# Similarity Methods in the Analysis for Laminar Forced Convection on a Horizontal Plate

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*Abstract:* - The problem of laminar forced convection on a horizontal plate has been analyzed through three similarity transformation methods which depend on the finite group of transformations. Comparisons between these methods are performed and the results are found to be in very good agreement that the new systematic method by Adnan et al [2], [3], is the simplest and more general. Furthermore four transformed similarity equations of the problem under consideration were obtained very easily under this method. General form of absolute invariants and the corresponding new variables of certain groups are determined. The Formulation of the New Systematic Method is explained in appendix -A.

*Key-Words:* - Similarity Transformation Methods, laminar forced convection

## 1 Introduction

Solving nonlinear partial differential equation is a fundamental task and of great importance. The similarity methods to find the symmetry group and similarity representations of nonlinear partial differential equations is one of the most powerful tools in mathematical physics. Morgan [21] and Michal [16] are pioneers in developing the similarity solutions of partial differential equations under the appropriate one-parameter group of transformations. Birkhoff [6] applied one-parameter group of transformation to obtain similarity solutions of some problems in the fluid mechanics. Later on Manohar [15], Hansen [10] and Ames [4], [5] extend the methods to special forms of n-parameter groups.

A variety methods and theories were presented by Moran and his co-workers [17], [18], [19] and [20], for reducing systems of partial differential equations.

Most recently, Al-salihi et al [2], [3],, proposed a method simpler than the systematic group formalism Moran and Gaggioli [18], (called the new systematic method).

The objective of this paper is to derive similarity equations for laminar forced convection on a horizontal plate using three methods of similarity. The outline of the proposed method is given in Appendix – A.

## 2 Problem Formulation

External flows involve a flow which is essentially infinite in extent over the outer surface of a body. Flow over an isothermal flat plate aligned with the flow is one example for such a flow.

The basic governing partial differential equations for description of mass, momentum, and energy conservation of two-dimensional laminar steady-state forced convection boundary layers are given below, noting that the variable physical properties and viscous thermal dissipation are ignored [23],

$$u_x+v_y=0 \tag{1}$$

$$uu_x+vu_y=U_\infty\frac{dU_\infty}{dx}+\nu u_{yy} \tag{2}$$

$$uT_x+vT_y=\frac{\nu}{\Pr}T_{yy} \tag{3}$$

with the boundary conditions

$$y=0: \quad u=v=0, \quad T=T_w \tag{4}$$

$$y\to\infty: \quad u=U_\infty, \quad T=T_\infty \tag{5}$$

where $U_\infty, T_w$ and $T_\infty$ are the velocity component beyond the boundary layer, temperature on the wall and temperature beyond the boundary layer, respectively.

Introducing a stream function $\psi(x,y)$ defining velocity components $u=\psi_y,\ v=-\psi_x$, and dimensionless temperature $\theta(x,y)=\frac{T-T_\infty}{\theta_w}$, where $\theta_w=T_w-T_\infty$. Equations (2) and (3) reduce to

$$\psi_y\psi_{xy}-\psi_x\psi_{yy}-U_\infty\frac{dU_\infty}{dx}-\nu\psi_{yyy}=0 \qquad (6)$$

$$\psi_y(\theta\frac{d\theta_w}{dx}+\theta_w\theta_x)-\psi_x\theta_y\theta_w-\frac{\nu}{\Pr}\theta_w\theta_{yy}=0 \qquad (7)$$

while equation (1) is identically satidsfied. Further the boundary conditions read as,

$$y=0:\quad \psi_y=-\psi_x=0,\quad \theta=1 \qquad (8a)$$
$$y\to\infty:\quad \psi_y=U_\infty,\quad \theta=0 \qquad (8b)$$

## 3 Determination of the Absolute Invariants (Invariants of Group):

Consider the one-parameter group of transformations given by

$$G_1=\begin{cases} S_1{:}\ \bar{x}_i=C^{x_i}(a)x_i+K^{x_i}(a) & i=1,\dots,n \quad n\geq 2 \\ \bar{u}^j=C^{u^j}(a)\,u^j+K^{u^j}(a) & j=1,\dots,m \quad m\geq 1 \end{cases} \qquad (9)$$

acting on base space (x,u) which has infinitesimal generator of the form (35), with

$$\xi_i=\frac{\partial\bar{x}_i}{\partial a}|_{a=ao}=\frac{\partial C^{x_i}}{\partial a}|_{a=ao}x_i+\frac{\partial K^{x_i}}{\partial a}|_{a=ao}=\alpha_i x_i+\beta_i$$

$$\zeta^j=\frac{\partial\bar{u}^j}{\partial a}|_{a=ao}=\frac{\partial C^{u^j}}{\partial a}|_{a=ao}u^j+\frac{\partial K^{u^j}}{\partial a}|_{a=ao}=\alpha_{(n+j)}u^j+\beta_{(n+j)}$$

We put

$$\alpha_i=\frac{\partial C^{x_i}}{\partial a}|_{a=a_o},\quad \beta_i=\frac{\partial K^{x_i}}{\partial a}|_{a=a_o},$$
$$\alpha_{(n+j)}=\frac{\partial C^{u^j}}{\partial a}|_{a=a_o},\quad \beta_{n+j}\frac{\partial K^{u^j}}{\partial a}|_{a=a_o}$$

where $a_0$ denotes the value of a which yields the identity element of the group, such group has $n+m-1$ absolute invariants, $n-1$ of them for the subgroup $S_1$. Utilizing the systematic technique of determination a complete set of absolute invariants, Moran and Gaggioli [18], i.e.

The function $\lambda(x_1,\dots,x_n,u^1,\dots,u^m)$ is an absolute invariant of the group $G_1$ with generator $X$ if and only if

$$X(\lambda)=\sum_{i=1}^{n}(\alpha_i x_i+\beta_i)\frac{\partial\lambda}{\partial x_i}+\sum_{j=1}^{m}\alpha_{n+j}x_i+\beta_{n+j}\frac{\partial\lambda}{\partial u^j}=0 \qquad (10)$$

However, to determining the absolute invariants of $G_1$, it is sufficient to solve the equation (10). This is possible via well-known characteristics techniques for solving linear PDE,

$$\frac{dx_1}{\alpha_1x_1+\beta_1}=\dots=\frac{dx_n}{\alpha_nx_n+\beta_n}=\frac{du^1}{\alpha_{n+1}u^1+\beta_{n+1}}=\dots=\frac{du^m}{\alpha_{n+m}u^1+\beta_{n+m}}$$

Thus, a set of absolute variables are determining by the nontrivial solutions of simultaneous equations, these simultaneous equations also suggest other possible sets of transformations and corresponding invariants. Such invariants generate new variables (similarity variables). We suppose $x_k$ is the independent variable to be eliminated. There are many sets of invariants and corresponding new variables:

**Set 1:** If $\alpha_k\neq 0$ the invariants of $G_1$, are

$$\Omega_\gamma=\begin{cases} A\,(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\alpha_\gamma}{\alpha_k}}(x_\gamma+\frac{\beta_\gamma}{\alpha_\gamma}) & if \quad \alpha_\gamma\neq 0 \\ A\ln(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\beta_\gamma}{\alpha_k}}x_\gamma & if \quad \alpha_\gamma=0 \end{cases}$$

$$g^j=\begin{cases} B\,(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\alpha_{n+j}}{\alpha_k}}(u^j+\frac{\beta_{n+j}}{\alpha_{n+j}}) & if \quad \alpha_{n+j}\neq 0 \\ A\ln(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\beta_{n+j}}{\alpha_k}}u^j & if \quad \alpha_\gamma=0 \\ B\,u^j & if \quad \alpha_{n+j}=\beta_{n+j}=0 \end{cases}$$

which generate the following new variables:

$$\left.\begin{aligned}&\eta_\gamma = A\,(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\alpha_\gamma}{\alpha_k}}(x_\gamma+\frac{\beta_\gamma}{\alpha_\gamma}),\quad u^j=R_1 \text{ and}\\ &\eta_\gamma = A\,\ln(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\beta_\gamma}{\alpha_k}}x,\quad u^j=R_1\end{aligned}\right\}\quad (11)$$

where

$$R_1=\begin{cases}\frac{1}{B}(x_k+\frac{\beta_k}{\alpha_k})^{\frac{\alpha_{n+j}}{\alpha_k}}F^j(\eta_1,\ldots,\eta_{n-1})\text{-}\frac{\beta_{n+j}}{\alpha_{n+j}} & if\quad \alpha_{n+j}\neq 0\\ \frac{1}{B}\,\mathrm{Ln}(x_k+\frac{\beta_k}{\alpha_k})^{\frac{\alpha_{n+j}}{\alpha_k}}F^j(\eta_1,\ldots,\eta_{n-1}) & if\quad \alpha_\gamma=0\\ \frac{1}{B}F^j(\eta_1,\ldots,\eta_{n-1}) & if\quad \alpha_{n+j}=\beta_{n+j}=0\end{cases}$$

**Set 2:** If $\alpha_k=0$ the invariants of $G_1$, are

$$\Omega_\gamma=\begin{cases}A\,e^{-\frac{\alpha_\gamma}{\alpha_k}x_k}(x_\gamma+\frac{\beta_\gamma}{\alpha_\gamma}) & if\quad \alpha_\gamma\neq 0\\ A\quad(\beta_k x_\gamma\text{-}\beta_\gamma x_k) & if\quad \alpha_\gamma=0\end{cases}$$

$$g^j=\begin{cases}B\,e^{-\frac{\alpha_{n+j}}{\alpha_k}x_k}(u^j+\frac{\beta_{n+j}}{\alpha_{n+j}}) & if\quad \alpha_{n+j}\neq 0\\ B\,(\beta_{n+j}x_k\text{-}\beta_k u^j) & if\quad \alpha_\gamma=0\\ B\,u^j & if\quad \alpha_{n+j}=\beta_{n+j}=0\end{cases}$$

which generate the following new variables:

$$\left.\begin{aligned}&\eta_\gamma = A\,e^{-\frac{\alpha_\gamma}{\beta_k}x_k}(x_\gamma+\frac{\beta_\gamma}{\alpha_\gamma}),\quad u^j=R_2 \text{ and}\\ &\eta_\gamma = A\,\ln(x_k+\frac{\beta_k}{\alpha_k})^{-\frac{\beta_\gamma}{\alpha_k}}x,\quad u^j=R_2\end{aligned}\right\}\quad (12)$$

where

$$R_2=\begin{cases}\frac{1}{B}e^{\frac{\alpha_{n+j}}{\beta_k}x}F^j(\eta_1,\ldots\eta_{n-1})\text{-}\frac{\beta_{n+j}}{\alpha_{n+j}} & if\quad \alpha_{n+j}\neq 0\\ \frac{1}{\beta_k}(\beta_{n+j}x_k-\frac{1}{B}F^j(\eta_1,\ldots\eta_{n-1})) & if\quad \alpha_{n+j}=0\\ \frac{1}{B}F^j(\eta_1,\ldots\eta_{n-1}) & if\quad \alpha_{n+j}=\beta_{n+j}=0\end{cases}$$

where $\gamma=1,\ldots,n-1$ $(\gamma\neq k)$ and $j=1,\ldots,m$.

## 4 Morgan's Method

First step in Morgan's method is define the simple group of transformations and carry out the transformation on the differential equations (6)-(7), to show that it is invariant under this transformations.

Consider the linear group of transformation (one-parameter group)

$$\bar{x}=A^{\alpha_1}x,\quad \bar{y}=A^{\alpha_2}y,\quad \bar{\psi}=A^{\alpha_3}\psi,$$
$$\bar{\theta}=A^{\alpha_4}\theta,\quad \bar{\theta}_w=A^{\alpha_5}\theta_w,\quad \bar{U}=A^{\alpha_6}U \qquad (13)$$

Substitute of (13) into (6)-(7), we get

$$A^{\alpha_1+2\alpha_2-2\alpha_3}(\bar{\psi}_{\bar{y}}\bar{\psi}_{\bar{x}\bar{y}}-\bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{y}\bar{y}}) -A^{\alpha_1-2\alpha_6}\bar{U}_\infty\frac{d\bar{U}_\infty}{d\bar{x}}-A^{3\alpha_2-\alpha_3}\nu\bar{\psi}_{\bar{y}\bar{y}\bar{y}}=0$$

$$A^{\alpha_1+\alpha_2-\alpha_3-\alpha_4-\alpha_5}[\bar{\psi}_{\bar{y}}(\bar{\theta}\frac{d\bar{\theta}_w}{d\bar{x}}+\bar{\theta}_w\bar{\theta}_{\bar{x}}) -\bar{\psi}_{\bar{x}}\bar{\theta}_{\bar{y}}\bar{\theta}_{\bar{w}}]-\frac{\nu}{\mathrm{Pr}}A^{2\alpha_2-\alpha_4-\alpha_5}\bar{\theta}_w\bar{\theta}_{\bar{y}\bar{y}}=0$$

Which be absolute invariant under (13) if;

$$\alpha_1+2\alpha_2-2\alpha_3=\alpha_1-2\alpha_6=3\alpha_2-\alpha_3=0$$
$$\alpha_1+\alpha_2-\alpha_3-\alpha_4-\alpha_5=2\alpha_2-\alpha_4-\alpha_5=0$$

or more general, conformally invariant if;

$$\begin{aligned}&\alpha_1+2\alpha_2-2\alpha_3=\alpha_1-2\alpha_6=3\alpha_2-\alpha_3,\\ &\alpha_1+\alpha_2-\alpha_3-\alpha_4-\alpha_5=2\alpha_2-\alpha_4-\alpha_5\end{aligned}\qquad (14)$$

the equations in (14), solved to give

$$\frac{\alpha_2}{\alpha_1}=1-m,\qquad \frac{\alpha_3}{\alpha_1}=m,$$
$$\frac{\alpha_6}{\alpha_1}=2m\text{-}1,\qquad \frac{\alpha_4}{\alpha_1}+\frac{\alpha_5}{\alpha_1}=n \qquad (15)$$

where $m$, $n$ are arbitrary constants.

Second step is determining the invariants by trial and/or inspection. By eliminating the parameter $A$ from (13), and make use (15) we obtain,

$$\bar{y}\bar{x}^{m-1}=yx^{m-1},\qquad \bar{\psi}\bar{x}^{-m}=\psi x^{-m},$$
$$\bar{\theta}\bar{x}^{-\frac{\alpha_4}{\alpha_1}}=\theta x^{-\frac{\alpha_4}{\alpha_1}},\qquad \bar{\theta}_w\bar{x}^{-\frac{\alpha_5}{\alpha_1}}=\theta_w x^{-\frac{\alpha_5}{\alpha_1}},$$
$$\bar{U}_\infty\bar{x}^{1-2m}=U_\infty x^{1-2m}$$

These combined variables enable us to derive new variables as

$$\left.\begin{array}{ll} \eta = yx^{m-1}, & \psi = F(\eta)x^{m}, \\ \theta = G(\eta)x^{\frac{\alpha_4}{\alpha_1}}, & \theta_w = C_1 x^{\frac{\alpha_5}{\alpha_1}}, \\ U_\infty = C_2 x^{2m-1} & \end{array}\right\} \quad (16)$$

We test whether the auxiliary conditions are expressible, without inconsistency, in terms of these new variables. This is, if $\alpha_4 = 0$. Put $\alpha_4 = 0$ in (16) we get the similarity variables which transform invariantly, the auxiliary conditions as well as the differential equations,

$$(1-2m)[C_2^2 - F'^2] - mFF'' - \nu F''' = 0 \quad (17)$$

$$nGF' - mFG' - \frac{\nu}{\Pr} G'' = 0 \quad (18)$$

With the boundary conditions

$$\begin{array}{llll} \eta = 0: & F = F' = 0, & G = 1 & \\ \eta \to \infty: & F' = C_2, & G = 0 & \end{array} \quad (19)$$

It seen that in this procedure there is no systematic approach to determine the absolute invariants and that the new variables generated by these absolute invariants were found for the differential equations alone, this leads to wastage of efforts if the auxiliary conditions are inexpressible in terms of such new variables.

# 5 Systematic Group Formalism Moran and Gaggioli [18]:

Drawbacks in the previous section were overcome by [18], [20]. Invariant analysis consider for auxiliary conditions as well as the differential equations and then absolute invariants determined by a systematic procedure.

Consider again, eqs. (6)-(7). First step is also define somewhat general group of transformation of the form (9) and show that the eqs. (6)-(7) is invariant under this transformations.

Consider the one-parameter group of transformation,

$$G = \begin{cases} S = \begin{cases} \bar{x} = C^x(a)x + K^x(a) \\ \bar{y} = C^y(a)y + K^y(a) \end{cases} \\ \bar{\psi} = C^\psi(a)\psi + K^\psi(a) \\ \bar{\theta} = C^\theta(a)\theta + K^\theta(a) \\ \bar{\theta}_w = C^{\theta_w}(a)\theta_w + K^{\theta_w}(a) \\ \bar{U}_\infty = C^{U_\infty}(a)U_\infty + K^{U_\infty}(a) \end{cases} \quad (20)$$

Where $C$'s and $K$'s are real-valued and at least differentiable in the real argument $a$. It is straightforward to show that (6)-(7) are invariant under $G$, by substitute (20) into (6)-(7), yield

$$\begin{aligned} &(\bar{\psi}_{\bar{y}}\bar{\psi}_{\bar{x}\bar{y}} - \bar{\psi}_{\bar{x}}\bar{\psi}_{\bar{y}\bar{y}}) - \bar{U}_\infty \frac{d\bar{U}_\infty}{d\bar{x}} - \nu\bar{\psi}_{\bar{y}\bar{y}\bar{y}} = \\ &\frac{1}{C^x}\left(\frac{C^\psi}{C^y}\right)^2 (\psi_y\psi_{xy} - \psi_x\psi_{yy}) - \\ &\frac{(C^{U_\infty})^2}{C^x} U_\infty \frac{dU_\infty}{dx} - \frac{C^\psi}{(C^y)^2}\nu\, \psi_{yyy} + R_1 \end{aligned} \quad (21)$$

$$\begin{aligned} &\bar{\psi}_{\bar{y}}(\bar{\theta}\frac{d\bar{\theta}_w}{d\bar{x}} + \bar{\theta}_w\bar{\theta}_{\bar{x}}) - \bar{\psi}_{\bar{x}}\bar{\theta}_{\bar{y}}\bar{\theta}_{\bar{w}} - \frac{\nu}{\Pr}\bar{\theta}_w\bar{\theta}_{yy} \\ &= \frac{C^\psi C^{\theta_w} C^\theta}{C^x C^y}[\psi_y(\theta\frac{d\bar{\theta}_w}{d\bar{x}} + \theta_w\theta_x) - \psi_x\theta_y\theta_w] \\ &- \frac{\nu}{\Pr}\frac{C^\theta C^{\theta_w}}{(C^y)^2}\theta_w\theta_{yy} + R_2 \end{aligned} \quad (22)$$

*Where*

$$R_1 = \frac{C^{U_\infty}}{C^x} K^{U_\infty} \frac{dU_\infty}{dx},$$

$$R_2 = \frac{C^\psi}{C^x C^y}[C^{\theta_w}K^\theta(\psi_y\theta\frac{d\theta_w}{dx}) + C^\theta K^{\theta_w}\theta_w[\psi_y\theta_x - \psi_x\,\theta_y]] - \frac{\nu}{\Pr}\frac{C^\theta}{(C^y)^2}K^{\theta_w}\theta_w\,\theta_{yy}$$

The conformal invariance of (21)-(22) implies

$$\begin{aligned} &R_1 = R_2 = 0,\ \frac{1}{C^x}\left(\frac{C^\psi}{C^y}\right)^2 = \frac{(C^{U_\infty})^2}{C^x} = \frac{C^\psi}{(C^y)^3} = H_1(a), \\ &\frac{C^\psi C^\theta C^{\theta_w}}{C^x C^y} = \frac{C^\theta C^{\theta_w}}{(C^y)^2} = H_2(a) \end{aligned} \quad (23)$$

The vanishing of $R_1$, $R_2$ and invariant of boundary conditions (8), implying that

$$K^\theta = K^{\theta_w} = K^{U_\infty} = 0, \qquad C^\theta = 1 \quad (24)$$

In view of (23), and invoking the results (24), we get

$$C^{U_\infty}=\frac{C^\psi}{C^y}=\frac{C^x}{(C^y)^2} \quad \text{and} \quad C^\psi=\frac{C^x}{C^y} \qquad (25)$$

By substituting from (24) and (25) into (20) we get the class of group.

$$G'=\begin{cases} S'=\begin{cases}\bar{x}=C^\psi C^y\ x+K^x(a)\\ \bar{y}=C^y\ y\end{cases}\\ \bar{\psi}=C^\psi\ \ \psi+K^\psi(a)\\ \bar{\theta}=\theta\\ \bar{\theta}_w=C^{\theta_w}\theta_w\\ \bar{U}_\infty=\dfrac{C^\psi}{C^y}U_\infty\end{cases}$$

Next step is finding the absolute invariants of $G'$ via systematic technique depend on invoked a basic theorem from group theory, i.e., $\eta(x,y)$ is an absolute invariant of subgroup $S'$ if it satisfies

$$(a_1\,x+b_1)\frac{\partial\eta}{\partial x}+a_2\,y\frac{\partial\eta}{\partial y}=0 \qquad (26)$$

Where $a_1=\frac{d(C^\psi C^y)}{da}\big|_{a=a_0}, b_1=\frac{dK^x}{da}\big|_{a=a_0}$, and $a_2=\frac{dC^y}{da}\big|_{a=a_0}$.
Eq. (26) has a solution in the form

$$\eta=\pi^{-\frac{a_2}{a_1}}y,\quad \pi=(x+\frac{b_1}{a_1})\ .$$

By a similar analysis the absolute invariant of the dependent variables $\psi$, $\theta$, $\theta_w$ and $U_\infty$ are

$$\pi^{-\frac{a_3}{a_1}}(\psi+\frac{b_3}{a_3}),\ \ \theta,\ \ \pi^{-\frac{a_5}{a_1}}\theta_w,\ \ \pi^{-\frac{a_6}{a_1}}U_\infty$$

These absolute variables generate the similarity variables of the problem of the form

$$\psi=\pi^{\frac{a_3}{a_1}}F(\eta)-\frac{b_3}{a_3},\qquad \theta=G(\eta),$$

$$\theta_w=C_1\,\pi^{\frac{a_5}{a_1}},\qquad U_\infty=C_2\,\pi^{\frac{a_6}{a_1}}$$

Using these similarity variables, (6)-(8) becomes,

$$\Gamma_1\,[(\frac{a_3}{a_1}-\frac{a_2}{a_1})F'^2-\frac{a_3}{a_1}\eta FF'']-\nu\,\Gamma_2\,F'''-\Gamma_3=0 \qquad (27)$$

$$\Gamma_4[\frac{a_5}{a_1}GF'-\frac{a_3}{a_1}FG']-\frac{\nu}{\Pr}\Gamma_5 G''=0 \qquad (28)$$

with transformed boundary conditions

$$\begin{aligned}\eta=0:&\ F=F'=0,\quad G=1\\ \eta\to\infty:&\ F'=\Gamma_6,\quad\ \ G=0\end{aligned} \qquad (29)$$

Where

$$\Gamma_1=\pi^{\frac{2a_3}{a_1}-\frac{2a_2}{a_1}-1},\quad \Gamma_2=\pi^{\frac{a_3}{a_1}-\frac{3a_2}{a_1}},\quad \Gamma_3=C_2^2\frac{a_6}{a_1}\pi^{\frac{2a_6}{a_1}-1}$$

$$\Gamma_4=\pi^{\frac{a_5}{a_1}+\frac{a_3}{a_1}-\frac{a_2}{a_1}-1},\quad \Gamma_5=\pi^{\frac{a_5}{a_1}-\frac{2a_2}{a_1}},\quad \Gamma_6=\pi^{\frac{a_6}{a_1}+\frac{a_2}{a_1}-\frac{a_3}{a_1}}$$

For (27)-(28) to be reduced to an expression in terms of the similarity variables; it is necessary that $\Gamma_1,\ldots,\Gamma_5$ should be constants or functions of $\eta$ alone and $\Gamma_6$ should be constants This is, if

$$\frac{a_2}{a_1}=\frac{1}{2},\quad \frac{a_3}{a_1}=\frac{3}{4},\quad \frac{a_4}{a_1}=\frac{1}{2},\quad and\quad \frac{a_6}{a_1}=\frac{1}{2}$$

With help these relations, (27)-(28)

$$(\frac{1}{2}F'^2-\frac{3}{4}\eta FF'')-\nu\ \ F'''-\frac{C_2^2}{4}=0$$

$$\frac{1}{2}GF'-\frac{3}{4}FG']-\frac{\nu}{\Pr}G''=0$$

Which represent the similarity equations of the problem, with boundary conditions of form (19).

## 6 New Systematic method

The drawbacks in the previous two methods were overcome by Adnan et al [2], [3]. In this method, group of the form (20) will be defined again in beginning of the analysis which has sets of absolute invariants and corresponding new variables of the form (11) and (12). The method has only one step, expressing of the basic equations (6)-(7) along with the boundary conditions (8) in terms of those new variables.

**Case 1:** For problem in hand and make use (11), we will invoke one set of new variables; as example

$$\eta=A\,\pi^{\frac{\alpha_2}{\alpha_1}}\ (y+\frac{\beta_2}{\alpha_2}),$$

$$\psi=\frac{1}{B_3}\pi^{\frac{\alpha_3}{\alpha_1}}F(\eta)-\frac{\beta_3}{\alpha_3},\qquad \theta=\frac{1}{B_4}\pi^{\frac{\alpha_4}{\alpha_1}}G(\eta),$$
$$\theta_w=\frac{1}{B_5}\pi^{\frac{\alpha_5}{\alpha_1}}k_1-\frac{\beta_5}{\alpha_5},\qquad U_\infty=\frac{1}{B_6}\pi^{\frac{\alpha_6}{\alpha_1}}k_2-\frac{\beta_6}{\alpha_6} \tag{30}$$

Where $\pi=(x+\frac{\beta_1}{\alpha_1})$. We assumed $A=B_3=B_4=B_5=B_6=1$. Eqs. (6)-(7), becomes

$$\mathrm{K}_1[(\alpha_3-\alpha_2)F'^2-\alpha_3\eta FF''] + \mathrm{K}_2(\beta_6-\alpha_6\mathrm{C}_2)-\nu\,\alpha_1\mathrm{K}_3F'''=0 \tag{31}$$

$$-\mathrm{K}_4F'+\mathrm{K}_5[(\alpha_4+\alpha_5)GF'-\alpha_3FG'] + \mathrm{K}_6[\alpha_3FG'-\alpha_4GF']-\frac{\nu}{\mathrm{Pr}}[\mathrm{K}_7-\mathrm{K}_8]G''=0 \tag{32}$$

with the transformed boundary conditions

$\eta=0\ (for\quad y=0,\quad \beta_2=0)$:
$F=F'=0,\qquad G=\mathrm{K}_9$
$\eta\to\infty\ (for\quad y\to\infty)$:
$F'=\mathrm{K}_{10},\qquad G-\mathrm{K}_{11}=0$

Where

$$\mathrm{K}_1=\pi^{\frac{2\alpha_3}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}-1},\quad \mathrm{K}_2=C_2\pi^{\frac{2\alpha_6}{\alpha_1}-1},\quad \mathrm{K}_3=\pi^{\frac{\alpha_3}{\alpha_1}-\frac{3\alpha_2}{\alpha_1}}$$
$$\mathrm{K}_4=C_1\frac{\alpha_5^2\beta_4}{\alpha_4}\pi^{\frac{\alpha_5}{\alpha_1}+\frac{\alpha_3}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-1},$$
$$\mathrm{K}_5=C_1\alpha_5\pi^{\frac{\alpha_5}{\alpha_1}+\frac{\alpha_4}{\alpha_1}+\frac{\alpha_3}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-1},\quad \mathrm{K}_6=\beta_5\pi^{\frac{\alpha_4}{\alpha_1}+\frac{\alpha_3}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-1}$$
$$\mathrm{K}_7=\alpha_1C_1\pi^{\frac{\alpha_5}{\alpha_1}+\frac{\alpha_4}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}},\quad \mathrm{K}_8=\alpha_1\beta_5\pi^{\frac{\alpha_4}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}},$$
$$\mathrm{K}_9=(1+\frac{\beta_4}{\alpha_4})\pi^{-\frac{\alpha_4}{\alpha_1}},\quad \mathrm{K}_{10}=\pi^{\frac{-\alpha_3}{\alpha_1}+\frac{\alpha_2}{\alpha_1}}(\pi^{\frac{\alpha_6}{\alpha_1}}C_2-\frac{\beta_6}{\alpha_6}),$$
$$\mathrm{K}_{11}=\frac{\beta_4}{\alpha_4}\pi^{-\frac{\alpha_4}{\alpha_1}}$$

For (31)-(32) to be reduced to an expression in terms of the new variables, it is necessary that $\mathrm{K}_1,\ldots,\mathrm{K}_8$ should be constants or functions of $\eta$ alone and $\mathrm{K}_9.\mathrm{K}_{10},\mathrm{K}_{11}$ should be constants. This is, only whenever:

$$\beta_2=\beta_4=\beta_5=\beta_6=0$$
$$\frac{2\alpha_3}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}-1=0,\qquad \frac{2\alpha_6}{\alpha_1}-1=0,$$
$$\frac{\alpha_3}{\alpha_1}-\frac{3\alpha_2}{\alpha_1}=0,\qquad \frac{\alpha_5}{\alpha_1}+\frac{\alpha_3}{\alpha_1}-\frac{\alpha_2}{\alpha_1}-1,$$
$$\frac{\alpha_5}{\alpha_1}-\frac{2\alpha_2}{\alpha_1}=0,\qquad -\frac{\alpha_3}{\alpha_1}+\frac{\alpha_2}{\alpha_1}+\frac{\alpha_6}{\alpha_1}=0 \tag{33}$$

Which be solved to give

$$\frac{\alpha_2}{\alpha_1}=\frac{1}{4},\quad \frac{\alpha_3}{\alpha_1}=\frac{3}{4},\quad \frac{\alpha_5}{\alpha_1}=\frac{1}{2},\quad \frac{\alpha_6}{\alpha_1}=\frac{1}{2}\quad and\quad \alpha_4=0$$

Thus, with help these relations, the new variables (30) become similarity variables and then the corresponding similarity equations of (6)-(7), take similar form to those reported in the previous section.

**Note that:** most the relations in (23)-(25) and values of the coefficients $a$'s and $b$'s in previous section may be find out easily by (33). And so it is possible to deduce the group under which the problem invariant if needed without going to deduce it separately.

Moreover, by simple manipulation to Eqs. (31)-(32), one can derive another form of similarity variables. For example; multiply (28) and (29) by $\pi^{1-2\frac{\alpha_6}{\alpha_1}}$ and $\pi^{1-\frac{\alpha_5}{\alpha_1}-\frac{\alpha_4}{\alpha_1}-\frac{\alpha_3}{\alpha_1}+\frac{\alpha_2}{\alpha_1}}$ respectively, and repeating the process we obtain

$$\frac{\alpha_2}{\alpha_1}=1-m,\quad \frac{\alpha_3}{\alpha_1}=m,\quad \frac{\alpha_5}{\alpha_1}=n,\quad and\quad \frac{\alpha_6}{\alpha_1}=2m-1$$

Substitute these relations in (30), we get new form of the similarity variables and then similarity equations corresponding to (6)-(7) similar to those reported in Section 4.

**Case 2:** Similarly, make use (12); we can invoke another set of new variables, as

$$\eta=A\,e^{-\frac{\alpha_2}{\beta_1}x}\,(y+\frac{\beta_2}{\alpha_2})$$
$$\psi=\frac{1}{B_3}e^{\frac{\alpha_3}{\beta_1}x}F(\eta)-\frac{\beta_3}{\alpha_3},\qquad \theta=\frac{1}{B_4}e^{\frac{\alpha_4}{\beta_1}x}G(\eta)-\frac{\beta_4}{\alpha_4}, \tag{34}$$
$$\theta_w=\frac{1}{B_5}e^{\frac{\alpha_5}{\beta_1}x}k_1-\frac{\beta_5}{\alpha_5},\qquad U_\infty=\frac{1}{B_6}e^{\frac{\alpha_6}{\beta_1}x}k_2-\frac{\beta_6}{\alpha_6}$$

Assume $A=B_3=B_4=B_5=B_6=1$. Eqs. (6)- (7), becomes

$$C_2[K_1 - C_2\alpha_6]+K_2[(\alpha_3-\alpha_2)F'^2-\alpha_3\eta FF'']-K_3\beta_1\nu F'''=0 \quad (35)$$

$$K_4F'+K_5[(\alpha_4+\alpha_5)GF'-\alpha_3FG'] \\ +K_6[\alpha_3FG'+\alpha_4GF']-[\frac{\nu}{Pr}K_7-K_8]\beta_1G''=0 \quad (36)$$

With the transformed boundary conditions

$$\eta=0 \ (for \quad y=0, \quad \beta_2=0): \\ F=F'=0, \qquad G=K_9 \\ \eta\to\infty \ (for \quad y\to\infty): \\ F'=K_{10}, \qquad G-K_{11}=0$$

Where

$$K_1=e^{\frac{\alpha_6}{\beta_1}x}\beta_6, \quad K_2=e^{(\frac{2\alpha_3}{\beta_1}-2\frac{\alpha_2}{\beta_1}-2\frac{\alpha_6}{\beta_1})x},$$

$$K_3=e^{(\frac{\alpha_3}{\beta_1}-3\frac{\alpha_2}{\beta_1}-2\frac{\alpha_6}{\beta_1})x}, \ K_4=C_1\frac{\alpha_5\beta_4}{\alpha_4}e^{(\frac{\alpha_3}{\beta_1}-\frac{\alpha_2}{\beta_1}+\frac{\alpha_5}{\beta_1})x}$$

$$, \ K_5=C_1e^{(\frac{\alpha_3}{\beta_1}-\frac{\alpha_2}{\beta_1}+\frac{\alpha_6}{\beta_1}+\frac{\alpha_4}{\beta_1})x},$$

$$K_6=C_1\frac{\beta_5}{\alpha_5}e^{(\frac{\alpha_4}{\beta_1}+\frac{\alpha_3}{\beta_1}-\frac{\alpha_2}{\beta_1})x}, \qquad K_7=C_1e^{(-2\frac{\alpha_2}{\beta_1}+\frac{\alpha_5}{\beta_1}+\frac{\alpha_4}{\beta_1})x},$$

$$K_8=\frac{\beta_5}{\alpha_5}e^{(-2\frac{\alpha_2}{\beta_1}+\frac{\alpha_4}{\beta_1})x}, \qquad K_9=(1+\frac{\beta_4}{\alpha_4})e^{-\frac{\alpha_4}{\beta_1}x},$$

$$K_{10}=e^{(\frac{\alpha_2}{\beta_1}-\frac{\alpha_3}{\beta_1})x}(e^{\frac{\alpha_6}{\beta_1}x}k_2-\frac{\beta_6}{\alpha_6}), \qquad K_{11}=\frac{\beta_4}{\alpha_4}e^{-\frac{\alpha_4}{\beta_1}x}.$$

For (35)-(36) to be reduced to an expression in terms of the new variables, it is necessary that $K_1,...,K_8$ should be constants or functions of $\eta$ alone and $K_9,K_{10},K_{11}$ should be constants. This is, only whenever:

$$\beta_2=\beta_4=\beta_5=\beta_6=\alpha_4=0, \\ \frac{\alpha_2}{\beta_1}=-m, \ \frac{\alpha_3}{\beta_1}=m, \ \frac{\alpha_5}{\beta_1}=-2m \quad and \quad \frac{\alpha_6}{\beta_1}=2m$$

Where $m$ is arbitrary constant. Using these relations, (32)-(33), become

$$-2mC_2^2+2mF'^2-m\eta FF''-\beta_1\nu F'''=0 \quad (37)$$

$$-2mGF'-mFG'-\frac{\nu}{Pr}\beta_1C_1G''=0 \quad (38)$$

With the transformed boundary conditions (19).

Moreover, by simple manipulation to Eqs. (35)-(36), one can derive another form of similarity variables. For example; multiply (36) by $e^{(\frac{\alpha_2}{\alpha_1}-\frac{\alpha_5}{\alpha_1}-\frac{\alpha_3}{\alpha_1})x}$, and repeating the process we obtain same results reported above with $\frac{\alpha_5}{\beta_1}$ is arbitrary constant $n$ (say). i.e., (37)-(38), become

$$-2mC_2^2+2mF'^2-m\eta FF''-\beta_1\nu F'''=0$$

$$nGF'-mFG'-\frac{\nu}{Pr}\beta_1C_1G''=0$$

With the transformed boundary conditions (19).

## 6 Conclusion

General form of absolute invariants and the corresponding new variables to group of the form (9) are determined. The results obtained here reveal that such new variables are applicable directly to partial differential equations, specially arising in engineering and applied science.

Three methods of similarity analysis which depend on the finite group of transformations have been carried out to present similarity equations of the problem of laminar forced convection on a horizontal plate.

The analysis and comparisons carried out here show that the effectiveness of the new systematic method in obtaining similarity equations for the problem. The results are found to be in agreement and thus the new systematic method is the simpler and more general. Four sets of similarity equations are obtained by this method and, it is found that, these sets of similarity equations include all similarity equations found by other methods.

The Formulation of the New Systematic Method is explained in appendix -A below.

It is hoped that the method presented here can be used effectively in the situations where the differential equations are more complicated.

## 7 Acknowledgment

AKA (Permenant Address: Department of Mathematics, Al-Baida University, Al-Baida, Republic of Yemen) acknowledges with thanks the financial support received from Al-Baida University and Government of Republic of Yemen. He also thanks to Indian Council for

Culture and Relations (ICCR) for providing Scholarship to pursue research in India and to UGC (India) for UGC-SAP-DRS (Phase-II) provided to the Department of Mathematics, Sardar Patel University, Valabh Vidyanagar under which the work was carried out.

## Appendix (A)

## Group of Transformation:

Consider the situation of a system $E$ of $N$ differential equations of $k$-th-order in $n$ independent variables $x=(x_1,...,x_n)$ and $m$ dependent variables $u(x)=(u^1(x),...,u^m(x))$, given by

$$E_\mu(x_1,...,x_n,u^1,...,u^m,\partial u,...,\partial^k u)=0 \qquad (39)$$

For each $\mu=1,2,...,N$, with the boundary conditions

$$B_\nu=0 \quad \text{on} \quad \omega_\nu=0, \quad \nu=1,2,...,s. \qquad (40)$$

and consider the one-parameter group

$$G_1 = \begin{cases} S_1: \bar{x}_i = f(x_1,...,x_n;a) & i=1,...,n \quad n\geq 2 \\ \bar{u}^j = g^j(u^1,...,u^m;a) & j=1,...,m \quad m\geq 2 \end{cases}$$

acting on base space $(x,u)$ which has infinitesimal form given by

$$G_1 = \begin{cases} S_1: \bar{x}_i = x_i + \varepsilon\, \xi_i & i=1,...,n \quad n\geq 2 \\ \bar{u}^j = u^j + \varepsilon\, \zeta^j & j=1,...,m \quad m\geq 2 \end{cases} \qquad (41)$$

with infinitesimal generators of the form

$$X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \zeta^j \frac{\partial}{\partial u^j} \qquad (42)$$

By definition, $G_1$ has $n+m-1$ functionally independent absolute invariants of the form

$$\left.\begin{array}{l} \Omega_1(x_1,...,x_n),...,\Omega_{n-1}(x_1,...,x_n) \\ g^j(x_1,...,x_n,u^1,...,u^m), \qquad j=1,...,m \end{array}\right\} \qquad (43)$$

## Formulation of the New Systematic Method:

It is well-known that any DE of order $k$ be invariant under one-parameter group $G_1$, if it is invariant under the $k$-th extended group $G_1^{(k)}$ of $G_1$. Therefore, the system $E$ is invariant under the group represented by (41), if it is invariant under the $k$-th extended group, $G_1^{(k)}$, of (41). According to [25], the necessary and sufficient conditions of this system to be invariance under an extended group is that this system satisfy a system of homogeneous linear partial differential equations

$$X^k E_\mu(x,u,\partial u,...,\partial^k u)=0$$

for each $\mu=1,...,N$, which has $p-1$ differential invariants, where $p=n+m+l$ and $l$ is number of the derivatives thereof up to the $k$-th order. We therefore add the following invariants to list in (43):

$$\hat{g}^\gamma(x_1,...,x_n,u^1,...,u^m,....), \qquad \gamma=1,...,l,$$

It is well known that an arbitrary function $\varphi$ (general form of $E$) obtained by equating an $p-1$ absolute invariants of $G_1^{(k)}$ to zero is invariant under $G_1^{(k)}$; that is,

$$\varphi(\lambda_1,...,\lambda_{p-1})=0$$

Where $\lambda_1,...,\lambda_{p-1}$ are $\Omega_1,...,\Omega_{n-1}$, $g's$ and $\hat{g}'s$ respectively. Since $\varphi$ be invariant under $G_1^{(k)}$, then according to Morgan's theorem, $\varphi$ can be expressed in terms of new variables $\eta_1,...,\eta_{n-1}$; as the independent variables and $F_1,...,F_m$ as dependent variables such that

$$\begin{cases} \eta_\gamma = \Omega_\gamma & \gamma=1,...,n-1 \\ F^j(\eta_1,...,\eta_{n-1}) = g^j & j=1,...,m \end{cases} \qquad (44)$$

The result be,

$$\varphi(\lambda_1,...,\lambda_{p-1}) = \tilde{\varphi}(\eta_1,...,\eta_{n-1};F_1,...,F_m,\partial F_j,...,\partial^k F_j)=0$$

We are easily led to the following result:

**Lemma 1:** $E_\mu$ has the form $\varphi_\mu$ if and only if $E_\mu$ is expressible in terms of the new invariants $(\eta_1,...,\eta_{n-1};F_1,...,F_m)$.

As a result, we will establish the central results which are the basis of our method.

**Theorem 1**: If a system $E$ is expressed in terms of $n+m-1$ new variables (44) of $G_1$, then it is invariant under this group.

**Proof:** Suppose that $E_\mu\ (\mu=1,...,N)$ are expressible in terms of the $(n+m-1)$ new variables, (44). Then, by Lemma (1), $E_\mu$ have the form $\varphi_\mu$ and so

$$E_\mu\,(\mathrm{x,u},u_i^j,...,u_{i_1,...,i_k}^j)=\varphi_\mu(\eta_1,...,\eta_{n-1},F_1,...,F_m)=0$$

For invariant $E_\mu$, it is enough to show that $X^{(k)}E_\mu=0$ for each $\mu=1,...,N$,

Let $z_1,...,z_p$ are the variables $x_1,...,x_n$, functions $u^1,...,u^m$, and the derivatives thereof up to the $k$-th order; and $\chi_1,...\chi_p$ are $\xi_1,...,\xi_n,\zeta^1,...,\zeta^m,\zeta_{[i]}^j,...,\zeta_{[i_1,..j_k]}^j$ respectively. Then

$$\begin{aligned}X^{(k)}E_\mu&=X^{(k)}\varphi\\&=\chi_1(z_1,...,z_p)\frac{\partial\varphi}{\partial z_1}+....+\chi_p(z_1,...,z_p)\frac{\partial\varphi}{\partial z_p}\\&=\chi_1(z_1,...,z_p)[\frac{\partial\varphi}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial z_1}+...+\frac{\partial\varphi}{\partial\lambda_{p-1}}\frac{\partial\lambda_{p-1}}{\partial z_1}]+...\\&\quad+\chi_p(z_1,...,z_p)[\frac{\partial\varphi}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial z_p}+...+\frac{\partial\varphi}{\partial\lambda_{p-1}}\frac{\partial\lambda_{p-1}}{\partial z_p}]\end{aligned}$$

$$\begin{aligned}&=\frac{\partial\varphi}{\partial\lambda_1}[\chi_1\frac{\partial\lambda_1}{\partial z_1}+...+\chi_p\frac{\partial\lambda_{p-1}}{\partial z_p}]+...\\&\quad+\frac{\partial\varphi}{\partial\lambda_{p-1}}[\chi_1\frac{\partial\lambda_{p-1}}{\partial z_1}+...+\chi_p\frac{\partial\lambda_{p-1}}{\partial z_p}]\end{aligned}$$

$$=\frac{\partial\varphi}{\partial\lambda_1}[0]+...+\frac{\partial\varphi}{\partial\lambda_{p-1}}[0]=0$$

Since $\lambda$'s satisfying $X^{(k)}\lambda_\gamma=0\ (\gamma=1,...,p\text{-}1)$.

Hence $E$ is invariant under the given group. According to this theorem if the system $E$ express in terms of those new variable, then it is invariant under the group and such new variable are similarity variables.

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