cdot\nabla B_k)\nabla a_k - (\boldsymbol{v}\cdot\nabla a_k)\nabla B_k, \\
\partial_t\boldsymbol{v} &= \nabla\partial_t\phi + \partial_t s\,\nabla\psi + s\,\nabla\partial_t\psi \\
&\quad +\partial_t B_k\,\nabla a_k + B_k\,\nabla\partial_t a_k.
\end{aligned}$$

where the symbol $\sum_{k=1}^{3}$ is omitted for terms including both a_k and B_k. Adding the last two and using (93) and (94), we obtain

$$\begin{aligned}
\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} &= \nabla(\partial_t\phi + s\,\partial_t\psi + B_k\,\partial_t a_k) \\
&\quad +(\mathrm{D}_t s)\,\nabla\psi + (\mathrm{D}_t B_k)\,\nabla a_k.
\end{aligned}$$

Last two terms vanish due to (93) and (94), and the first can be replaced by $-\nabla(\frac{1}{2}\,v^2 + h)$ by (92). Thus, this equation reduces to the following Euler's equation:

$$\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla\big(\tfrac{1}{2}\,v^2 + h\,\big).$$

Namely, the set of equations (91)∼(94) is a solution of the Euler equation.

Recently, a new Eulerian variational problem was investigated by [11], with fixing both ends of a path line in the variational calculus. This was based on the idea that the Eulerian variation should coincide with that of the Lagrangian description. What they obtained was the velocity $\boldsymbol{v}$ equivalent to the following:

$$\begin{aligned}
\boldsymbol{v} &= \nabla\phi + s\,\nabla\psi + \sum_{k=1}^{2} B_k\,\nabla a_k, \\
&\quad \tfrac{1}{2}\,v^2 + h + \partial_t\phi + s\,\partial_t\psi + \sum_{k=1}^{2} B_k\,\partial_t a_k = 0, \\
&\quad \mathrm{D}_t s = 0, \qquad \mathrm{D}_t\psi = 0, \\
&\quad \mathrm{D}_t B_k = 0, \qquad \mathrm{D}_t a_k = 0, \qquad (k = 1, 2).
\end{aligned}$$

Thus, the number of potentials B_k can be reduced from three to two. This was made possible because the density ρ is connected with the derivatives $\partial x^i/\partial a_j$ by the mass conservation condition of (25) [10].

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