Binormal Evolution of Curves with Prescribed Velocity

O. J. GARAY and A. PÁMPANO
University of the Basque Country UPV/EHU
Department of Mathematics
Aptdo. 644, 48080 Bilbao
SPAIN
oscarj.garay@ehu.es, alvaro.pampano@ehu.es

Abstract: At the end of the 19th century, Kirchhoff studied dynamical problems involving vortex flows of inviscid incompressible fluids focusing on flows having the shape of a vortex tube (*vortex filaments).* In 1906, Da Rios, a student of Levi-Civita, analyzed the motion of a vortex filament and obtained the remarkable equation describing its evolution, which, under mild conditions, is equivalent to the so called *binormal evolution equation*. Motivated by this, in this work we use fundamental facts of the theory of submanifolds to analyze the evolution of curves under binormal flows with curvature dependent velocity in pseudo-riemannian 3-space forms. The compatibility conditions for these systems are given by the Gauss-Codazzi equations, which here are expressed with respect to a geodesic coordinate system in terms of the Frenet curvatures of the evolving curves. Then, an existence result is derived from the Fundamental Theorem of submanifolds. Moreover, we show the connection between travelling wave solutions of the Gauss-Codazzi equations and the Frenet-Serret dynamics of curves. In fact, travelling wave solutions of the Gauss-Codazzi equations are shown to lead to the Euler-Lagrange equations of extremal curves for curvature dependent energies with a penalty on the total torsion and the length (*generalized Kirchhoff centerlines*). A characterization of generalized Kirchhoff centerlines in terms of Killing vector fields allows us to construct binormal evolution surfaces with prescribed velocity by using them as initial conditions for the evolution. Binormal surfaces obtained in this way evolve without change in shape. Finally, we particularize the previous findings to three significant cases which give rise to Hasimoto surfaces, Hopf tubes, and constant mean curvature surfaces

Key–Words: binormal flow, curve evolution, Frenet-Serret dynamics, extremal curves, submanifolds, real space forms.

1 Introduction

In 1906 Da Rios [15] modeled the movement of a thin vortex filament in a viscous fluid by the motion of a curve propagating in $\mathbb{R}^3$ according to

$$\frac{\partial x}{\partial t} = \frac{\partial x}{\partial s} \times \frac{\partial^2 x}{\partial s^2} \,, \tag{1}$$

which is known as the *localized induction equation,* (LIE). For notation consistency along this paper, LIE is often to be written as

$$x_t = x_s \times x_{ss} \,. \tag{2}$$

Hasimoto discovered [9] that LIE is equivalent to the *non-linear Schrödinger equation* (NLS) which is a well known example of soliton equation. If t represents time, $\gamma^t(s) := x(s,t)$ describes a curve evolving in $\mathbb{R}^3$ according to (1). Moreover, if the initial vortex filament is arc-length parametrized, then (2) can be written in terms of B, the Frenet binormal to the curve $\gamma^t(s)$

$$x_t = \kappa B \,,$$

and the curve is said to be evolving by the *binormal flow*. It locally determines a regular surface in $\mathbb{R}^3$ which is usually called a *Hasimoto surface*.

On the other hand, by choosing a geodesic coordinate system in an isometrically immersed surface (U, x) of $\mathbb{R}^3$, one can check that the following equation is satisfied on U

$$x_t = \phi \, x_s \times x_{ss} \,, \tag{3}$$

where ϕ is a function which depends on the surface metric coefficients with respect to (U, x). If $\phi = 1$, we have the Hasimoto surfaces. The following question naturally arises: given an arbitrary ϕ, when can we find a parametrized surface, (U, x), such that (3) is satisfied?

Along this paper a geometric generalization of the LIE will be studied within more general ambient spaces. To be more precise, let $M_r^3(\rho)$ denote a Riemannian or Lorentzian real space form (for details, see next section) then the following equation will be considered

$$x_t = f(s,t) \, x_s \times \widetilde{\nabla}_{x_s} x_s \,, \tag{4}$$

where, $x : U \subset \mathbb{R}^2 \to M_r^3(\rho)$, $f(s,t)$ is a $C^\infty(U)$ function and $\widetilde{\nabla}$ denotes the Levi-Civita connection on $M_r^3(\rho)$. Of course, if $f = 1$ and $M_r^3(\rho) = \mathbb{R}^3$, then we get the original LIE. Actually, we will be more interested in evolution equations of the type

$$x_t = f(|\widetilde{\nabla}_{x_s} x_s|)\, x_s \times \widetilde{\nabla}_{x_s} x_s \,, \qquad (5)$$

where the initial curve $\gamma(s) := x(s,0)$ is assumed to be arc-length parametrized. Then, so is $\gamma^t(s) := x(s,t)$ for all t and (5) becomes

$$x_t = \mathcal{Q}\,(\kappa) B \,, \qquad (6)$$

κ and B being, respectively, the Frenet curvatures and binormals of the evolving curves, and $\mathcal{Q}$ a certain smooth function. Here, surfaces in $M_r^3(\rho)$ evolving by (6) will be referred to as *binormal evolution surfaces with velocity $\mathcal{Q}$.*

In section 2 we revise fundamental formulas and equations of the theory of submanifolds, which are used in section 3 to find compatibility conditions for (4) in terms of the geometric invariants of the evolving curves. These conditions are basically a simplified version of Gauss and Codazzi equations for surfaces with respect to a geodesic coordinate system, and we will see that they extend to $M^n(\rho)$ backgrounds the classical *Da Rios equations* for LIE in $\mathbb{R}^3$, [15]. Then, an existence result and a few geometric evolution invariants are given. To end this section, we use the *complex wave function* introduced by Hasimoto [9] to find a non-linear equation which is equivalent to Codazzi equations (it reduces to NLS in case of LIE in $\mathbb{R}^3$). In order to construct explicit examples of binormal evolution surfaces, in section 4 we introduce what we call the *generalized Kirchhoff centerlines* as a family of curves representing particle trajectories under an specific Frenet-Serret dynamics. Then, in section 5 we find travelling wave solutions of the Codazzi-Da Rios conditions and show that they correspond to generalized Kirchhoff centerlines what allows us to construct binormal evolution surfaces spanned by filament motions involving no change in shape. Finally, in section 6 some applications to concrete choices of the evolution surface velocity are given. For the sake of brevity and simplicity, long computations will be omitted along this paper.

2 Preliminaries

For more details in this section, one may wish to consult [7]. Let $M_r^n(\rho)$ be a complete, connected, simply connected, pseudo-Riemannian n-manifold with constant sectional curvature ρ (a *pseudo-Riemannian space form*) with metric $\langle , \rangle$ and Levi-Civita connection $\widetilde{\nabla}$. If $\gamma : I \to M_r^n(\rho)$ is a smooth immersed curve in $M_r^n(\rho)$, $\dot\gamma(t)$ will represent its velocity vector $\frac{d\gamma(t)}{dt}$ and the covariant derivative of a vector field $X(t)$ along γ will be denoted by $\frac{DX(t)}{dt}$. A non-null curve can be parametrized by the arc-length and this natural parameter is called *proper time*. For a non-null immersed curve, the *first Frenet curvature*, or simply, *the curvature,* is defined as $\kappa_1 = \sqrt{\varepsilon_2 \langle \frac{D\dot\gamma(t)}{dt}, \frac{D\dot\gamma(t)}{dt} \rangle}$, where ε_2 denotes the causal character of $\frac{D\dot\gamma(t)}{dt}$. A geodesic is a constant speed curve whose tangent vector is parallel propagated along itself, i.e. a curve whose tangent, $\dot\gamma(t) = T(t)$, satisfies the equation $\frac{DT(t)}{dt} = 0$. Obviously, geodesics have zero curvature. As it can be shown, extremals of the variational problem we will be concerned in section 4 can be considered to live in a 3-space so we take $n = 3$, so from now on we restrict ourselves to pseudo-Riemannian 3-space forms, $M_r^3(\rho)$. When working in three dimensions, there are only two different options for the index r. The case $r = 0$, denoted simply by $M^3(\rho)$, represents the Riemannian 3-space forms (the Euclidean space, $\mathbb{R}^3$, if $\rho = 0$; the 3-dimensional sphere, $\mathbb{S}^3(\rho)$, when $\rho > 0$; and the hyperbolic space, $\mathbb{H}^3(\rho)$, when ρ is negative). The other case is $r = 1$ and now $M_1^3(\rho)$ represents the Lorentzian 3-space forms (the Minkowski flat space, $\mathbb{L}^3$, when $\rho = 0$; the de-Sitter 3-space, $\mathbb{S}_1^3(\rho)$, for positive ρ; and the anti-de-Sitter 3-space, $\mathbb{H}_1^3(\rho)$, for $\rho < 0$). The cases $r = 2$ and $r = 3$ are equivalent to $r = 1$ and $r = 0$, respectively.

Consider the Euclidean pseudo-space $\mathbb{E}_\nu^m$. That is, $\mathbb{R}^m$ endowed with the canonical metric of index ν, denoted by $\langle \cdot, \cdot \rangle$, and the Levi-Civita connection, denoted by $\bar\nabla$. Then, pseudo-Riemannian 3-space forms can be isometrically immersed in $\mathbb{E}_\nu^4$, the 4-dimensional Euclidean pseudo-space in a standard way, [7]. As usual, the cross product of two vector fields X, Y in $M_r^3(\rho)$, denoted by $X \times Y$, is defined so that $\langle X \times Y, Z \rangle = det(X, Y, Z)$ for any other vector field Z of $M_r^3(\rho)$.

An immersed curve in a pseudo-Riemannian manifold $M_r^3(\rho)$ is called a *Frenet curve of rank m*, $2 \le m \le 3$, if m is the highest integer for which there exists an orthonormal frame defined along γ, $\{e_1(t) = \dot\gamma(t), e_2(t), e_3(t)\}$ and non-negative smooth functions on γ, $\kappa_i(t)$, $t \in I$, $1 \le i \le m-1$, called the *Frenet curvatures*, such that the *Frenet-Serret equations* are satisfied (for more details see [8]). Let γ be a unit speed non-geodesic curve contained in $M_r^3(\rho)$ with non-null velocity $\dot\gamma = T$. If it also has non-null acceleration $\frac{D\dot\gamma}{ds}$, then γ is a Frenet curve of rank 2 or 3 and the classical standard Frenet frame along γ is given by $\{T = \dot\gamma, N = \frac{\varepsilon_2}{\kappa_1} \widetilde{\nabla}_T T, B\}$, and B is chosen so that $det(T,N,B)=1$. From now on, the first and second Frenet curvatures $\{\kappa_1,\kappa_2\}$ will be denoted by $\{\kappa,\tau\}$, and will be referred to as the *curvature* and *torsion* of γ in $M_r^3(\rho)$, respectively. Then, the *Frenet equations* can be written as

$$\begin{aligned} \frac{DT}{ds} &= \widetilde{\nabla}_T T = \varepsilon_2 \kappa N\,, \\ \frac{DN}{ds} &= \widetilde{\nabla}_T N = -\varepsilon_1 \kappa T + \varepsilon_3 \tau B\,, \\ \frac{DB}{ds} &= \widetilde{\nabla}_T B = -\varepsilon_2 \tau N\,, \end{aligned} \tag{7}$$

where ε_i, $1\leq i\leq 3$, denote the causal character of T, N and B, respectively, and the following relations hold

$$T=\varepsilon_1 N\times B\,,\; N=\varepsilon_2 B\times T\,,\; B=\varepsilon_3 T\times N\,. \tag{8}$$

In a pseudo-Riemannian space form any local geometrical scalar defined along Frenet curves can always be expressed as a function of their curvatures and derivatives.

Now, for a given isometric immersion of a surface, $x: N_\nu^2 \to M_r^3(\rho)$, $\nu\in\{0,1\}$, we denote by ∇ the Levi-Civita connection of the immersion (N_ν^2,x). As it is also customary, for a surface N_ν^2 in any 3-dimensional space form $M_r^3(\rho)$, we require the first fundamental form to be non-degenerate. Take X,Y,Z,W tangent vector fields to N_ν^2 and choose ξ a normal vector field to N_ν^2. Then the *formulas of Gauss and Weingarten* are [7]

$$\begin{aligned} \bar{\nabla}_X Y &= \widetilde{\nabla}_X Y - \rho\langle X,Y\rangle x \\ &= \nabla_X Y + h(X,Y) - \rho\langle X,Y\rangle x\,, \end{aligned} \tag{9}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X^\perp \xi\,. \tag{10}$$

where x is the position vector, h denotes the second fundamental form of N_ν^2 in $M_r^3(\rho)$, and $D^\perp$ denotes the connection on the normal bundle of N_ν^2.

Denoting by R and $\widetilde{R}$ the Riemann curvature tensors associated to ∇ and $\widetilde{\nabla}$, respectively, we have

$$\widetilde{R}(X,Y)Z = \rho(<Y,Z>X - <X,Z>Y)\,, \tag{11}$$

while the *equations of Gauss and Codazzi* are given respectively by, [7]

$$\begin{aligned} \langle \widetilde{R}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle \\ &\quad -\langle h(X,W),h(Y,Z)\rangle \\ &\quad +\langle h(X,Z),h(Y,W)\rangle\,, \end{aligned} \tag{12}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z)\,, \tag{13}$$

where

$$\begin{aligned} (\nabla h)(X,Y,Z) &= D_X^\perp h(Y,Z) - h(\nabla_X Y,Z) \\ &\quad - h(Y,\nabla_X Z). \end{aligned} \tag{14}$$

We will often resort to the standard abuse of notation and identification tricks in submanifold theory.

3 Binormal Evolution Surfaces

Given a smooth map $x: U\subset\mathbb{R}^2 \to M_r^3(\rho)$, $x(s,t)$, we want to investigate the following evolution problem

$$x_t = f(|\widetilde{\nabla}_{x_s} x_s|)\, x_s \times \widetilde{\nabla}_{x_s} x_s\,, \tag{15}$$

where f is a suitable smooth function. If $x(s,t)$ describes the evolution of $\gamma(s)=x(0,s)$ under (15), we assume that the *initial condition* $\gamma(s)$ is a unit speed Frenet curve of rank 2 or 3, and we denote by $\gamma^t(s) := x(s,t)$ the evolving copy of γ at time t which will be called *the filament at time* t. It is easy to show that if s is the proper time for $\gamma(s)$, that is the arc-length parameter, then so is for every t. In fact, we have

$$\frac{\partial}{\partial t}\langle x_s(s,t),x_s(s,t)\rangle = 2\,\langle \widetilde{\nabla}_{x_s} x_t, x_s\rangle = 0\,, \tag{16}$$

where the last equality is obtained from (15). That is $\langle x_s(s,t),x_s(s,t)\rangle$ does not depend on "time" t, so since $\langle x_s(s,0),x_s(s,0)\rangle = \langle \frac{d\gamma}{ds},\frac{d\gamma}{ds}\rangle = \langle T,T\rangle = \varepsilon_1$, then so is for every t. Thus, (15) is length-preserving evolution.

From now on, we will assume that s is the arc-length parameter and that $\widetilde{\nabla}_{x_s}x_s$ is non-null everywhere. Then for any fix t we may consider the associated Frenet frame $\{T=\dot\gamma,N,B\}(s,t)$ on $x(s,t)=\gamma^t(s)$ described in (7). We are going to assume also that f is never zero so that $x(s,t)$ defines an immersed surface in $M_r^3(\rho)$. Take $P(u)$ as any solution of the following differential equation

$$\dot{P}(u) := \frac{dP}{du} = \varepsilon_2\varepsilon_3 u f(u)\,. \tag{17}$$

Since our curves γ^t are arc-length parametrized, we can combine (7), (8) and (15) to obtain

$$\begin{aligned} x_t &= f(\kappa)x_s \times \widetilde{\nabla}_{x_s}x_s = f(\kappa)\,T\times\frac{DT}{ds} \\ &= \varepsilon_2\kappa f(\kappa)\; T\times N = \varepsilon_2\varepsilon_3\kappa f(\kappa)B \\ &= \dot{P}\,(\kappa)B\,. \end{aligned} \tag{18}$$

This means that $\gamma(s)=x(s,0)$ evolves by the binormal flow with velocity $\dot P(\kappa)$. The corresponding immersed surface (U,x) in $M_r^3(\rho)$ swept out by $\gamma(s)$ will be denoted S_γ and called a *binormal evolution surface* with initial condition γ (and velocity $\dot P$). If every filament curve γ^t is a closed curve, then the binormal evolution surface S_γ will be called *binormal evolution tube*. As $\gamma^t(s)=x(s,t)$ is not a geodesic in $M_r^3(\rho)$ then $\widetilde{\nabla}_{x_s} x_s$ is not null and the unit Frenet normal to $\gamma^t(s)$, $N(s,t)$, is parallel to the unit normal to S_γ, ξ, for s sufficiently small. This means that our filaments $\gamma^t(s)$ are geodesics in S_γ for any t and that (18) can be written as

$$x_t = \dot{P}(\kappa) N_\gamma \,, \tag{19}$$

where $\{T, N_\gamma\}$ is the Frenet frame along γ in S_γ. Hence, $x(s,t)$ are geodesics of S_γ evolving also by a normal flow within S_γ.

Let $x : U \to M_r^3(\rho)$ denote the immersion of the surface $x(U) = S_\gamma \equiv N_\nu^2$ in $M_r^3(\rho)$ with local orientation determined by the unit normal vector ξ. As before, denote by $\{T(s,t), N(s,t), B(s,t)\}$ the Frenet frame of $\gamma^t(s)$ and choose the following local adapted frame on S_γ

$$\begin{aligned} e_1 = x_s = T \,, \quad e_2 = \frac{x_t}{\dot{P}} = B \,, \\ e_3 = \xi = T \times B = -\varepsilon_2 N \,. \end{aligned} \tag{20}$$

With respect to the local parametrization of S_γ given by x

$$x(s,t) = \gamma^t(s) \,, \tag{21}$$

we have that the coefficients of the metric are (reparametrizing the geodesics if needed) $g_{11} = \langle x_s, x_s \rangle = \varepsilon_1$, $g_{12} = g_{21} = \langle x_s, x_t \rangle = 0$ and $g_{22} = \langle x_t, x_t \rangle = \varepsilon_3 \dot{P}^2$. That is, with respect to the parametrization (21) the metric of S_γ can be written as

$$g = \varepsilon_1 ds^2 + \varepsilon_3 \dot{P}^2 dt^2 . \tag{22}$$

Using the metric coefficients, g_{ij}, one may compute the Christoffel symbols of the Levi-Civita connection of (22) with respect to this parametrization (see, for instance [7], Proposition 1.1). Combining this with the Gauss and Weingarten formulas, (9), (10), and Gauss and Codazzi equations, (12), (13), all the geometric relevant information about the immersion (U,x) can be expressed in terms of the chosen parametrization (21). This requires bringing in some computational stuff and very long calculations whose details are omitted here (for technical background see, for example, [7]).

On the other hand, the second fundamental form can be considered as a quadratic form given by $h(X) := \langle A_\xi X, X \rangle$, and it admits the following expression with respect to the parametrization (21)

$$\varepsilon_2 h = -\kappa ds^2 + 2\tau \dot{P} ds\, dt + \varepsilon_2 \dot{P}^2 h_{22} dt^2 \,, \tag{23}$$

where $\kappa(s,t)$, $\tau(s,t)$ denote the curvature and torsion of the curves $\gamma^t(s)$ and h_{22} is given by

$$\begin{aligned} h_{22} =& \langle \widetilde{\nabla}_{e_2} e_2, e_3 \rangle \\ =& \frac{1}{\kappa} \{ \varepsilon_3 \frac{\dot{P}_{ss}}{\dot{P}} - \varepsilon_2 \tau^2 + \varepsilon_1 \varepsilon_3 \rho \} \,. \end{aligned} \tag{24}$$

Also, it can be shown that Gauss and Codazzi equations boil down to

$$\kappa_t = -2\dot{P}_s \tau - \tau_s \dot{P} \,, \tag{25}$$

$$\varepsilon_3 \tau_t = \tag{26}$$

$$\left(\frac{1}{\kappa} \left(\varepsilon_2 \dot{P}_{ss} + \varepsilon_1 \dot{P}(\kappa^2 - \varepsilon_1 \varepsilon_3 \tau^2 + \varepsilon_2 \rho) - \varepsilon_1 \kappa P \right) \right)_s .$$

Finally, it can be shown that a binormal evolution surface, with velocity $\dot{P}$ (18), S_γ, parametrized by (21), $x : U \to M_r^3(\rho)$, satisfies the PDE system

$$x_{ss} = -\frac{\kappa}{\dot{P}} x_s \times x_t - \varepsilon_1 \rho x \,, \tag{27}$$

$$x_{ts} = \frac{\dot{P}_s}{\dot{P}} x_t - \frac{\tau \dot{P}}{\kappa} x_{ss} - \varepsilon_1 \frac{\tau \dot{P}}{\kappa} \rho x \,, \tag{28}$$

$$\begin{aligned} x_{tt} =& -\varepsilon_1 \varepsilon_3 \dot{P} \dot{P}_s x_s - \varepsilon_2 \frac{h_{22} \dot{P}^2}{\kappa} x_{ss} \\ & - \left(\frac{\varepsilon_2 \varepsilon_1 h_{22}}{\kappa} + \varepsilon_3 \right) \dot{P}^2 \rho x + \frac{\dot{P}_t}{\dot{P}} x_t \,, \end{aligned} \tag{29}$$

where h_{22} is given in (24). Now, we have

Proposition 1. *Let $G : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. For any pair of functions $\kappa(s,t)$, $\tau(s,t)$ satisfying*

$$\kappa_t = -2G_s \tau - \tau_s G \,, \tag{30}$$

$$\tau_t = \varepsilon_1 \varepsilon_3 \kappa G_s + \varepsilon_2 (h_{22} G)_s \,. \tag{31}$$

where

$$h_{22} = \frac{1}{\kappa} \{ \varepsilon_3 \frac{G_{ss}}{G} - \varepsilon_2 \tau^2 + \varepsilon_1 \varepsilon_3 \rho \} \,, \tag{32}$$

there exists an isometric immersion $x : U \to M_r^3(\rho)$ foliated by a family of geodesics $\gamma^t(s) = x(s,t)$ evolving by

$$x_t(s,t) = G(s,t) B(s,t) \,,$$

$B(s,t)$ being the Frenet unit binormals defined on $\gamma^t(s)$, i.e., (U,x) is a binormal evolution surface with velocity $G(s,t)$. Moreover, if $P : \mathbb{R} \to \mathbb{R}$ is smooth and G is chosen so that $G(s,t) = \frac{dP}{du}(\kappa(s,t))$, then (U,x) evolves by (18).

Proof: Substituting $\dot{P}$ by G in (27)-(29) we obtain PDE system for which (30) and (31) are the compatibility conditions. Thus, the fundamental theorem of submanifolds says that given functions $\kappa(s,t)$, $\tau(s,t)$ and $G(s,t)$ smoothly defined on a connected domain U and satisfying (30)-(31), there exists a solution of

(27)-(29) determining a smooth isometric immersion (U, x) of a surface in $M_r^3(\rho)$ whose metric is given by

$$g = \varepsilon_1 ds^2 + \varepsilon_3 G^2 dt^2 \,, \tag{33}$$

and the second fundamental form by

$$\varepsilon_2 h = \kappa ds^2 - 2G\tau ds \otimes dt - \varepsilon_2 h_{22} G^2 dt^2 \,, \tag{34}$$

where h_{22} is defined in (32). Consider the coordinate curves $\gamma^t(s) := x(s,t)$. The first coefficient of the metric tells us that $\gamma^t(s)$ are arc-length parametrized $\forall t$, then we denote by $\{x_s = T(s,t), N(s,t), B(s,t)\}$ the Frenet frame along the coordinate curves $\gamma^t(s)$. By computing the Christoffel symbols we see that $\gamma^t(s)$ are geodesics. Then, combining Gauss equation (9) and Frenet formulas (7), we see that the unit normals $N(s,t)$ are perpendicular to the surface (U,x). Hence, $x_t = \lambda(s,t)B(s,t)$, but then the second coefficient of g implies that $G(s,t) = \lambda(s,t)$. □

The above parametrized surface (U,x) is foliated by geodesics having $\kappa(s,t)$ and $\tau(s,t)$ as curvature and torsion. If, in addition, U is simply connected, then the immersion (U,x) would be unique (up to rigid motions in $M_r^3(\rho)$). Moreover, for binormal evolution tubes (closed filaments) length, total torsion and curvature energy are invariant. More precisely

Proposition 2. *With the previous notation, let S_γ be a Binormal evolution surface with velocity $\dot{P}$, having by initial condition a Frenet curve of rank* 2 *or* 3 *in $M_r^3(\rho)$, $\gamma(s)$, parametrized by proper time. Denote by $x(s,t)$ the parametrization of S_γ determined by* (15) *and assume that $s \in [0,1]$ and all filament curves $\gamma^t(s) = x(s,t)$ are C^4-closed in* $[0,1]$. *Then, length of $\gamma^t(s)$, the total torsion $\int_o^1 \tau \, ds$, and $\int_o^1 P(\kappa)\, ds$, are independent of t.*

Proof: Assume that $s \in [0,1]$ and that for every t, $x(s,t)$ is C^4-closed in $[0,1]$. If the initial condition $\gamma(s)$ is parametrized by proper time, then so are the filaments $\gamma^t(s)$, so length is preserved. The invariance of the total torsion $\int_o^1 \tau \, ds$ is a direct consequence of (26). Finally, multiplying (25) by $\dot{P}$, we have

$$\begin{aligned} \frac{d}{dt}\int_0^1 P(\kappa)\, ds &= \int_0^1 \kappa_t \dot{P}\, ds \\ &= -\int_0^1 \frac{d}{ds}(\dot{P}^2\tau)\, ds = 0\,. \end{aligned} \tag{35}$$

Thus, $\int_o^1 P(\kappa)\, ds$ attains the same value at every filament curve. □

Corollary 3. *Under the conditions of the above proposition, if a copy of the filament is a closed minimizer for the energy $\int_\gamma P(\kappa)$, then so is any other copy of the filament.*

To end this section, we rewrite the Gauss-Codazzi equations (25) and (26) in terms of the *complex wave function*. The *Hasimoto transformation* [9] maps any curve $\gamma(s)$ with positive curvature $\kappa > 0$ and torsion τ into its complex wave function Ψ defined by

$$\Psi(s,t) = \kappa(s,t) e^{i\int_{s_o}^s \tau(s^*,t)ds^*} \,. \tag{36}$$

Moreover, the curve can be recovered (up to congruences in $M_r^3(\rho)$) from its complex wave function Ψ, in terms of its curvature and torsion, by taking

$$\kappa = \langle \Psi, \Psi \rangle^{\frac{1}{2}} \,, \tag{37}$$

$$\tau = Im(\frac{\Psi_s}{\Psi})\,. \tag{38}$$

Using this transformation and a choice of a suitable s_0 (such that, $\int \tau_t ds^*|_{s_0} = 0$), we can see that the Gauss-Codazzi equations (25) and (26) are equivalent to

$$\begin{aligned} \dot{P}_{ss}(1-\varepsilon_2\varepsilon_3) &= i\Psi_t + \left(\dot{P}\frac{\Psi}{|\Psi|}\right)_{ss} \\ &+ \varepsilon_1\varepsilon_3\left(\frac{\dot{P}}{|\Psi|}\left(|\Psi|^2 + \varepsilon_3\rho\right) - P\right)\Psi\,. \end{aligned} \tag{39}$$

Thus, we have that each solution of (39) gives rise to two functions (37) and (38), such that the only curves having them as curvature and torsion (up to rigid motions in $M_r^3(\rho)$) give a foliation of a binormal evolution surface with velocity $\dot{P}$, (18). The converse is clear, so we have

Proposition 4. $x(s,t)$ *evolves by* (18)*, if and only if, the complex wave function Ψ evolves by* (39)*.*

4 Generalized Kirchhoff Centerlines

Remember that we are assuming for all our curves that $\dot{\gamma}(t)$ and $\frac{D\dot{\gamma}}{dt}$ are not lightlike vectors along the curve. We shall denote by $\Omega_{p_op_1}$ the space of smooth immersed curves of $M_r^3(\rho)$ joining two points of it, i.e.

$$\begin{aligned} \Omega_{p_op_1} = \{\beta : [0,1] \to M_r^3(\rho), \frac{d\beta}{dt}(t) \neq 0, \\ \forall t \in [0,1], \beta(i) = p_i, i \in \{0,1\}\}\,, \end{aligned} \tag{40}$$

where $p_i \in M_r^3(\rho), i \in \{0,1\}$, are arbitrary fixed points of $M_r^3(\rho)$. As before the arc-length or natural parameter is represented by $s \in [0,L]$, L being the length of γ. We are going to consider *curvature energy functionals* acting on $\Omega_{p_op_1}$ of the following form

$$\begin{aligned} \Theta(\gamma) &= \int_\gamma \mathcal{F}(\kappa) + \mu\tau + \lambda \\ &= \int_0^L (\mathcal{F}(\kappa)(s) + \mu\tau(s) + \lambda)\, ds\,, \end{aligned} \tag{41}$$

where $\mathcal{F}(u)$ is a $C^\infty(\mathbb{R})$ function and $\mu, \lambda \in \mathbb{R}$. By computing the first variation formula for (41) acting on $\Omega_{p_o p_1}$ (under arbitrary boundary conditions) and using the Frenet frame (7), the following proposition can be shown

Proposition 5. *The Euler-Lagrange equations for the curvature energy functional $\Theta(\gamma) = \int_\gamma \mathcal{F}(\kappa) + \mu\tau + \lambda$, acting on $\Omega_{p_o p_1}$ can be written as*

$$\begin{aligned} \varepsilon_1\varepsilon_2\mu\kappa\tau - \varepsilon_1\varepsilon_2\kappa(\mathcal{F}+\lambda) &- \varepsilon_2(\varepsilon_3\tau^2 - \varepsilon_1\kappa^2)\dot{\mathcal{F}} \\ + \dot{\mathcal{F}}_{ss} + \varepsilon_1\rho\dot{\mathcal{F}} &= 0\,, \qquad (42) \\ 2\varepsilon_3\tau\dot{\mathcal{F}}_s + \varepsilon_3\tau_s\dot{\mathcal{F}} - \varepsilon_1\mu\kappa_s &= 0\,, \qquad (43) \end{aligned}$$

where the subscript s denotes the derivative with respect to s, and $\dot{\mathcal{F}} = \frac{d\mathcal{F}}{d\kappa}$.

For reasons that will be made clear later, along this paper curves satisfying above equations (42) and (43) will be called *generalized Kirchhoff centerlines*. A vector field W along γ, which infinitesimally preserves unit speed parametrization is said to be a *Killing vector field along* γ if it evolves in the direction of W without changing shape, only position. In other words, the following equations must hold

$$W(v)(\bar{t},0) = W(\kappa)(\bar{t},0) = W(\tau)(\bar{t},0) = 0\,, \quad (44)$$

($v = |\dot{\gamma}|$ being the speed of γ) and this is independent on the choice of the tangent variation of γ to W. The following proposition extends a result of [12] to the pseudo-riemannian case

Proposition 6. *Let $M_r^3(\rho)$ be a complete, simply connected, pseudo-Riemannian space form and γ a non-null immersed curve in $M_r^3(\rho)$. A vector field W on γ is a Killing vector field along γ, if and only if, it extends to a Killing field $\tilde{W}$ on $M_r^3(\rho)$.*

Finally, by using a family of variation formulas computed along the derivation of the Euler-Lagrange equations, we are able to characterize generalized Kirchhoff centerlines in terms of vector fields along the curve γ.

Proposition 7. *The vector field $\mathcal{I} = \varepsilon_1\varepsilon_3\mu T + \dot{P}B$ is a Killing vector field along γ, if and only if, γ is a generalized Kirchhoff centerline.*

5 Travelling Wave Solutions

Here, *travelling wave* is understood to be a function $u(x,t)$ of the form $u(x,t) = f(x - \eta t)$, $\eta \in \mathbb{R}$ for some smooth function f. We are going to analyze travelling wave solutions of the Gauss-Codazzi equations (25) and (26) associated to a binormal evolution surface with respect to the parametrization $x(s,t) = \gamma^t(s)$. Define $\iota = s - \varepsilon_1\varepsilon_3\mu t$ and take $\kappa(s,t) = \kappa(\iota), \tau(s,t) = \tau(\iota)$. Differentiating the Gauss-Codazzi equations (25) and (26) we get

$$\begin{aligned} \varepsilon_1\varepsilon_3\mu\kappa_\iota &= \dot{P}\tau_\iota + 2\dot{P}_\iota\tau\,, \qquad (45) \\ -\varepsilon_1\varepsilon_2\mu\kappa\tau &= \dot{P}_{\iota\iota} + \varepsilon_1\varepsilon_2\dot{P}\kappa^2 - \varepsilon_2\varepsilon_3\dot{P}\tau^2 \qquad (46) \\ &\quad + \varepsilon_1\rho\dot{P} - \varepsilon_1\varepsilon_2\kappa P - \varepsilon_2\varepsilon_3 c\kappa\,, \end{aligned}$$

for some $c \in \mathbb{R}$. Then, calling $\lambda = \varepsilon_1\varepsilon_3 c$, it is easy to verify that (45) and (46) are precisely the Euler-Lagrange equations, (42) and (43), for $\Theta(\gamma) = \int_\gamma P(\kappa) + \mu\tau + \lambda$. In other words, γ must be the a generalized Kirchhoff centerline. Hence, the next proposition shows how to construct solutions of binormal evolution surfaces in $M_r^3(\rho)$

Proposition 8. *Travelling wave solutions of Gauss-Codazzi equations (25) and (26) correspond to the curvature and torsion of generalized Kirchhoff centerlines. Moreover, generalized Kirchhoff centerlines evolve following* (18) *by isometries of $M_r^3(\rho)$ and slippery.*

Proof: The first part has been just stated. As for the second one, consider $\gamma(s)$ a generalized Kirchhoff centerline. Then, by Proposition 7, $\mathcal{I} = \varepsilon_1\varepsilon_3\mu T + \dot{P}B$ is a Killing vector field along γ. Denote by V the Killing vector field on $M_r^3(\rho)$ which extends $\mathcal{I}$ (Proposition 6) and denote by $\{\phi_t\}, t \in \mathbb{R}$, the 1-parameter group of isometries associated to V, i.e., the flow of V. Define the surface $y(s,t) := \phi_t(\gamma(s))$. Since $\{\phi_t\}, t \in \mathbb{R}$ are isometries, we have $y_t = \varepsilon_1\varepsilon_3\mu T + \dot{P}B$ and we see that, after reparametrizations, $y(s,t)$ evolve by (18). □

So travelling wave solutions of the Gauss-Codazzi equations represent binormal evolution surfaces S_γ, where the initial condition γ is a generalized Kirchhoff centerline. In this case γ evolves by rigid motions and slippery. Observe that the last assertion can be also derived combining Proposition 6 and Proposition 7. On the other hand, solutions evolving by congruences correspond to travelling wave solutions with $\mu = 0$ and it is a straightforward computation to verify that the Gauss-Codazzi equations (25) and (26) are equivalent to the Euler-Lagrange equation for $\Theta(\gamma) = \int_\gamma P(\kappa) + \lambda$. Then, γ evolves under (18) by isometries of $M_r^3(\rho)$, if and only if, γ is an extremal of $\Theta(\gamma) = \int_\gamma P(\kappa) + \lambda$. So, we get a foliation of the binormal evolution surface by critical points of Θ, that are also geodesics of the surface S_γ.

6 Applications

Let us consider now different choices of the velocity in the evolution equation

$$x_t = f(\kappa)\, x_s \times \widetilde{\nabla}_{x_s} x_s = \dot{P}(\kappa) B \,. \tag{47}$$

6.1 Hasimoto Surfaces in $M_r^3(\rho)$.

We first choose $f(u) = 1$. In other words, we are going to consider the evolution in $M_r^3(\rho)$ of a unit speed Frenet curve, $\gamma(s)$, of rank 2 or 3 under LIE, (2). Let $x(s,t)$ describe the evolution of $\gamma(s)$ under LIE. Since our curves γ are arc-length parametrized, LIE can be simplified in terms of the *binormal flow*

$$\frac{\partial x}{\partial t} = \varepsilon_2 \varepsilon_3 \kappa B \tag{48}$$

and the corresponding evolution surfaces are known as *Hasimoto surfaces*. Then (25) and (26) reduce to

$$\kappa_t = -\varepsilon_2\varepsilon_3(2\kappa_s \tau + \kappa \tau_s)\,, \tag{49}$$

$$\tau_t = \varepsilon_2(\varepsilon_2 \frac{\kappa_{ss}}{\kappa} - \varepsilon_3 \tau^2 + \frac{1}{2}\varepsilon_1 \kappa^2 + \varepsilon_1\varepsilon_2\rho)_s\,. \tag{50}$$

Notice that if we were considering evolution under LIE in the standard Euclidean case, then $\varepsilon_i = 1$, for $i = 1, 2, 3$, and (49), (50) would be precisely the *da Rios equations*, [15]. In other words, in the Euclidean case da Rios equations are nothing but the Gauss-Codazzi equation of Hasimoto surfaces expressed with respect to the geodesic coordinate system (21). By this reason the Gauss-Codazzi equations of the general case, (49) and (50), will be referred to as the *Codazzi-da Rios equations in space forms*.

On the other hand, now we have $\dot{P} = \varepsilon_2\varepsilon_3\kappa$, so that the complex wave equation corresponding to (49) and (50) boils down to

$$\begin{aligned} |\Psi|_{ss}(1 - \varepsilon_2\varepsilon_3) &= i\,\varepsilon_2\varepsilon_3\Psi_t + \Psi_{ss} \\ &\quad + \varepsilon_1\varepsilon_3\left(\frac{|\Psi|^2}{2} + \varepsilon_3\rho\right)\Psi\,. \end{aligned} \tag{51}$$

So we see that in the Euclidean space, $M_r^3(\rho) = \mathbb{R}^3$, (51) is nothing but the focusing nonlinear Schrödinger equation, while in the Minkowsy 3-space case, $M_r^3(\rho) = \mathbb{L}^3$, we obtain the defocusing nonlinear Schrödinger equation if $\varepsilon_2\varepsilon_3 = 1$.

For Hasimoto surfaces the energy Θ given in (41) is nothing but $\Theta(\gamma) = \int_\gamma \kappa^2 + \mu\tau + \lambda$. In $\mathbb{R}^3$, extremals of this functional are known to be centerlines of a Kirchhoff elastic rods, [13]. The converse is also true [10]. In other words, in $\mathbb{R}^3$ travelling wave solutions of the Gauss-Codazzi-Da Rios equations (49) and (50) determine Hasimoto surfaces S_γ whose initial conditions γ are centerlines of Kirchhoff rods. They evolve under LIE by rigid motions and slippery in $\mathbb{R}^3$, [13]. In [11] the notion of Kirchhoff elastic rods is extended to riemannian space forms, $M^3(\rho)$, and it is shown that centerlines of Kirchhoff elastic rods provide solutions to the Euler-Lagrange equations for $\Theta(\gamma) = \int_\gamma \kappa^2 + \mu\tau + \lambda$ in $M^3(\rho)$. This motivates our definition after Proposition 5. Moreover, classical elasticae in $M^3(\rho)$ evolve by rigid motions and correspond to soliton solutions of LIE. We also remark that centerlines of Kirchhoff elastic rods can be identified with magnetic trajectories of Killing magnetic fields in $M^3(\rho)$, [3].

6.2 Hopf Cylinders and Pure Binormal Evolution in $M^3(\rho)$.

Consider now the binormal evolution equation given by

$$x_t = B \tag{52}$$

in $M^3(\rho)$. In this case Θ given in (41) is nothing but

$$\Theta(\gamma) = \int_\gamma \kappa + \mu\tau + \lambda\,.$$

Critical curves for Lagrangians of the form $\Theta_{\eta\mu\lambda}(\gamma) = \int_\gamma \eta\kappa(s) + \mu\tau(s) + \lambda$, where $\eta, \mu, \lambda \in \mathbb{R}$, have been used to construct models of spinning relativistic particles, both massive and massless, in Lorentzian backgrounds, [14]. If the ambient space is a riemannian space form, $M^3(\rho)$, it is known that a curve $\gamma \in M^3(\rho)$ is critical for $\Theta_{\eta\mu\lambda}$, if and only if, γ is a Lancret helix in $M^3(\rho)$ [1] (i.e., a curve making constant angle with a fix unit Killing field in $M^3(\rho)$). Hence, Lancret curves evolve under (52) by congruence and slippery. The most interesting case is when $M^3(\rho) = \mathbb{S}^3(\rho)$ in which case Lancret curves are geodesics of Hopf tubes. Also, one can find a huge family of closed Lancret helices in $\mathbb{S}^3(\rho)$ giving rise, therefore, to compact binormal evolutions surfaces in $\mathbb{S}^3(\rho)$ verifying (52). However, these surfaces will show self-intersections, in general. Moreover the $\mathbb{S}^3(\rho)$ background is the only case in which the variational problem associated to the *total curvature energy*, $\mu = \lambda = 0$, makes sense, [1]. Assuming without loss of generality $\rho = 1$, critical curves of $\int_\gamma \kappa$ in $\mathbb{S}^3(1)$ are characterized by having torsion $\tau^2 = 1$. Hence, horizontal lifts via the Hopf map of arbitrary curves β of $\mathbb{S}^2(\frac{1}{2})$ evolve under $x_t = B$ by rigid motions and the corresponding binormal surface is a Hopf tube of $\mathbb{S}^3(1)$ shaped on β. In this case we can find binormal Hopf Tori with no self-intersections. Lorentzian versions of these results can be checked in [4].

6.3 Constant Mean Curvature Binormal Evolution Surfaces

The most important extrinsic invariant for a surface is, probably, the mean curvature, [7]. This motivates our interest in studying binormal evolution surfaces with constant mean curvature, $-H$.

Let $S_\gamma \subset M_r^3(\rho)$ be a binormal evolution surface and consider the coordinate system given in (21). Then, using (23), the mean curvature of a binormal evolution surface S_γ admits the following expression with respect to (21)

$$H = \frac{1}{2\kappa\dot{P}}(\dot{P}[\varepsilon_1\varepsilon_2\kappa^2 + \varepsilon_2\varepsilon_3\tau^2 - \varepsilon_1\rho] - \dot{P}_{ss})\,. \quad (53)$$

Thus, we have that S_γ has constant mean curvature H, if and only if,

$$\dot{P}[\varepsilon_1\varepsilon_2\kappa^2 + \varepsilon_2\varepsilon_3\tau^2 - \varepsilon_1\rho] - \dot{P}_{ss} = 2\kappa\dot{P}H\,, \quad (54)$$

for a fixed real number $H \in \mathbb{R}$. Now, we focus on binormal evolution surface such that the initial filament curve γ evolves by rigid motions. We know that this is equivalent to the fact that γ is critical for the energy (41) with $\mu = 0$, that is, for $\Theta(\gamma) = \int_\gamma (P(\kappa) + \lambda) ds$. So, the Euler-Lagrange equations (42) and (43) for $\mu = 0$ must be verified. Then, combining (43) and (54) we get

Proposition 9. *Assume that a binormal evolution surface S_γ is obtained from an initial filament γ under evolution by isometries of $M_r^3(\rho)$. Then, S_γ has constant mean curvature, H, if and only if, the initial filament γ is critical for the curvature energy $\Theta(\gamma) = \int_\gamma (\kappa - \varepsilon_1\varepsilon_2 H)^{\frac{1}{2}}\, ds$.*

As an immediate consequence, we obtain for minimal surfaces $H = 0$ (maximal, if the surface is spacelike)

Corollary 10. *Under the above conditions, a binormal evolution surface S_γ is minimal (maximal), if and only if, γ is critical for $\Theta(\gamma) = \int_\gamma \sqrt{\kappa}\, ds$.*

Actually, constant mean curvature surfaces in $\mathbb{R}^3$ which are either rotational or helicoidal have been known for some time. A classical result says that surfaces of revolution in $\mathbb{R}^3$ with constant mean curvature are precisely the Delaunay surfaces, i.e., surfaces of revolution swept out by the roulette of a conic: the plane, cylinder, sphere, the catenoid, the unduloid and nodoid. Helicoidal surfaces in $\mathbb{R}^3$ with constant mean curvature have been classified in [6]. What Proposition 9 is telling us is that the "profile" curves, γ, which span constant mean curvature S_γ evolving by congruences in $M_r^3(\rho)$, are critical curves for the curvature energy $\Theta(\gamma) = \int_\gamma \sqrt{\kappa - \varepsilon_1\varepsilon_2 H}\, ds$, where H is a constant. The variational problem associated to $\Theta(\gamma) = \int_\gamma \sqrt{\kappa}\, ds$ in $\mathbb{R}^3$ has been investigated by Blaschke, [5], as an special case of a Radon problem. Blaschke proved that extremals in $\mathbb{R}^3$ are a special family of Lancret helices and, in particular, extremals in $\mathbb{R}^2$ are catenaries. A more detailed analysis of this variational problem in $M_r^3(\rho)$ will be done in a forthcoming work.

7 Conclusions

In this work, we analyze the evolution of curves under binormal flows with curvature dependent velocity in pseudo-riemannian 3-space forms, using fundamental facts of the theory of submanifolds. The compatibility conditions for these systems are given by the Gauss-Codazzi equations which are expressed in terms of the curvature and torsion of the evolving curves. Then, an existence result is derived from the Fundamental Theorem of submanifolds. On the other hand, travelling wave solutions of the Gauss-Codazzi equations are shown to be equivalent to the Euler-Lagrange equations of extremal curves for curvature dependent energies under two constraints on the total torsion and the length (generalized Kirchhoff centerlines). A characterization of generalized Kirchhoff centerlines in terms of Killing vector fields give us the key to construct binormal evolution surfaces by using generalized Kirchhoff centerlines as initial conditions for the evolution. Binormal surfaces obtained in this way evolve without changing shape by congruences and slippery. Finally, we particularize the previous findings to three notable choices of the velocity which give rise to Hasimoto surfaces, Hopf tubes, and constant mean curvature surfaces, respectively

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