One-dimensional Fractional Quasi-static Thermoelasticity Problem for a Half-space

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Abstract: - This paper is devoted to the analytical solution of the space-time fractional heat conduction equation and associated thermoelasticity problem for a half-space in one-dimensional case. Finite Riesz fractional derivative and Caputo derivative are considered. Laplace transform with respect to time and sin-Fourier transform with respect to spatial coordinate are used. Numerical results are also presented.

Key-Words: - fractional heat conduction equation, Caputo derivative, Riesz derivative, Laplace transform, Fourier transform, generalized Mittag-Leffler function, rapidly heated boundary, quasi-static thermoelasticity problem.

1 Introduction

Continuum mechanics is suited to consider the media with spatially uniform characteristics. If the materials are heterogeneous and nonregular, then transport processes do not obey the lows of classical mechanics. Particle and energy transport (diffusion process, thermal conduction), occurring in porous materials, amorphous semiconductors, percolation clusters, polymer films, are called as anomalous or fractional processes through its association with fractional calculus [1], [2], [3]. The most acceptable and simple mathematical apparatus for description of the anomalous diffusion (or anomalous heat conduction) is partial differential equations with space-time fractional derivatives. The order of time derivative is specified by the quantity α, which is characterized by set topology [4]. In [5], on the basis of the experimental analysis of the basic sediments, the values of the exponent α are given in range $0{,}66 < \alpha < 0{,}909$. The space fractional derivative enters into the transfer equation in case of replacing Gauss distribution of the classical Brownian motion with more general stable Levy distribution [6], [7].

In recent years increasing interest has been shown in initial-value problems for fractional differential equation (FDE). For instance, in the paper [8] Cauchy problem for multidimensional space-time FDE was considered. Weitzner and Zaslavsky [9] analyzed the kinetic equation with fractional Riesz derivative [10]. The solutions of the time-fractional diffusion equations with Caputo derivatives are presented in [11], [12], [13], [14], [15], [16]. The basic methods used in the above mentioned papers are integral transforms [17], variable separation method [12], numerical methods [6].

Fractional heat conduction equation appeared as generalization of the Fourier law and standard heat conduction equation. The time-nonlocal constitutive equation for the heat flux was considered in [18], [19].

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\,\mathrm{grad}\,T(\tau)\,d\tau\,,$$

$0 < \alpha < 1$,

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}\,\mathrm{grad}\,T(\tau)\,d\tau\,,$$

$1 < \alpha < 2$.

This equations and energy conservation law yield the time-fractional heat conduction equation [20], [21], [22], [23]. Space-nonlocal law for heat flux is also considered. It is used in anomalous heat conduction models [11].

Boundary value problems for the FDEs with space fractional Riesz and Riesz-Feller derivatives are of particular interest [6], [24]. Adomian's decomposition method was generalized for the solving of FDE with Caputo fractional derivatives by Momani [25]. Ciesielski and Leszczynski [6] developed a numerical approximation of Riesz-Feller fractional derivative.

In the present paper we consider the one-dimensional heat conduction equation with space and time fractional derivatives. The boundary value problem is solved using the Laplace integral transform and sin-Fourier transform. Also we study the corresponding quasi-static thermoelasticity problem for the half-space.

2 Problem Formulation

Let us consider a half-space with rapidly heated boundary. It is assumed that the initial conditions for temperature are uniform. Then the equations for corresponding initial-boundary value problem of quasi-static thermoelasticity read [11], [26]

heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} - a\frac{\partial^\beta T}{\partial {x_1}^\beta} = 0, \quad 0 < \alpha < 2, \quad 1 < \beta < 2,$$

$x_1 > 0$, $t > 0$, (1)

equilibrium equations in terms of displacements

$$\mu\frac{\partial^2 u_i}{\partial x_j^2} + (\lambda + \mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} = \gamma\frac{\partial T}{\partial x_i}, \quad (2)$$

stress-strain-temperature relation

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\left(\lambda \operatorname{div}\vec{u} - \gamma T\right),$$

$i, j = 1,2,3$ (3)

initial conditions

$$T(x_1, 0) = 0, \; 0 < \alpha \le 1,$$

$$T(x_1, 0) = \left.\frac{\partial T(x_1, t)}{\partial t}\right|_{t=0} = 0, \; 1 < \alpha < 2, \quad (4)$$

boundary condition

$$T(0, t) = P(t). \quad (5)$$

Here $T(x_1, t)$ is the temperature, $\vec{u} = (u_1, u_2, u_3)$ is the displacement vector, λ, μ – Lame constants, δ_{ij} – Kronecker symbol, a is the thermal diffusivity coefficient, $\gamma = 3K\alpha_T$, α_T is the linear expansion coefficient, K is the bulk modulus, σ_{ij} is the components of stress, $\partial^\alpha / \partial t^\alpha$ is the Caputo fractional derivative [14], $\partial^\beta / \partial x^\beta$ is the finite Riesz fractional derivative ([10], [27]), $P(t)$ is the prescribed jump function.

It is suitable to define the finite Riesz fractional derivative by following formula [27]

$$\frac{\partial^\beta \varphi(x)}{\partial x^\beta} = \frac{\mathrm{I}_{0+}^{2-\beta}\dfrac{d^2\varphi}{dx^2} + \mathrm{I}_{-}^{2-\beta}\dfrac{d^2\varphi}{dx^2}}{2\cos\dfrac{(2-\beta)\pi}{2}}, \; 1 < \beta < 2, \; (6)$$

Where

$$\mathrm{I}_{0+}^{\nu}\varphi(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x - \zeta)^{\nu-1}\varphi(\zeta)\,d\zeta,$$

$\mathrm{I}_{-}^{\nu}\varphi(x) = \dfrac{1}{\Gamma(\nu)}\displaystyle\int_x^\infty (\zeta - x)^{\nu-1}\varphi(\zeta)\,d\zeta$ are the Riemann-Liouville fractional integrals, $\nu > 0$.

Within the framework of one-dimensional elasticity problem we have

$\dfrac{\partial u_i}{\partial x_2} = 0$ and $\dfrac{\partial u_i}{\partial x_3} = 0$, $i = 1,2,3$.

Put into action the displacement potential $\Phi(x_1, t)$ [28]. Then from the equations (2), (3) we obtain

$$u_i = \frac{\partial \Phi}{\partial x_i}, \; \sigma_{11} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0,$$

$$\sigma_{22} = \sigma_{33} = -2\mu\frac{\partial^2\Phi}{\partial x_1^2} = -2\mu m T, \; m = \frac{\gamma}{\lambda + 2\mu}.$$

(7)

Thus, in order to derive the stresses in domain under consideration it is necessary to determine the temperature from the equation (1) , initial conditions (4) and boundary conditions (5).

3 Problem Solution

3.1 Heat conduction equation

The Laplace transform rule for the fractional Caputo derivative has the form [14]

$$\mathrm{L}\left[\frac{\partial^\alpha f(t)}{\partial t^\alpha}\right](s) = s^\alpha \,\mathrm{L}\left[f(t)\right] - \sum_{k=0}^{n-1} f^{(k)}\left(0^+\right)s^{\alpha-1-k},$$
$n - 1 < \alpha < n$.

By applying the Laplace transform to equation (1) and condition (5) under the assumption of initial conditions (4), we get the following boundary value problem

$$a\frac{\partial^\beta \bar{T}}{\partial x_1^\beta} - s^\alpha \bar{T} = 0, \qquad (8)$$

$$\bar{T}(0,s) = \bar{P}(s), (9)$$

where $\bar{f}(x_1,s) = \mathrm{L}[f(x_1,t)](s)$, L is the Laplace transform operator, defined by the integral

$$\mathrm{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt.$$

In order to obtain the $\sin$-Fourier transform

$$\mathrm{F}_s[\varphi](\zeta) = \int_0^\infty \varphi(x)\sin\zeta x dx$$

for the finite Riesz derivative we write the following relations [10]

$$\mathrm{F}_s\left(\mathrm{I}_{0+}^\nu \varphi\right) = \zeta^{-\nu}\left(\sin\frac{\nu\pi}{2}\mathrm{F}_c\,\varphi + \cos\frac{\nu\pi}{2}\mathrm{F}_s\,\varphi\right),$$

$$\mathrm{F}_c[\varphi](\zeta) = \int_0^\infty \varphi(x)\cos\zeta x dx,$$

$$\mathrm{F}_s\left(\mathrm{I}_{-}^\nu \varphi\right) = \zeta^{-\nu}\left(-\sin\frac{\nu\pi}{2}\mathrm{F}_c\,\varphi + \cos\frac{\nu\pi}{2}\mathrm{F}_s\,\varphi\right)$$

, $0 < \nu < 1$. (10)

By applying the $\sin$-Fourier transform to the formula (6) and taking into account (10), we obtain

$$\mathrm{F}_s\left[\frac{\partial^\beta \varphi(x)}{\partial x^\beta}\right](\zeta) = -\zeta^\beta\,\mathrm{F}_s[\varphi] + \zeta^{\beta-1}\varphi(0),$$

$1 < \beta < 2$. (11)

Using the sin-Fourier transformation, relation (11) and boundary condition (9), equation (8) can be converted to the following form

$$\zeta^\beta\,\mathrm{F}_s\left[\bar{T}(x_1,s)\right] + \frac{s^\alpha}{a}\mathrm{F}_s\left[\bar{T}(x_1,s)\right] = \zeta^{\beta-1}\bar{P}(s).$$

Solving the latter equation yields

$$\mathrm{F}_s\left[\bar{T}(x_1,s)\right](\zeta) = \frac{\zeta^{\beta-1}a\bar{P}(s)}{a\zeta^\beta + s^\alpha}. \quad (12)$$

In order to obtain desired temperature function it is necessary to apply the inverse operators F_s^{-1}, L^{-1} to the right-hand side of equation (12).

Denote

$$f(t) = \mathrm{L}^{-1}\left[\frac{1}{a\zeta^\beta + s^\alpha}\right].$$

Then

$$\mathrm{L}^{-1}\left[\frac{\bar{P}(s)}{a\zeta^\beta + s^\alpha}\right] = \int_0^t f(\tau)P(t-\tau)d\tau$$

(13)

To invert the Laplace transform the following formula [13] is used

$$\mathrm{L}^{-1}\left[\frac{1}{s^\alpha + b}\right] = t^{\alpha-1}E_{\alpha,\alpha}\left(-bt^\alpha\right), \qquad b > 0,$$

$\alpha > 0$, (14)

where $E_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function [29].

Inverting the integral transforms in formula (12) according to relations (13), (14) we get

$$T(x_1,t) =$$
$$= \frac{2a}{\pi}\int_0^\infty \zeta^{\beta-1}\left(\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-a\zeta^\beta\tau^\alpha\right)P(t-\tau)d\tau\right)\sin\zeta x_1 d\zeta$$

$x_1 > 0$, $t > 0$. (15)

This expression gives the solution of the problem (1)-(5). The improper integral in the right-hand side of (15) converges, but its computation gives additional difficulties. The series representing function $E_{\alpha,\alpha}\left(-a\zeta^\beta\tau^\alpha\right)$, is badly summarized for arguments of large magnitude. The fact of the matter is that every term of alternate series, which represent function $E_{\alpha,\alpha}\left(-a\zeta^\beta\tau^\alpha\right)$, has a large magnitude whereas the value of Mittag-Leffler function (the sum of this series) is a small number. Similar problem appear during the numerical evaluation of the Wright function [30]. We used the algorithm for numerical evaluation of the Mittag-Leffler function presented in [18].

3.2 Numerical implementation

Consider the half-space with rapidly heated boundary at initial time, that is

$$P(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (16)$$

For numerical calculations we introduce the following nondimensional parameters:

$$\tau = \left(\frac{a}{L^\beta}\right)^{1/\alpha} t, \qquad \xi = \frac{x_1}{L}, \qquad \theta = \frac{T}{T_0},$$

$$\Sigma_{22} = -\frac{\sigma_{22}}{2\mu m T_0},$$

where L is the typical length scale, T_0 is the initial temperature.

Distributions of nondimensional temperature θ and stress Σ_{22} are shown in Figures 1-5 for various values of α and β.

Temperature-time (or stress-time) dependence for $\beta = 1.99$, $\xi = 1$ is presented in fig. 1. Here the temperature is defined by its development during the all previous period of heat conduction. For small α the temperature equilibrium is reached much faster then if $\alpha \to 2$. In fig. 2 the temperature wave front when $\alpha \to 2$ and diffusion processes otherwise are observed. Li and Wang [31] established a connection between anomalous heat conduction and anomalous diffusion. It was shown that subdiffusion ($0 < \alpha < 1$) implies an anomalous heat conduction with a convergent thermal conductivity, and superdiffusion ($1 < \alpha < 2$) implies an anomalous heat conduction with a divergent thermal conductivity. When $\alpha = 1$ normal (classical) heat conduction occurs, prescribed by Fourier's law. The same behavior of functions we can observe in figures 1, 2.

Temperature (stress) distributions in time and space for various values of β and $\alpha = 1$ are shown in fig. 3, 4. Here is present the effect of spatial non-locality. That is the variation of temperature depends not only on its values in the neighbourhood of a selected point, but also on its values in remote points. Therefore, significant differences in temperature (stress) distribution take place only for large values of time or spatial variable.

When $\alpha = 1$, $\beta = 2$ equation (1) turn into the classical heat conduction equation, based on the Fourier's law. Its solution under the initial conditions (4) and boundary condition (5), (16) has the well-known form [28]

$$T(x_1, t) = \operatorname{erfc}\left(\frac{x_1}{2\sqrt{at}}\right), \ t > 0.$$

In nondimensional variables

$$\theta(\xi, \tau) = \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right). \quad (17)$$

This function is marked by squares in fig. 4.

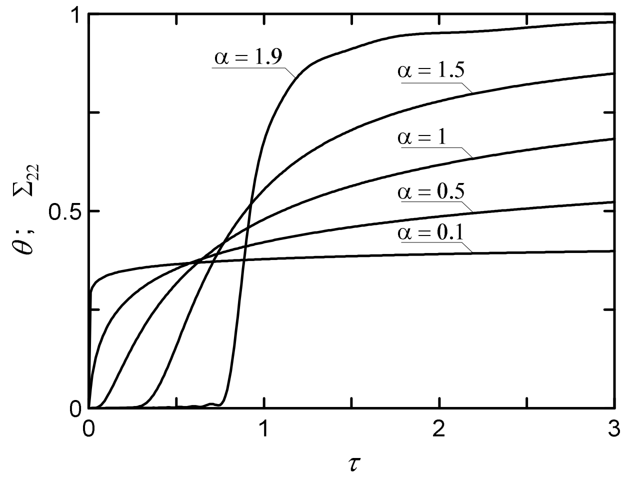

Fig. 1. Nondimensional temperature (stress) distribution in time at the point $\xi = 1$ for various values of α ($\beta = 1{,}99$).

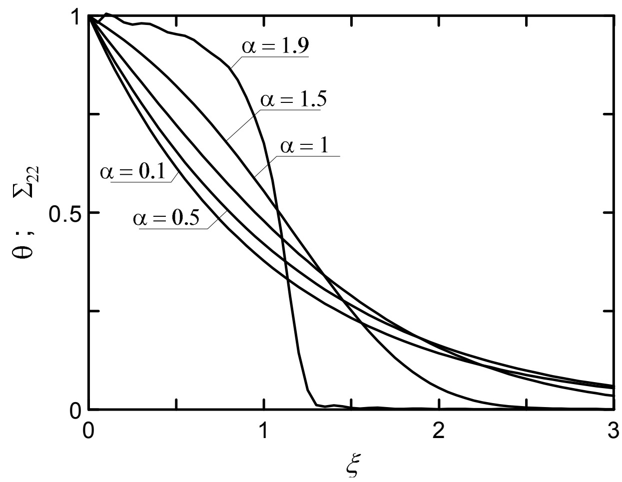

Fig. 2. Nondimensional temperature (stress) distribution in space for various values of α and $\tau = 1$, ($\beta = 1{,}99$).

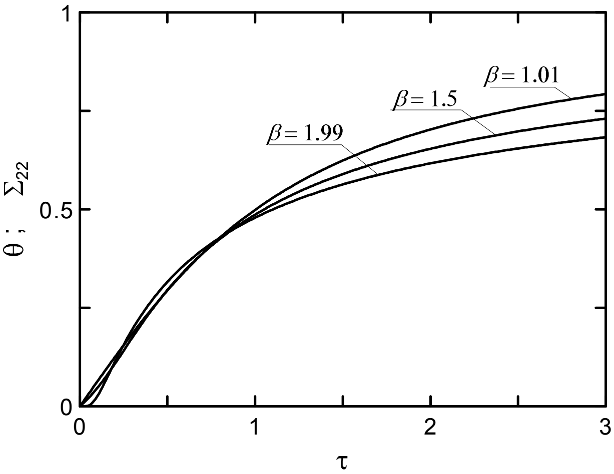

Fig. 3. Nondimensional temperature (stress) distribution in time at the point $\xi = 1$ for various values of β $(\alpha = 1)$.

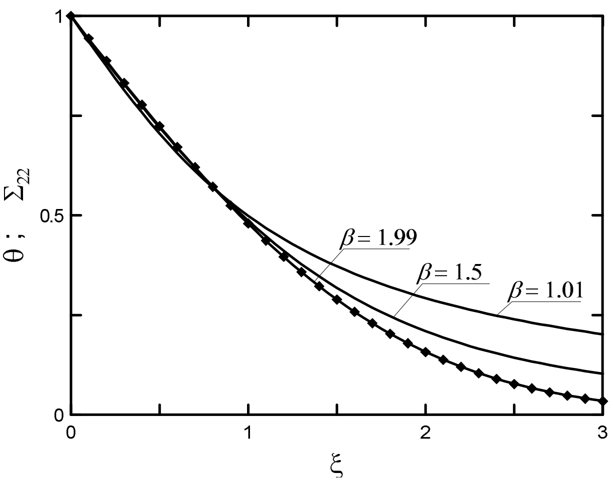

Fig. 4. Nondimensional temperature (stress) distribution in space for various values of β and $\tau = 1$ $(\alpha = 1)$.

In the article [32] time-fractional equation with the zero initial conditions and the constant boundary value of a function is considered. This problem corresponds to the equation (1) with conditions (4), (5), (16) for $\beta \to 2$ and $L \to \infty$. The solution of a problem in such case is presented in fig. 5 for $\alpha = 0{,}5; 1{,}0; 1.5$ and $\tau = 0.75$. These results are in good agreement with Povstenko's results.

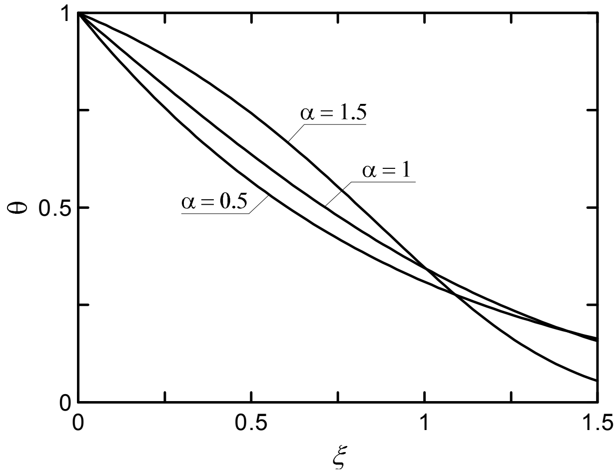

Fig. 5. Nondimensional temperature distribution in space for various values of α ($\tau = 0.75$).

4 Conclusion

In this article we obtain the solution of the one-dimensional boundary value problem for heat conduction equation with fractional Caputo and Riesz derivatives and corresponding quasi-static thermoelasticity problem. The solution satisfies the appropriate boundary conditions. In the limit case ($\alpha = 1$, $\beta \to 2$) it coincides with the solution to the classical heat conduction equation.

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