Two-phase Flow of a Third Grade Fluid Between Parallel Plates

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*Abstract:*The two-phase flow of a third grade fluid between parallel plates is considered in three different cases. The Homotopy Analysis Method (HAM) is used to solve the nonlinear differential equations and the solutions up to second order of approximation are provided in case of Couette, Poiseuille and Couette-Poiseuille flow. The velocity profile is used to study qualitatively the effect of the physical parameters and in particular, of the fluids' material constants.

Keywords: fluid mechanics, homotopy analysis method, non-Newtonian fluid, third grade fluid, two-phase flow

1 Introduction

There are many industrial and manufacturing processes, for example oil industry or polymer production,where immiscible fluids flow in contact. This wide applicatin of flow of *n* adjacent fluids had stimulated the study of velocity profile in the last century [1][2]. The effect of viscosity in laminar flows, both in a planar channel and in a horizontal pipeof two immiscible fluids has been studied in [3].

Geophysical issues like flow of lava, snow avalanches and mud slides or issues related to medicine (for example blood and mucus) are also topics of intensive research[4]. The problem of mass transfer in continuous streching surfaces, used for example inpaper production and plastic films are discussed in [5], [6] and [7].

To study fluid flow in all cases listed above, as well as in many other technological applications presents challenges. These fluids do not follow the assumption of a linear relation between the stress and rate of strain at a point, or in other words, they are non-Newtonian fluids [8]. Despite these difficulties, many studies have been conducted in the area of heat and mass transfer in non – Newtonian fluids,(e.g. [9]).

The constitutive relations for non-Newtonian fluids are complicated and to create a constitutive model several different approachis have been used. As mentioned in [10] out of the many constitutive models of non-Newtonian fluids, one that has the support of experimentalist and theoreticians was first given in details by [11].

The particular class of interest to us in this paper is non-Newtonian fluids that have the following stress constitutive assumption (incompressible fluid):

$$\boldsymbol{S}= \boldsymbol{f}(\boldsymbol{A}_1,\boldsymbol{A}_2,\ldots\boldsymbol{A}_n), \qquad (1)$$

Where $\boldsymbol{S}$ is the Couchy stress and $\boldsymbol{A}_1$, $\boldsymbol{A}_2,\ldots\boldsymbol{A}_n$ are the Rivlin-Erickson tensors. One particular subclass with the following stress function

$$\boldsymbol{S}= \boldsymbol{f}(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \mu, \boldsymbol{A}_1,\boldsymbol{A}_2, \boldsymbol{A}_3), \qquad (2)$$

is referred to as a third grade fluid. There are some similarities between fluids of second and higher than second grade, as discussed in [12]. Third grade fluids provide more realistic models for researchers, but at the same time add difficulties in the solution process. The newly developed method, known as HAM [13], [14]has led to much improved solutions of several problems in fluid mechanics.

2 Formulation of the Problem

The aim of this work is to study two-layer flow of a non-Newtonian third grade fluid, without taking into consideration the interfacial instabilities. Considering the fully developed stage of steady laminar flow of two fluids located between two large parallel plates, three cases are examined:

1. *Plane Couette flow* – Two immiscible, incompressible fluids (1) and (2) of density $\rho_1, \rho_2 (\rho_1 > \rho_2)$ and viscosity μ_1, μ_2 flow between two parallel plates. The flow is induced by the motion of the upper plate which moves with constant speed *U*, directed along the $x-$axes. The lower plate is stationary. The origin of the Cartesian coordinates is taken to be on the plane of symmetry of the flow. The distance between the two plates is 2*b*.(Fig.1)

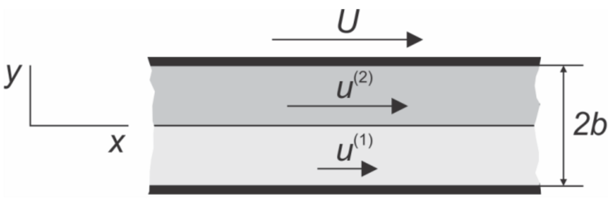

Fig.1 Two-layer Couette flow in a gap between infinite parallel plates driven by the upper plate.

2. *Poiseuille flow* - the fluid is between two stationary infinite plates, under constant pressure gradient (in the *x*-direction).

3. *Couette- Poiseuille flow* - the flow is driven by the upper plate which moves with constant velocity *U* and a pressure gradient is also applied.

In each of these cases no-slip conditions are satisfied; the gap is small compared to the plates' dimensions; the fluids are incompressible and the fluid below has density higher than the fluid above.
The constitutive equations for the two fluids are constructed based on the law of conservation of mass and momentum,

$$\nabla\, \mathbf{V} = 0$$
$$\rho \frac{D\mathbf{V}}{Dt} = \nabla \mathbf{S} - \nabla\, p + \mathbf{f}. \tag{3}$$

Where $\mathbf{V}$ is the velocity vector, ρ is the density of the fluid, $\frac{D}{Dt}$ is the material derivative, p is the pressure, and $\mathbf{f}$ represents the body forces.
The extra-stress tensor is

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 \left(\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1 \right) + \beta_3 \left(\mathrm{tr}\, \mathbf{A}_1^2 \right) \mathbf{A}_2, \tag{4}$$

where μ is the viscosity, α and β are material constants, $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivlin-Erickson tensors, defined as

$$\mathbf{A}_1 = \nabla\, \mathbf{V} + \left(\nabla\, \mathbf{V} \right)^T$$
$$\mathbf{A}_n = \frac{D}{Dt} \mathbf{A}_{n-1} + \mathbf{A}_{n-1} \left(\nabla\, \mathbf{V} \right) + \left(\nabla\, \mathbf{V} \right)^T ; n \geq 2. \tag{5}$$

In the expression for the extra-stress tensor $\beta_1 = 0$, according to [10].These equations can be written for each of the two fluids under consideration.
The flow is steady, fully-developed and the velocity and extra stress are

$$\mathbf{V} = \left[u(y), 0, 0 \right]; \quad \mathbf{S} = \mathbf{S}(y), \tag{6}$$

and (3) can be rewritten in the form

$$\rho \begin{bmatrix} 0 \\ 0 \end{bmatrix} = div \begin{bmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{bmatrix} - grad \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}, \tag{7}$$

or

$$\frac{dS_{xy}}{dy} - \frac{\partial p}{\partial x} = 0$$
$$\frac{dS_{yy}}{dy} - \frac{\partial p}{\partial y} = 0. \tag{8}$$

Introducing a generalized pressure $\hat{p} = p - S_{yy}$, the equations take the form

$$\frac{dS_{xy}}{dy} - \frac{\partial \hat{p}}{\partial x} = 0$$
$$\frac{\partial \hat{p}}{\partial y} = 0. \tag{9}$$

From the independence of the generalized pressure of *y*, it follows that

$$\frac{dS_{xy}}{dy} = \frac{d\hat{p}}{dx}. \quad (10)$$

The expression for the stress is

$$S_{xy} = \mu\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3; \beta = \beta_2 + \beta_3. \quad (11)$$

Each of the three cases discussed in this paper will begin with the last two equations applied for the corresponding fluid and physical situation.

3 Homotopy Analysis Method (HAM)

HAM has been applied successfully in the last years for solving nonlinear differential equations in different areas and in particularly in fluid mechanics [15],[16],[17]. One special area of application of this method is to solve equations arising when non-Newtonian fluids are studied. The method is used to find a solution in case of porous medium, [18],[19] thin fluid flows[20], [21], and Couette/Poiseuille flows[22].

The basic idea of HAM is described in[14]. A zero-order deformation equation is constructed,

$$\begin{aligned}&(1-q)\mathscr{L}\left[\phi(\vec{r},\, t;q) - f_0(\vec{r},\, t)\right]\\&= qhH(\vec{r},\, t)N\left[\phi(\vec{r},\, t;q)\right],\end{aligned} \quad (12)$$

where $q \in [0,1]$ is an embedding parameter, h is a non-zero auxiliary parameter , $H(\vec{r},\, t)$ is a non-zero auxiliary function and $\mathscr{L}$ is an auxiliary linear operator, $f_0(\vec{r},\, t)$ is an initial guess for $f(\vec{r},\, t)$, $\phi(\vec{r},\, t;q)$ is an unknown function. In case $q=0$ and $q=1$, $\phi(\vec{r},\, t;q)$ deforms from the initial guess $\phi(\vec{r},\, t;0) = f_0(\vec{r},\, t)$ to the solution $\phi(\vec{r},\, t;1) = f(\vec{r},\, t)$. Expanding $\phi(\vec{r},\, t;q)$ in a Taylor series with respect to the embedding parameter, one has

$$\begin{aligned}&\phi(\vec{r},\, t;q) = f_0(\vec{r},\, t) + \sum_{k=1}^{\infty} f_k(\vec{r},\, t) q^k\\&f_k(\vec{r},\, t) = \frac{1}{k!}\left.\frac{\partial^k\phi(\vec{r},\, t;q)}{\partial q^k}\right|_{q=0}.\end{aligned} \quad (13)$$

If the linear operator, the initial guess, the auxiliary parameter and the auxiliary functions are properly chosen so that the above series converges at $q=1$, one of the solutions of the non-linear equations is

$$f(\vec{r},\, t) = f_0(\vec{r},\, t) + \sum_{k=1}^{\infty} f_k(\vec{r},\, t). \quad (14)$$

4 The HAM Solution for the Current Problem

Here the HAM is employed to find solutions of nonlinear differential equations that describe the flow of a third grade non-Newtonian fluid in case of Couette, Poseuielle and Couette – Poseuielle flow. Skipping the details, since they can be found in[13], [14] and other sources, we will give only a brief idea how HAM was applied in this particular set of problems. The following homotopy was considered for each of the two layers, where the superscript indicates the layer number (lower fluid is labeled1):

$$\begin{aligned}&\tilde{\mathscr{H}}\left(\tilde{u}^{(1)};q,h_1,h_2,H\right) =\\&(1-q)\mathscr{L}\left[\tilde{u}^{(1)}(y;q,h_1,h_2,H) - u_0^{(1)}(y)\right]\\&-qh_1HN\left[\tilde{u}^{(1)}(y;q,h_1,h_2,H)\right]\\&\tilde{\mathscr{H}}\left(\tilde{u}^{(2)};q,h_1,h_2,H\right) =\\&(1-q)\mathscr{L}\left[\tilde{u}^{(2)}(y;q,h_1,h_2,H) - u_0^{(2)}(y)\right]\\&-qh_2HN\left[\tilde{u}^{(2)}(y;q,h_1,h_2,H)\right].\end{aligned} \quad (15)$$

In this expression H is a non-zero auxiliary function, h_1 and h_2 are non-zero auxiliary parameters, $q \in [0,1]$ is an embedding parameter and $\tilde{u}^{(1)}(y;q,h_1,h_2,H)$, $\tilde{u}^{(2)}(y;q,h_1,h_2,H)$ are solutions of equations $\tilde{\mathscr{H}}\left[\tilde{u}^{(1)}(y;q,h_1,h_2,H)\right] = 0$ and $\tilde{\mathscr{H}}\left[\tilde{u}^{(2)}(y;q,h_1,h_2,H)\right] = 0$ respectively. Function $\tilde{u}^{(1)}(y;q,h_1,h_2,H)$ will vary from the initial approximation $\tilde{u}^{(1)}(y;0) = u_0^{(1)}(y)$ to the exact solution $\tilde{u}^{(1)}(y;1) = u^{(1)}(y)$ as the embedding parameter varies from zero to one. Same is applied for $\tilde{u}^{(2)}(y;q,h_1,h_2,H)$. Auxiliary linear, $\mathscr{L}\left[\tilde{u}^{(1)};q,h_1,h_2,H\right]$ and nonlinear, $N\left[\tilde{u}^{(1)};q,h_1,h_2,H\right]$ operators are chosen in a proper manner, following the rules of HAM. The linear operator is $\mathscr{L}\left[\tilde{u}^{(1)};q,h_1,h_2,H\right] = \frac{\partial\tilde{u}^{(1)}(y;q,h_1,h_2,H)}{\partial y}$ applied to the lower fluid, and similar in case of the upper fluid. For simplicity the auxiliary function is chosen to be $H=1$.We have to keep in mind that only a proper choice of the parameters h_1, h_2 will ensure convergence of the solution.

4.1 Couette flow

Using (10)and (11), in the absence of a pressure gradient, i.e., $\frac{d\hat{p}}{dx}=0$, the governing equations are

$$\begin{aligned}\frac{du^{(1)}}{dy}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{du^{(1)}}{dy}\right)^3&=\frac{A}{\mu^{(1)}}\\ \frac{du^{(2)}}{dy}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{du^{(2)}}{dy}\right)^3&=\frac{A}{\mu^{(2)}}.\end{aligned} \tag{16}$$

In these equations $\beta^{(1)}, \beta^{(2)}$ are positive constants representing the material constants of the lower and upper fluid respectively, A is the constant of integration. The following boundary conditions reflect the fact that the lower fluid has zero velocity at $y=-b$, the upper fluid has velocity U at $y=b$, the velocities and stress of the two fluids at $y=0$ are equal:

$$\begin{aligned}&u^{(1)}(-b)=0\\ &u^{(2)}(b)=U\\ &u^{(1)}(0)=u^{(2)}(0)\\ &S_{xy}^{(1)}(0)=S_{xy}^{(2)}(0).\end{aligned} \tag{17}$$

The zero order deformation equations for the two fluids are

$$\begin{aligned}&(1-q)\mathscr{L}\left[\tilde{u}^{(1)}(y;q)-u_0^{(1)}(y)\right]=\\ &h_1q\left[\frac{\partial\tilde{u}^{(1)}(y;q)}{\partial y}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{\partial\tilde{u}^{(1)}(y;q)}{\partial y}\right)^3-\frac{A}{\mu^{(1)}}\right]\\ &(1-q)\mathscr{L}\left[\tilde{u}^{(2)}(y;q)-u_0^{(2)}(y)\right]=\\ &h_2q\left[\frac{\partial\tilde{u}^{(2)}(y;q)}{\partial y}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{\partial\tilde{u}^{(2)}(y;q)}{\partial y}\right)^3-\frac{A}{\mu^{(2)}}\right].\end{aligned} \tag{18}$$

In case $q=0$, the right side of these equations is zero and we have

$$\begin{aligned}&\mathscr{L}\left[\tilde{u}^{(1)}(y;0)-u_0^{(1)}(y)\right]=0\\ &\mathscr{L}\left[\tilde{u}^{(2)}(y;0)-u_0^{(2)}(y)\right]=0.\end{aligned} \tag{19}$$

According to the property of linear operator [14] it follows

$$\begin{aligned}\tilde{u}^{(1)}(y;0)&=u_0^{(1)}(y)\\ \tilde{u}^{(2)}(y;0)&=u_0^{(2)}(y).\end{aligned} \tag{20}$$

In case $q=1$, the equations are transformed into nonlinear differential equations, equivalent to the governing equations

$$\begin{aligned}\frac{\partial\tilde{u}^{(1)}(y;1)}{\partial y}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{\partial\tilde{u}^{(1)}(y;1)}{\partial y}\right)^3-\frac{A}{\mu^{(1)}}&=0\\ \frac{\partial\tilde{u}^{(2)}(y;1)}{\partial y}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{\partial\tilde{u}^{(2)}(y;1)}{\partial y}\right)^3-\frac{A}{\mu^{(2)}}&=0,\end{aligned} \tag{21}$$

where

$$\begin{aligned}\tilde{u}^{(1)}(y;1)&=u^{(1)}(y)\\ \tilde{u}^{(2)}(y;1)&=u^{(2)}(y).\end{aligned} \tag{22}$$

As the parameter varies from zero to one, the function $\tilde{u}(y;q)$ varies from the initial approximation $\tilde{u}(y;0)=u_0(y)$ to the exact solution $\tilde{u}(y;1)=u(y)$ together with A that varies from A_0 to A, where $A=A_0+\sum_{i=1}^{m}A_i$.

According to the basic idea of HAM one has freedom to choose not only the auxiliary function and nonlinear operator, but the auxiliary parameters *h₁*, *h₂* and the initial approximation of the solution of the equation. The proper choice of these parameters will ensure the existence of solution of the zero order differential equation, subjected to the initial conditions for parameter $q\in[0,1]$. Next the $m^{\text{-th}}$ derivative of $\tilde{u}(y;q)$ can be expressed in a Taylor series

$$\begin{aligned}\tilde{u}^{(1)}(y;q)&=u_0^{(1)}(y)+\sum_{m=1}^{\infty}u^{(1)}{}_m(y;q)q^m\\ \tilde{u}^{(2)}(y;q)&=u_0^{(2)}(y)+\sum_{m=1}^{\infty}u^{(1)}{}_m(y;q)q^m,\end{aligned} \tag{23}$$

where in case $q=0$ the derivatives are

$$\begin{aligned}u^{(1)}{}_m(y;q)&=\frac{1}{m!}\frac{\partial^m\tilde{u}^{(1)}(y;q)}{\partial q^m}\bigg|_{q=0}\\ u^{(2)}{}_m(y;q)&=\frac{1}{m!}\frac{\partial^m\tilde{u}^{(2)}(y;q)}{\partial q^m}\bigg|_{q=0}.\end{aligned} \tag{24}$$

The boundary conditions are

$$\begin{aligned} &u_0^{(1)}(-b;q)=0 \\ &u_0^{(1)}(0;q)=u_0^{(2)}(0;q) \\ &u_0^{(2)}(b;q)=U\ . \end{aligned} \tag{25}$$

The zero order solution will be in case of no differentiation for q when $q=0$. These solutions are

$$\begin{aligned} u_0^{(1)} &= \frac{A_0}{\mu^{(1)}}(y+b); \\ u_0^{(2)} &= \frac{A_0}{\mu^{(2)}}(y-b)+U, \end{aligned} \tag{26}$$

$$A_0=\frac{\mu^{(1)}\mu^{(2)}U}{b\left(\mu^{(1)}+\mu^{(2)}\right)}.$$

After differentiating the zero order deformation equation with respect to q and equating q to zero, the first order deformation equations take the form

$$\begin{aligned} &\mathscr{L}\left[u_1^{(1)}-0\right]=h_1\,N\left(\tilde{u}^{(1)}(y;q)\right)\Big|_{q=0} \\ &\mathscr{L}\left[u_1^{(2)}-0\right]=h_2\,N\left(\tilde{u}^{(2)}(y;q)\right)\Big|_{q=0}, \end{aligned} \tag{27}$$

where

$$\begin{aligned} N\left(\tilde{u}^{(1)}(y;q)\right) &= \frac{\partial \tilde{u}^{(1)}(y;q)}{\partial y}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{\partial \tilde{u}^{(1)}(y;q)}{\partial y}\right)^3 \\ &-\frac{A_0}{\mu^{(1)}}-\frac{A_1}{\mu^{(1)}} \\ N\left(\tilde{u}^{(2)}(y;q)\right) &= \frac{\partial \tilde{u}^{(2)}(y;q)}{\partial y}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{\partial \tilde{u}^{(2)}(y;q)}{\partial y}\right)^3 \\ &-\frac{A_0}{\mu^{(2)}}-\frac{A_1}{\mu^{(2)}}. \end{aligned} \tag{28}$$

Integration with respect to y of the first order deformation equation will lead to the first order approximation solution

$$\begin{aligned} u_1^{(1)} &= \left[\frac{2h_1A_0^3\beta^{(1)}}{\mu^{(1)^4}}-h_1\frac{A_1}{\mu^{(1)}}\right]y+E_{1u^{(1)}} \\ u_1^{(2)} &= \left[\frac{2h_2A_0^3\beta^{(2)}}{\mu^{(2)^4}}-h_2\frac{A_1}{\mu^{(2)}}\right]y+E_{1u^{(2)}}, \end{aligned} \tag{29}$$

with the help of boundary conditions

$$\begin{aligned} &u_1^{(1)}(-b;q)=0 \\ &u_1^{(1)}(0;q)=u_1^{(2)}(0;q) \\ &u_1^{(2)}(b;q)=0, \end{aligned} \tag{30}$$

the constants of integration can be found to be

$$\begin{aligned} E_{1u^{(1)}} &= \left[\frac{2h_1A_0^3\beta^{(1)}}{\mu^{(1)^4}}-h_1\frac{A_1}{\mu^{(1)}}\right]b \\ E_{1u^{(1)}} &= -\left[\frac{2h_2A_0^3\beta^{(2)}}{\mu^{(2)^4}}-h_2\frac{A_1}{\mu^{(2)}}\right]b, \end{aligned} \tag{31}$$

where

$$\begin{aligned} A_1 &= 2kA_0^3\left(\frac{h_1\beta^{(1)}}{\mu^{(1)^4}}+\frac{h_2\beta^{(2)}}{\mu^{(2)^4}}\right) \\ k &= \frac{\mu^{(1)}\mu^{(2)}}{h_2\mu^{(1)}+h_1\mu^{(2)}}. \end{aligned} \tag{32}$$

The first approximations of the velocities of the two fluids can be written as

$$\begin{aligned} u_1^{(1)} &= h_1\left[\frac{2A_0^3\beta^{(1)}}{\mu^{(1)^4}}-\frac{A_1}{\mu^{(1)}}\right](y+b) \\ u_1^{(2)} &= h_2\left[\frac{2A_0^3\beta^{(2)}}{\mu^{(2)^4}}-\frac{A_1}{\mu^{(2)}}\right](y-b). \end{aligned} \tag{33}$$

To find the second approximation one has to differentiate the first order deformation equation with respect to q and set $q=0$. The result is

$$\begin{aligned} &\mathscr{L}\left[u_2^{(1)}-u_1^{(1)}\right]=h_1\frac{\partial N\left(\tilde{u}^{(1)}(y;q)\right)}{\partial q}\Bigg|_{q=0} \\ &\mathscr{L}\left[u_2^{(2)}-u_1^{(2)}\right]=h_2\frac{\partial N\left(\tilde{u}^{(2)}(y;q)\right)}{\partial q}\Bigg|_{q=0}. \end{aligned} \tag{34}$$

Integrating with respect to y, the second order approximation of the velocities is

$$\begin{aligned} u_2^{(1)} &= (1+h_1)u_1^{(1)}+\frac{6h_1\beta^{(1)}A_0^2}{\mu^{(1)^3}}\left[\frac{2h_1A_0^3\beta^{(1)}}{\mu^{(1)^4}}-h_1\frac{A_1}{\mu^{(1)}}\right]y \\ &-h_1\frac{A_2}{\mu^{(1)}}y+E_{2u^{(1)}} \\ u_2^{(2)} &= (1+h_2)u_1^{(2)}+\frac{6h_2\beta_vA_0^2}{\mu^{(2)^3}}\left[\frac{2h_2A_0^3\beta^{(2)}}{\mu^{(2)^4}}-h_2\frac{A_1}{\mu^{(2)}}\right]y \\ &-h_2\frac{A_2}{\mu^{(2)}}y+E_{2u^{(2)}}. \end{aligned} \tag{35}$$

Appling the boundary conditions similar to (30), the constant of integration and the next term in the series representation of the constant A are

$$E_{2u^{(1)}}=\frac{6h_1\beta^{(1)}A_0^2}{\mu^{(1)^3}}\left[\frac{2h_1A_0^3\beta^{(1)}}{\mu^{(1)^4}}-h_1\frac{A_1}{\mu^{(1)}}\right]b-h_1\frac{A_2}{\mu^{(1)}}b$$

$$E_{2u^{(2)}}=-\frac{6h_2\beta_v A_0^2}{\mu^{(2)^3}}\left[\frac{2h_2A_0^3\beta^{(2)}}{\mu^{(2)^4}}-h_2\frac{A_1}{\mu^{(2)}}\right]b+h_2\frac{A_2}{\mu^{(2)}}b, \tag{36}$$

and

$$A_2=k\begin{Bmatrix}12A_0^5\left(\frac{h_1^2\beta^{(1)^2}}{\mu^{(1)^7}}+\frac{h_2^2\beta^{(2)^2}}{\mu^{(2)^7}}\right)\\-6A_0^2A_1\left(\frac{h_1^2\beta^{(1)}}{\mu^{(1)^4}}+\frac{h_2^2\beta^{(2)}}{\mu^{(2)^4}}\right)\end{Bmatrix}. \tag{37}$$

The second order approximations of the velocities of the two fluids are

$$u_2^{(1)}=(1+h_1)u_1^{(1)}+\left[\frac{12h_1^2A_0^5\beta^{(1)^2}}{\mu^{(1)^7}}-\frac{6h_1^2\beta^{(1)}A_0^2A_1}{\mu^{(1)^4}}-h_1\frac{A_2}{\mu^{(1)}}\right](y+b)$$

$$u_2^{(2)}=(1+h_2)u_1^{(2)}+\left[\frac{12h_2^2A_0^5\beta^{(2)^2}}{\mu^{(2)^7}}-\frac{6h_2^2\beta^{(2)}A_0^2A_1}{\mu^{(2)^4}}-h_2\frac{A_2}{\mu^{(2)}}\right](y-b). \tag{38}$$

4.2 Poiseuille flow

Since there is a pressure gradient $G=-\frac{d\hat{p}}{dx}$, the integration of equation (10) (applied for each layer), with the help of (11), will lead to the governing equations

$$\frac{du^{(1)}}{dy}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{du^{(1)}}{dy}\right)^3=-\frac{G}{\mu^{(1)}}y+\frac{B}{\mu^{(1)}}$$

$$\frac{du^{(2)}}{dy}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{du^{(2)}}{dy}\right)^3=-\frac{G}{\mu^{(2)}}y+\frac{B}{\mu^{(2)}}, \tag{39}$$

where B is a constant of integration. The stress for the two fluids at the boundary $y=0$ is the same $S_{xy}^{(1)}=S_{xy}^{(2)}$, which condition equates the constants of integration. The remaining boundary conditions (except the above mentioned) are

$$u^{(1)}(-b)=0$$
$$u^{(2)}(b)=0 \tag{40}$$
$$u^{(1)}(0)=u^{(2)}(0).$$

Following HAM, the zero order deformation equations for the two fluids are

$$(1-q)\mathscr{L}\left[\tilde{u}^{(1)}(y;q)-u_0^{(1)}(y)\right]=h_1q\left[\frac{\partial\tilde{u}^{(1)}(y;q)}{\partial y}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{\partial\tilde{u}^{(1)}(y;q)}{\partial y}\right)^3+\frac{G}{\mu^{(1)}}y-\frac{B}{\mu^{(1)}}\right] \tag{41}$$

$$(1-q)\mathscr{L}\left[\tilde{u}^{(2)}(y;q)-u_0^{(2)}(y)\right]=h_2q\left[\frac{\partial\tilde{u}^{(2)}(y;q)}{\partial y}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{\partial\tilde{u}^{(2)}(y;q)}{\partial y}\right)^3+\frac{G}{\mu^{(2)}}y-\frac{B}{\mu^{(2)}}\right].$$

In this expression $B=B_0+\sum_{i=1}^{m}B_i$ and $\tilde{u}^{(1)}(y;q)$, $\tilde{u}^{(2)}(y;q)$ are functions of the embedding parameter q and homotopy parameters h_1 and h_2.

There is a relation between the solutions in case $q=0$ and $q=1$ given by Taylor's expansion [14].

The zero order solutions are

$$u_0^{(1)}=\frac{B_0}{\mu^{(1)}}(y+b)-\frac{G}{2\mu^{(1)}}\left(y^2-b^2\right)$$
$$u_0^{(2)}=\frac{B_0}{\mu^{(2)}}(y-b)-\frac{G}{2\mu^{(2)}}\left(y^2-b^2\right), \tag{42}$$

$$B_0=\frac{Gb(\mu^{(1)}-\mu^{(2)})}{2\left(\mu^{(1)}+\mu^{(2)}\right)}.$$

These solutions satisfy the boundary conditions (applied for the zero order approximation). Using the first order deformation equations, similar to equation (27), where the non-linear operators are

$$N\left(\tilde{u}^{(1)}(y;q)\right)=\frac{\partial\tilde{u}^{(1)}(y;q)}{\partial y}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{\partial\tilde{u}^{(1)}(y;q)}{\partial y}\right)^3+\frac{G}{\mu^{(1)}}y-\frac{B_0}{\mu^{(1)}}-\frac{B_1}{\mu^{(1)}}$$

$$N\left(\tilde{u}^{(2)}(y;q)\right)=\frac{\partial\tilde{u}^{(2)}(y;q)}{\partial y}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{\partial\tilde{u}^{(2)}(y;q)}{\partial y}\right)^3+\frac{G}{\mu^{(2)}}y-\frac{B_0}{\mu^{(2)}}-\frac{B_1}{\mu^{(2)}}, \tag{43}$$

and integrating for y, together with the boundary conditions (applied for the first order approximation), the velocities of the two layers and the constant B_1 can be found to be

$$\begin{aligned} u_1^{(1)} &= -\frac{h_1\beta^{(1)}}{2\mu^{(1)^4}G}\left[\left(B_0-Gy\right)^4-\left(B_0+Gb\right)^4\right] \\ &\quad -h_1\frac{B_1}{\mu^{(1)}}\left(y+b\right) \\ u_1^{(2)} &= -\frac{h_2\beta^{(2)}}{2\mu^{(2)^4}G}\left[\left(B_0-Gy\right)^4-\left(B_0-Gb\right)^4\right] \\ &\quad -h_2\frac{B_1}{\mu^{(2)}}\left(y-b\right), \end{aligned} \tag{44}$$

$$\begin{aligned} B_1 &= \frac{k}{2Gb}\left[\frac{h_1\beta^{(1)}}{\mu^{(1)^4}}\left(B_0+Gb\right)^4-\frac{h_2\beta^{(2)}}{\mu^{(2)^4}}\left(B_0-Gb\right)^4\right] \\ &\quad -\frac{k}{2Gb}B_0^4\left(\frac{h_1\beta^{(1)}}{\mu^{(1)^4}}-\frac{h_2\beta^{(2)}}{\mu^{(2)^4}}\right). \end{aligned} \tag{45}$$

To find the second order approximation of the velocities it is necessary to differentiate again with respect to the embedding parameter. The result can be written in a form similar to (34). After integration for y and applying the boundary conditions (applied for the first and second order approximation) the second order approximation for the velocities is

$$\begin{aligned} u_2^{(1)} &= \left(1+h_1\right)u_1^{(1)}+\frac{2\beta^{(1)^2}h_1^2}{G\mu^{(1)^7}}\left[\left(B_0+Gb\right)^6-\left(B_0-Gy\right)^6\right] \\ &\quad -\frac{2\beta^{(1)}B_1h_1^2}{G\mu^{(1)^4}}\left[\left(B_0+Gb\right)^3-\left(B_0-Gy\right)^3\right]-h_1\frac{B_2}{\mu^{(1)}}\left(y+b\right) \\ u_2^{(2)} &= \left(1+h_2\right)u_1^{(2)}+\frac{2\beta^{(2)^2}h_2^2}{G\mu^{(2)^7}}\left[\left(B_0-Gb\right)^6-\left(B_0-Gy\right)^6\right] \\ &\quad -\frac{2\beta^{(2)}B_1h_2^2}{G\mu^{(2)^4}}\left[\left(B_0-Gb\right)^3-\left(B_0-Gy\right)^3\right]-h_2\frac{B_2}{\mu^{(2)}}\left(y-b\right). \end{aligned} \tag{46}$$

The constant B_2 is

$$B_2=\frac{2k}{Gb}\left\{\begin{aligned} &B_0^3B_1\left(\frac{h_1^2\beta^{(1)}}{\mu^{(1)^4}}-\frac{h_2^2\beta^{(2)}}{\mu^{(2)^4}}\right)-\frac{h_1^2B_1\beta^{(1)}}{\mu^{(1)^4}}\left(B_0+Gb\right)^3 \\ &+\frac{h_2^2B_1\beta^{(2)}}{\mu^{(2)^4}}\left(B_0-Gb\right)^3-\left(\frac{h_1^2\beta^{(1)^2}}{\mu^{(1)^7}}-\frac{h_2^2\beta^{(2)^2}}{\mu^{(2)^7}}\right)B_0^6 \\ &-\frac{h_2^2\beta^{(2)^2}}{\mu^{(2)^7}}\left(B_0-Gb\right)^6+\frac{h_1^2\beta^{(1)^2}}{\mu^{(1)^7}}\left(B_0+Gb\right)^6 \end{aligned}\right\} \tag{47}$$

4.3 Couette -Poiseuille flow

The flow in this case is driven by the upper plate, moving with velocity U and pressure gradient G. No-slip conditions are satisfied.

Starting again with equations (10) and (11), the governing equations for the two fluids are

$$\begin{aligned} \frac{du^{(1)}}{dy}+\frac{2\beta^{(1)}}{\mu^{(1)}}\left(\frac{du^{(1)}}{dy}\right)^3 &= -\frac{G}{\mu^{(1)}}y+\frac{C}{\mu^{(1)}} \\ \frac{du^{(2)}}{dy}+\frac{2\beta^{(2)}}{\mu^{(2)}}\left(\frac{du^{(2)}}{dy}\right)^3 &= -\frac{G}{\mu^{(2)}}y+\frac{C}{\mu^{(2)}}. \end{aligned} \tag{48}$$

The shear stress of the two fluids is equal at the boundary between the layers and the additional boundary conditions are

$$\begin{aligned} &u^{(1)}(-b)=0 \\ &u^{(2)}(b)=U \\ &u^{(1)}(0)=u^{(2)}\left(0\right). \end{aligned} \tag{49}$$

The zero order deformation equations for the two fluids are similar to equation (40) and zero order solutions are

$$\begin{aligned} u_0^{(1)} &= \frac{C_0}{\mu^{(1)}}\left(y+b\right)-\frac{G}{2\mu^{(1)}}\left(y^2-b^2\right) \\ u_0^{(2)} &= U+\frac{C_0}{\mu^{(2)}}\left(y-b\right)-\frac{G}{2\mu^{(2)}}\left(y^2-b^2\right), \end{aligned} \tag{50}$$

$$C_0=\frac{Gb(\mu^{(1)}-\mu^{(2)})}{2\left(\mu^{(1)}+\mu^{(2)}\right)}+\frac{Uk}{b}.$$

These solutions satisfy the boundary conditions given in equation (25). Using the first order deformation equations similar to equation (27), where the non-linear operators are similar to the operators given in equation (43) (different constant) and integrating for y, the first order solutions are

$$\begin{aligned} u_1^{(1)} &= -\frac{h_1\beta^{(1)}}{2\mu^{(1)^4}G}\left[\left(C_0-Gy\right)^4-\left(C_0+Gb\right)^4\right] \\ &\quad -h_1\frac{C_1}{\mu^{(1)}}\left(y+b\right) \\ u_1^{(2)} &= -\frac{h_2\beta^{(2)}}{2\mu^{(2)^4}G}\left[\left(C_0-Gy\right)^4-\left(C_0-Gb\right)^4\right] \\ &\quad -h_2\frac{C_1}{\mu^{(2)}}\left(y-b\right), \end{aligned} \tag{51}$$

where

$$C_1=-\frac{k}{2Gb}\left[\begin{aligned} &B_0^4\left(\frac{h_1\beta^{(1)}}{\mu^{(1)^4}}-\frac{h_2\beta^{(2)}}{\mu^{(2)^4}}\right)-\frac{h_1\beta^{(1)}}{\mu^{(1)^4}}\left(C_0+Gb\right)^4 \\ &+\frac{h_2\beta^{(2)}}{\mu^{(2)^4}}\left(C_0-Gb\right)^4 \end{aligned}\right]. \tag{52}$$

Following the same procedure, the second order approximation for the velocities is

$$u_2^{(1)} = (1+h_1)u_1^{(1)} + \frac{2\beta^{(1)^2}h_1^2}{G\mu^{(1)^7}}\left[(C_0+Gb)^6 - (C_0-Gy)^6\right] - \frac{2\beta^{(1)}C_1h_1^2}{G\mu^{(1)^4}}\left[(C_0+Gb)^3 - (C_0-Gy)^3\right] - h_1\frac{C_2}{\mu^{(1)}}(y+b)$$
$$u_2^{(2)} = (1+h_2)u_1^{(2)} + \frac{2\beta^{(2)^2}h_2^2}{G\mu^{(2)^7}}\left[(C_0-Gb)^6 - (C_0-Gy)^6\right] - \frac{2\beta^{(2)}C_1h_2^2}{G\mu^{(2)^4}}\left[(C_0-Gb)^3 - (C_0-Gy)^3\right] - h_2\frac{C_2}{\mu^{(2)}}(y-b), \quad (53)$$

where

$$C_2 = \frac{2k}{Gb}\left\{\begin{array}{l} C_0^3C_1\left(\frac{h_1^2\beta^{(1)}}{\mu^{(1)^4}} - \frac{h_2^2\beta^{(2)}}{\mu^{(2)^4}}\right) - \frac{h_1^2C_1\beta^{(1)}}{\mu^{(1)^4}}(C_0+Gb)^3 \\ + \frac{h_2^2C_1\beta^{(2)}}{\mu^{(2)^4}}(C_0-Gb)^3 - C_0^6\left(\frac{h_1^2\beta^{(1)^2}}{\mu^{(1)^7}} - \frac{h_2^2\beta^{(2)^2}}{\mu^{(2)^7}}\right) \\ + \frac{h_1^2\beta^{(1)^2}}{\mu^{(1)^7}}(C_0+Gb)^6 - \frac{h_2^2\beta^{(2)^2}}{\mu^{(2)^7}}(C_0-Gb)^6 \end{array}\right\} \quad (54)$$

5 Discussion

According to [14]h-curves provide a straightforward method to know the corresponding valid region for h. Choosing the value of h in the valid region ensures convergence of the corresponding solution series. It was also shown that different values of the parameter lead to convergence after different number of iterations. The convergence parameter could take in some cases values close to –1, or in others it can be as small as – 0.02.

It is notable from [19] that the value of h changes when the physical parameters associated with the problem change, i.e., the HAM method manifests advantage in cases when obtaining convergence of the series solution for some of the physical parameters was a problem, for example [23], while HAM solutions hold even for those values.

Recently, the convergence region has been determined based on the series expression of the solution and h-curve that reflect more than ten iterations. In these curves the convergence appears over an interval of the values of h. For example [24]determined suitable values of h using twenty iterations. The permissible values of h vary in the range $-0.95 < h < -0.5$, and convergence takes place after different numbers of iterations for different values of h.

In [25] we note the use of two optimal convergence-control parameters as well as a minimum of the square residual error to choose the proper value of the parameters.

We apply the HAM solution provided in this section to analyze and discuss qualitatively the effect of different physical parameters on the velocity profile. In each case the convergence parameter(s) were chosen based on the h_1, h_2-curves variation of the first derivative of the velocities of the two layers at$y$ = 0.A similar procedure was applied in[19]. As expected, the value(s) of h vary with the parameters and are different in most of the presented cases.

Fig.2 to Fig.5 present the velocity as a function of the distance between the plates in case of Couette flow. For each graph the value of the convergence parameter was chosen based on h-curves for $u^{(1)'}(0)$, $u^{(2)'}(0)$.

It follows that the variation in the profile is significant when the material constants $\beta^{(1)}$ and $\beta^{(2)}$ differ at a greater value. In case of Fig.2 b) and Fig.3 a) the difference in the material constants of two fluids is 0.7. This relation is depicted on Fig.4 where significant delay into the drag is observed in case b). In the other two cases Fig.2a) and Fig.3b) the velocity profile is close to the case of two identical fluids $\beta^{(1)} = \beta^{(2)}$ (Fig.5).

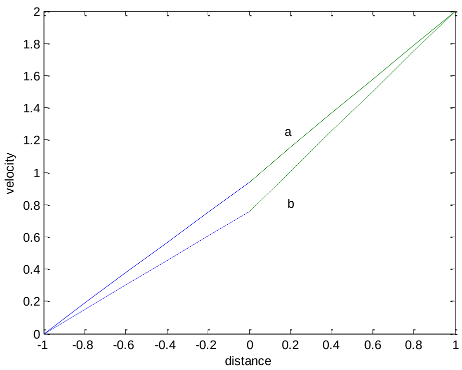

Fig.2 Velocity profile (Couette flow) in case $U = 2; b = 1; \mu^{(1)} = 1; \mu^{(2)} = 0.8; \beta^{(1)} = 0.8$ and

$\beta^{(2)} = 0.6;\ h_1 = h_2 = -0.2$ (a)

$\beta^{(2)} = 0.1;\ h_1 = h_2 = -0.3$ (b)

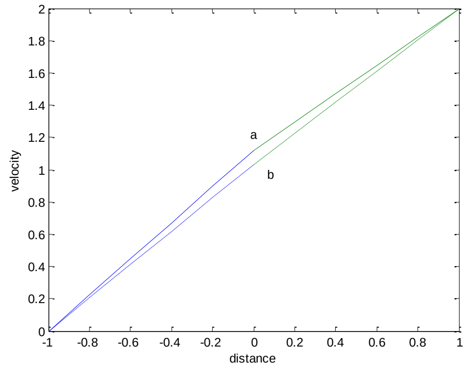

Fig.3 Velocity profile (Couette flow) in case $U=2; b=1; \mu^{(1)}=1; \mu^{(2)}=0.8; \beta^{(1)}=0.2$ and

$$\beta^{(2)}=0.9;\ h_1=h_2=-0.2\,(a)$$
$$\beta^{(2)}=0.4;\ h_1=h_2=-0.32\ \ (b)$$

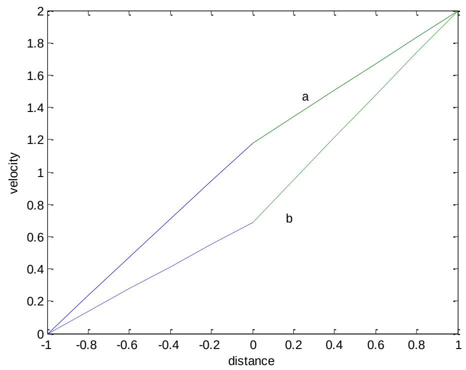

Fig.4 Velocity profile (Couette flow) in case $U=2; b=1; \mu^{(1)}=1; \mu^{(2)}=0.8$ and

$$\beta^{(1)}=0;\ \beta^{(2)}=0.9;\ h_1=h_2=-0.2\,(a)$$
$$\beta^{(1)}=0.8;\ \beta^{(2)}=0;\ h_1=h_2=-0.32\ \ (b)$$

The variation of the velocity profile with the variation of the pressure and material constants $\beta^{(1)}$, $\beta^{(2)}$ in case of Poiseuille flow is shown on Fig.6 to Fig.8. In all cases the value for the homotopy parameters is chosen following the criteria ensuring convergence.

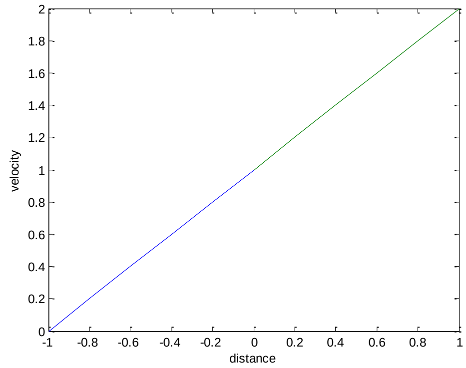

Fig.5 Velocity profile (Couette flow) in case $U=2; b=1; \mu^{(1)}=1; \mu^{(2)}=0.8\ h_1=h_2=-0.16$ and

$$\beta^{(1)}=\beta^{(2)}=0.8\,(a)\ \beta^{(1)}=\beta^{(2)}=0\ \ (b)$$

From Fig.6 the effect of the pressure gradient on the fluids' velocity is visible, as expected.In addition to this, the velocity profile is characterized with slight asymmetry ($\beta^{(2)}<\beta^{(1)}$).

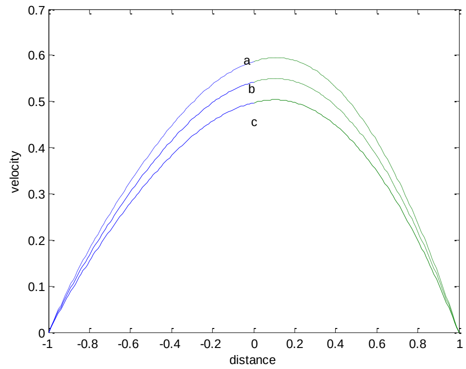

Fig.6 Velocity profile (Poisuille flow) in case $b=1; \mu^{(1)}=1; \mu^{(2)}=0.8;\ \beta^{(1)}=0.4; \beta^{(2)}=0$ and

$$G=1.2\ ;\ h_1=h_2=-0.33\,(a)$$
$$G=1.1;\ h_1=h_2=-0.375\,(b)$$
$$G=1\ ;\ h_1=h_2=-0.42\ \ (c)$$

Fig.7 shows the velocity profile for Newtonian/ non-Newtonian fluid in two cases (different values of the material constant). The non-Newtonian fluid appears to have a delay and the result is a significant shift in the velocity distribution.

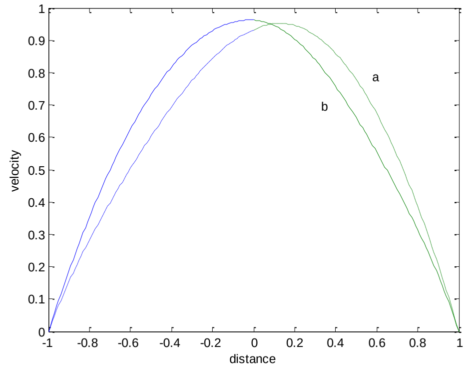

Fig.7 Velocity profile (Poiseuille flow) in case $G=2; b=1;\ \mu^{(1)}=1; \mu^{(2)}=0.8\ h_1=h_2=-0.1$ and $\beta^{(1)}=0.6; \beta^{(2)}=0$ (a) $\beta^{(1)}=0; \beta^{(2)}=0.4$ (b)

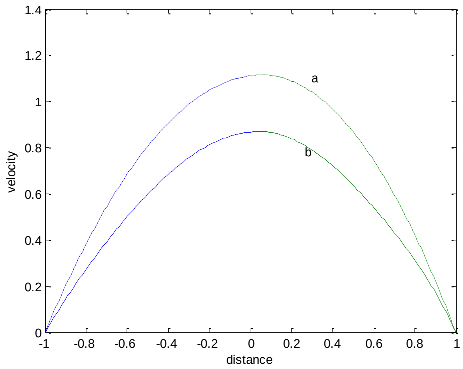

Fig.8 Velocity profile (Poisuille flow) in case $\beta^{(1)}=0; \beta^{(2)}=0$ (a) and $\beta^{(1)}=0.4; \beta^{(2)}=0.4$ (b) $G=2; b=1;\ \mu^{(1)}=1; \mu^{(2)}=0.8\ h_1=h_2=-0.1$

Fig.8 presents case of pair of Newtonian fluids and pair of non-Newtonian fluids with $\beta^{(1)}=\beta^{(2)}$.

Fig.9 to Fig.11 present the velocity profile for different values of the material constants and pressure gradient. Significant change of the profile with the change of the pressure gradient is presented on Fig.9, where the pressure gradient changes from a value *G* = 1 Fig.9 a) to *G* = 3 Fig.9c) and prevails on the effect of the drag velocity *U* = 2.

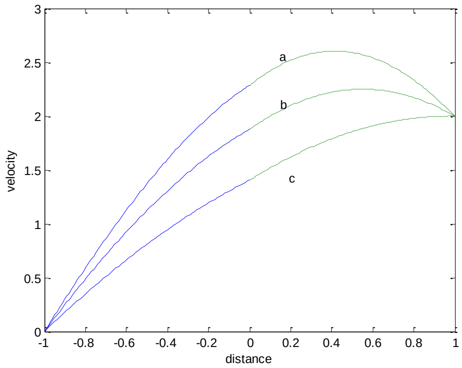

Fig.9 Velocity profile (Couette–Poiseuille flow)in case $U=2; b=1; \mu^{(1)}=1; \mu^{(2)}=0.8;\ \beta^{(1)}=0.4$ $\beta^{(2)}=0.6;\ h_1=h_2=-0.03$ and different values of pressure gradient $G=3$ (a); $G=2$ (b); $G=1$ (c)

The variation of the material constants $\beta^{(1)}$ and $\beta^{(2)}$ affects the velocity distribution, as it is shown on Fig.10 and Fig.11, even though not as significantly as the pressure gradient. There is a tendency of increasing values of velocity with the decrease of the material constants as one can see from Fig.10a), b) and c).

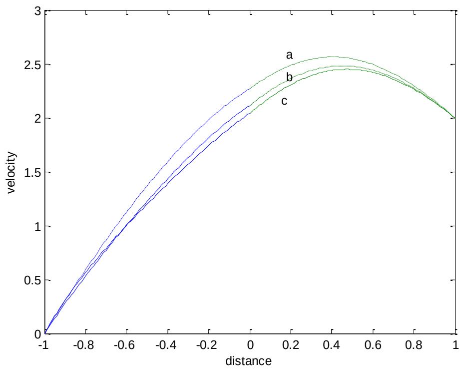

Fig.10 Velocity profile (Couette–Poiseuille flow) in case

$G=3; U=2; b=1; \mu^{(1)}=1; \mu^{(2)}=0.8; \beta^{(2)}=0.4$.

$h_1=h_2=-0.03$ and $\beta^{(1)}=0.2$ (a); 0.4 (b); 0.6 (c)

Fig.11 is included to emphasize the tendency of the velocity to increase with the decrease of the material constant value and in a vice-verse. As a result, the smooth transition at the boundary of the two fluids is disturbed as it can be seen from Fig.11 b).

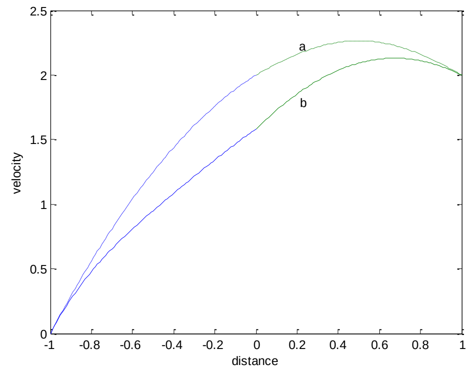

Fig.11 Velocity profile (Couette–Poiseuille flow) in case $U = 2; G = 2; b = 1; \mu_1 = 1; \mu_2 = 0.8$

$h_1 = h_2 = -0.05$ and

$\beta^{(1)} = 0; \beta^{(2)} = 1$ (a) and $\beta^{(1)} = 0.8; \beta^{(2)} = 0.2$ (b)

The solutions for each of the three cases were compared to the solution provided in [27] where Homotopy Perturbation Method is used. As expected, the solutions are identical in case of the $h = -1$. The problem with HPM is that for $h = -1$ the solutions diverge and therefore cannot be considered. As it is highlighted in many sources, the powerful tool of HAM is the homotopy parameter and the proper choice of this parameter will lead to convergent solutions.

The exact solutions for Poiseuille flow in one particular case of fluid parameters are given in [18] and the solutions are compared to those in [14]. The author came to the conclusion that despite similarity in the behavior, the results differ by a factor of 100. Qualitatively our results are different from the exact solutions for Poiseuille flow provided in [18]. The velocity profile and values onFig.8 b) appear to be close to those in [14].

6 Conclusions

Homotopy Analysis Method was successfully applied to obtain solutions of the governing equations for two-layer Couette, Poiseuille and Couette-Poiseuille fluid flow between parallel plates for a third-grade fluid. Solutions up to second order of approximation were obtained. A pair of convergence control parameters was used to ensure convergence of the solution.

The velocity profile was used to study qualitatively the effect of the physical parameters and in particular, of the fluids' material constants. The proper choice of the convergence parameters were based on constructing *h*-curves. Some of our results (single fluid flow) were compared to those given in [22] and [24]. The results presented in this work appear to be closer to the solutions given in [22].

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