

# Reliability of the past climate reconstructions based on the borehole measurements

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*Abstract:* - There were many past climate reconstructions based on the borehole measurements. The measured temperature-depth profiles both in rock and glaciers present the input data for solutions of the inverse problems for determination of the boundary conditions at the surface. However, the properties of such solutions have not been derived with mathematical point of view. We find out that the solution of this problem is not unique and stable. The uniqueness and stability properties take place for the inverse problems that assume solution in the form of the finite segments of the Fourier series for the temperature. We formulate the algorithm that provides reliable temperature reconstructions.

*Key-Words:* -Boreholes, Climate reconstruction, Heat and mass transfer, Inverse problems

## 1 Introduction

The climate global changes in many aspects influence on economy and global politics [1], [2]. That is why knowledge of the surface temperature changes is important problem. The development of methods for the past surface temperature reconstructions can give a key in understanding these changes. The systematic instrumental temperature measurements took place no more than two centuries. Thus, indirect estimations of the past temperatures present main information on the past climate. There are two different sources of information about paleotemperatures. These are the boreholes [3], [4] and the high-resolution proxy climate indicators, for example, the tree rings [5], [6], lake varved sediments [7], [8], corals and sclerosponges [9], speleothems [10], [11] etc. The most reliable data are kept in the measured borehole temperatures that present response to the surface temperature history.

The underground temperature distribution is mainly determined by two types of processes [3]. The first is the surface temperature changes and the second is the heat flux from the Earth that is subjected to the long-time geological processes. The surface temperature changes take place at relatively smaller time scale. Therefore, the measured temperature-depth profiles in the borehole contain information on the climatic changes at the surface. The seasonal temperature variations at the surface are noticeable at depth about 10-15 meters while the climatic oscillations reach several hundred meters

and more. Thus, the boreholes of several hundred meters can contain information on the past surface temperatures for several hundred years.

The heat and mass transfer in rocks is described by the one-dimensional thermal diffusivity equation [3]. The past surface temperature reconstruction is the inverse problem that contains additional re-determination condition. The measured temperature-depth profile presents such condition. We found out that this problem has not the unique and stable solution in general case.

There are several well-known methods of the past surface temperature reconstructions: the Monte-Carlo method [12], [13]; the least-squares inversion method [3]; the singular value decomposition method [14], [15] and the control method [16]. We show that these temperature reconstructions are not unique and stable algorithms with mathematical point of view. Some examples are considered in the paper to demonstrate the algorithm of the past surface temperature reconstructions.

## 2 Mathematical Model

The mathematical statement of the inverse problem consists of the one-dimensional thermal conductivity equation, the initial condition, the boundary condition at the bottom of borehole and the measured-temperature-depth profile. The former is used as the re-determination condition,  $\chi(z)$ , where  $z$  is vertical coordinate. Then the inverse

problem to find the temperature in the past is the solution of the following problem:

$$\begin{cases} T_t = a^2 T_{zz}, & 0 < t < t_f, \quad 0 < z < H, \\ T(0, t) = U_s + \mu(t), & 0 < t \leq t_f, \\ -k \cdot T_z(H, t) = q, & 0 < t \leq t_f, \\ T(z, 0) = U(z), & 0 < z < H, \\ T(z, t_f) = \chi(z), & 0 < z < H. \end{cases} \quad (1)$$

where  $H$  is the borehole depth,  $a^2$  is the thermal diffusivity,  $k$  is the thermal conductivity,  $q$  is the geothermal heat flux at the bottom of borehole,  $U(z)$  is the steady-state temperature profile associated with this flux,  $U_s$  is the initial temperature, which characterizes the average temperature that was on the surface in the past before the beginning of sharp temperature variations on the surface,  $\mu(t)$  is temperature variations on the surface in time with respect to its initial value  $U_s$  from the moment  $t=0$  to the time of measurements of the borehole temperature profile  $t_f$ ,  $\mu(0)=0$ .

Let us represent the borehole temperature  $T(z,t)$  in the form of the superposition of two temperature profiles: the steady-state temperature profile  $U(z)$  associated with the geothermal heat flow from the Earth and the residual temperature profile  $V(z,t)$  associated with temperature variations on the surface:

$$T(z,t) = U(z) + V(z,t). \quad (2)$$

Then, the steady-state temperature profile:

$$U(z) = U_s - (q/k) \cdot z \quad (3)$$

Let us denote  $\theta(z) = \chi(z) - U(z)$  is deviation of the measured temperature profile from the steady-state one. This deviation is associated with the surface temperature changes. Thus, the problem of finding surface temperature history is reduced to the solution of the problem:

$$\begin{cases} V_t = a^2 V_{zz}, & 0 < t < t_f, \quad 0 < z < H, \\ V(0, t) = \mu(t), & 0 < t \leq t_f, \\ V_z(H, t) = 0, & 0 < t \leq t_f, \\ V(z, 0) = 0, & 0 < z < H, \\ V(z, t_f) = \theta(z), & 0 < z < H. \end{cases} \quad (4)$$

### 3 Properties of Solution

Let us show that the inverse problem (4) in the general case has no the unique solution.

#### Lemma 1.

In addition to the trivial solution ( $V(z,t) \equiv 0$ ;  $\mu(t) \equiv 0$ ), the inverse problem

$$\begin{cases} V_t = a^2 V_{zz}, & 0 < t < t_f, \quad 0 < z < H, \\ V(0, t) = \mu(t), & 0 < t \leq t_f, \\ V_z(H, t) = 0, & 0 < t \leq t_f, \\ V(z, 0) = 0, & 0 < z < H, \\ V(z, t_f) = 0, & 0 < z < H. \end{cases} \quad (5)$$

has a nontrivial solution ( $V(z,t); \mu(t)$ ).

#### Proof.

Let us assume that  $\mu(0) = \mu(t_f) = 0$  and

$$\mu(t) = \sum_{m=1}^{\infty} \alpha_m \cdot \sin(\pi m t / t_f),$$

where  $\alpha_m$  are unknown coefficients. Let  $V^{(m)}(z,t)$  be a solution of the direct problem specified as

$$\begin{cases} V_t^{(m)} = a^2 V_{zz}^{(m)}, & 0 < t < t_f, \quad 0 < z < H, \\ V^{(m)}(0, t) = \alpha_m \sin(\pi m t / t_f), & 0 < t \leq t_f, \\ V_z^{(m)}(H, t) = 0, & 0 < t \leq t_f, \\ V^{(m)}(z, 0) = 0, & 0 < z < H. \end{cases} \quad (6)$$

The solution of this problem can be obtained in the form

$$V^{(m)}(z,t) = \alpha_m \cdot \sin(\pi m t / t_f) - \alpha_m \pi m / t_f \cdot \sum_{n=1}^{\infty} I_n e_n(z) \int_0^t \exp(-\lambda_n(t-\tau)) \cdot \cos(\pi m \tau / t_f) d\tau \quad (7)$$

where

$$I_n = \frac{1}{\|e_n(z)\|^2} \int_0^H e_n(z) dz = \frac{2}{H} \frac{1}{\sqrt{\lambda_n}},$$

$e_n = \sin(\sqrt{\lambda_n} z)$  and  $\lambda_n = a^2 \pi^2 / H^2 \cdot (n-1/2)^2$  are the eigen functions and eigen values, respectively, of the following Sturm–Liouville problem:

$$\begin{cases} a^2 Z''(z) + \lambda Z = 0, & 0 < z < H, \\ Z(0) = Z'(H) = 0. \end{cases} \quad (8)$$

The asymptotic behaviour of the eigenvalues is  $|\lambda_n| \sim C \cdot n^2, n \rightarrow \infty$ . Therefore, the series  $\sum_{n=1}^{\infty} 1/|\lambda_n|$  converges. Then, the set of the functions  $\{e^{\lambda_n t}\}_{n=1}^{\infty}$  is incomplete in  $L_2(0, t_f)$ . This is a corollary of

Müntz's theorem [17]. Thus, there is a nonzero function  $F(t)$  specified at  $t \in [0, t_f]$  such that  $F(t)$  is orthogonal to  $\{e^{\lambda_n t}\}_{n=1}^{\infty}$  in  $L_2(0, t_f)$ . Let us expand  $F(t)$  into the Fourier series at  $t \in [0, t_f]$ :

$$F(t) = \sum_{m=1}^{\infty} \beta_m \cdot \sin(\pi m t / t_f). \text{ Let us prove that}$$

$V(z, t) = \sum_{m=1}^{\infty} V^{(m)}(z, t)$  is a solution of the problem specified by Eqs. (5) and  $\mu(t) = F(t)$ ,  $\alpha_m = \beta_m$ . Indeed,  $V(z, t)$  satisfies the first of Eqs. (5), as well as the initial and boundary conditions. Let us verify the last condition in Eqs. (5):

$$V(z, t_f) = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \alpha_m m}{t_f} I_n e_n(z) \cdot \int_0^{t_f} \exp(-\lambda_n(t_f - \tau)) \cdot \cos(\pi m \tau / t_f) d\tau. \tag{9}$$

The integration of Eq. (9) by parts yields

$$V(z, t_f) = \sum_{n=1}^{\infty} I_n e_n(z) \lambda_n \exp(-\lambda_n t_f) \cdot \int_0^{t_f} \exp(\lambda_n \tau) \cdot \left( \sum_{m=1}^{\infty} \alpha_m \sin(\pi m \tau / t_f) \right) d\tau. \tag{10}$$

Since the inner series in Eq. 10 is identically equal to  $F(\tau)$  and is orthogonal to  $\{e^{\lambda_n \tau}\}_{n=1}^{\infty}$ ,  $V(z, t_f) = 0$ . Thus, we find the nontrivial solution  $V(z, t)$  and the lemma is proved.

Thus, the solution of the problem (4) without additional constraints is not unique.

### 4 Uniqueness and Stability

Let us assume that  $\mu(t) = \sum_{k=-m}^m \mu_k \cdot \exp(i2\pi k t / t_f)$  is a finite segment of the Fourier series. Let us show that in this case the uniqueness of the function  $\mu(t)$  can be proved.

Let us represent  $V(z, t)$  from problem (4) as  $V(z, t) = \mu(t) + W(z, t)$ , then the problem of finding surface temperature history can be represented in the form

$$\begin{cases} W_t + f(t) = a^2 W_{zz}, & 0 < t < t_f, \quad 0 < z < H, \\ W(0, t) = 0, & 0 < t \leq t_f, \\ W_z(H, t) = 0, & 0 < t \leq t_f, \\ W(z, 0) = 0, & 0 < z < H, \\ W(z, t_f) = s(z), & 0 < z < H. \end{cases} \tag{11}$$

Where  $f(t) = \mu'(t)$  and  $s(z) = \theta(z) - \mu(t_f)$ . The solution of the problem (11) is presented by the equation

$$W(z, t) = \sum_{n=1}^{\infty} I_n e_n(z) \int_0^t e^{-\lambda_n(t-\tau)} \cdot f(\tau) d\tau. \tag{12}$$

where  $e_n(z)$  and  $\lambda_n$  are the eigen functions and eigen values, respectively, of the Sturm–Liouville problem (8).

Since  $\mu(t)$  is a finite segment of the Fourier series,  $f(t)$  is a finite segment of the Fourier series too, and  $f(t) = \sum_{k=-m}^m f_k \cdot \exp(i2\pi k t / t_f)$ .

Since  $f(t) = \mu'(t)$ ;  $\mu(0) = 0$ , the function  $\mu(t)$  is uniquely determined from  $f(t) \in C[0, t_f]$ . If the uniqueness of the function  $f(t)$  is proved, then the uniqueness of the function  $\mu(t)$  can be proved too.

To prove uniqueness of the problem (11) it is sufficient to show that  $W(z, t) \equiv 0$  and  $f(t) \equiv 0$  if  $s(z) = 0$ .

It is known that  $\{e_n(z)\}_{n=1}^{\infty}$  is the complete orthonormalized set, and  $I_n \neq 0$ ,  $\lambda_n \in \mathbb{R}$ ,  $\lambda_n \rightarrow \infty$ ,  $n \rightarrow \infty$ . From the condition that  $W(z, t_f) = 0$ , it follows that

$$\forall n \in \mathbb{N}: \int_0^{t_f} e^{-\lambda_n(t_f - \tau)} \cdot f(\tau) d\tau = 0.$$

Thus, the integer function  $F(\lambda) = \int_0^{t_f} e^{\lambda \tau} \cdot f(\tau) d\tau$  has the infinite number of zeros. This is possible only for  $f(t) \equiv 0$ , because  $f(t)$  is a finite segment of a Fourier series. Therefore,  $\mu(t) \equiv 0$ , and the uniqueness property is proved.

Let us show that this solution is stable. Let two solutions  $W_1(z, t)$ ,  $f_1(t)$  and  $W_2(z, t)$ ,  $f_2(t)$  of the problem (11) correspond to the close re-determination functions  $s_1(z)$  and  $s_2(z)$ , respectively. Let us show that if  $f(t)$  is a finite

segment of a Fourier series, then these solutions are closed each other.

From Eq. (12) and the determination condition  $W(z, t_f) = s(z)$ , it follows that:

$$s(z) = \int_0^{t_f} K(z, \tau) f(\tau) d\tau. \tag{13}$$

where  $K(z, \tau) = \sum_{n=1}^{\infty} I_n e_n(z) e^{-\lambda_n(t_f - \tau)}$  is the kernel of linear operator.

Eq. (13) is the Fredholm integral equation of the first kind. This is a classical ill-posed problem. If  $f(t)$  is a finite segment of a Fourier series, the uniqueness theorem is proved for Eq. (13); i.e., from the condition  $s(z) = 0, z \in [0, H]$ , it follows that  $f(t) = 0, t \in [0, t_f]$ .

Since  $\{e_n(z)\}_{n=1}^{\infty}$  is an orthonormalized basis in  $L_2(0, H)$ ,  $f(t) = \sum_{k=-m}^m f_k \cdot \exp(i2\pi kt / t_f)$  then Eq. (13) is equivalent to the following system of linear equations:

$$s_n = I_n \sum_{k=-m}^m f_k \cdot \int_0^{t_f} e^{-\lambda_n(t_f - \tau)} e^{i2\pi k \tau / t_f} d\tau, \tag{14}$$

$n = 1, 2, 3, \dots$

where  $s_n = \frac{1}{\|e_n(z)\|^2} \int_0^H s(z) e_n(z) dz$ ;

$f_k, k = 0, \pm 1, \pm 2, \dots, \pm m$  are unknowns.

The number of the equations is infinite, whereas the number of unknowns is  $2m+1$ , therefore, the system in the general case has no solution at arbitrary  $s_n$  values. Thus, the problem under investigation is reduced to the solution of the system of the algebraic equations of the form:  $\mathbf{A}\mathbf{f}=\mathbf{s}$ , where

$$\mathbf{f} = \begin{bmatrix} f_{-m} \\ f_{-m+1} \\ \vdots \\ f_0 \\ \vdots \\ f_m \end{bmatrix}; \mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \\ \vdots \end{bmatrix}; \mathbf{A} = (\alpha_{pq}),$$

$$\alpha_{pq} = I_p \int_0^{t_f} e^{-\lambda_p(t_f - \tau)} e^{i2\pi q \tau / t_f} d\tau,$$

where  $p = 1, 2, \dots; q = 1, 2, \dots, 2m + 1$ .

We have proved the uniqueness theorem. Therefore, the homogeneous problem has only the trivial solution. Let us prove that this problem has the stability property in the following meaning. Let two solutions  $\mathbf{f}^{(1)}$  and  $\mathbf{f}^{(2)}$  correspond to two columns  $\mathbf{s}^{(1)}$  and  $\mathbf{s}^{(2)}$ , respectively, so that  $\|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}\| = \sup_{i \in N} |s_i^{(1)} - s_i^{(2)}|$ . Then these solutions  $\mathbf{f}^{(1)}$  and  $\mathbf{f}^{(2)}$  are close each other by the norm.

Let us find the image  $\text{Im A}$  of the linear operator  $A$  that specifies the transformation  $R^n \rightarrow C_0$ , where  $C_0$  is the space of the number sequences  $(s_1, \dots, s_n, \dots)$  converging to zero because the Fourier coefficients tend to zero. Let us denote  $\mathbf{A}$  as the matrix of the linear operator  $A$  in a certain basis.

**Statement.**

If  $\mathbf{e}_1, \dots, \mathbf{e}_n$  constitute a basis in  $R^n$ , then the vectors  $\mathbf{A}\mathbf{e}_1, \dots, \mathbf{A}\mathbf{e}_n$  constitute a basis in  $\text{Im A}$ .

**Proof.**

Let us consider an arbitrary vector  $\mathbf{z} \in \text{Im A}$ . Then  $\exists \mathbf{x} \in R^n$  such that  $\mathbf{A}\mathbf{x}=\mathbf{z}$ .

Let us expand the vector  $\mathbf{x}$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ :  $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$ . Then, since the operator  $A$  is linear,  $\mathbf{z} = \mathbf{A}\mathbf{x} = x_1\mathbf{A}\mathbf{e}_1 + x_2\mathbf{A}\mathbf{e}_2 + \dots + x_n\mathbf{A}\mathbf{e}_n$ . Therefore, such an expansion exists.

Let us prove the uniqueness of this expansion, i.e., linear independence of the elements  $\mathbf{A}\mathbf{e}_1, \dots, \mathbf{A}\mathbf{e}_n$ .

If  $\alpha_1\mathbf{A}\mathbf{e}_1 + \alpha_2\mathbf{A}\mathbf{e}_2 + \dots + \alpha_n\mathbf{A}\mathbf{e}_n = 0$  then  $\mathbf{A}(\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \dots + \alpha_n\mathbf{e}_n) = 0$ , owing to linearity; since the kernel of the operator is zero, we have  $\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \dots + \alpha_n\mathbf{e}_n = 0$ ; therefore  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$  because  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are linearly independent; thus, the statement is proved.

**Corollary.**

$\text{Im A}$  is a finite-dimensional ( $n$ - dimensional) subspace of  $C_0$ .

The linear operator  $A$  transforms  $R^n$  into  $V_n = \text{Im A}$  and has zero kernel. Therefore, the operator  $A$  has the inverse operator  $A^{-1}$  that specifies the transformation  $V_n \rightarrow R^n$  and is a linear bounded operator.

If  $\mathbf{A}\mathbf{f}^{(l)} = \mathbf{s}^{(l)}, l=1,2$ , then  $\mathbf{f}^{(l)} = \mathbf{A}^{-1}\mathbf{s}^{(l)}, l=1,2$ , therefore,  $\mathbf{f}^{(2)} - \mathbf{f}^{(1)} = \mathbf{A}^{-1}(\mathbf{s}^{(2)} - \mathbf{s}^{(1)})$ . Moreover, the estimation  $\|\mathbf{f}^{(2)} - \mathbf{f}^{(1)}\| \leq \|\mathbf{A}^{-1}\| \cdot \|\mathbf{s}^{(2)} - \mathbf{s}^{(1)}\|$  is valid for

the stability of the inverse problem because the operator  $A^{-1}$  is bounded.

Thus, the solutions of the inverse problem in the form of the segments of the Fourier series are close if the redetermination functions are close each other by the norm. Therefore, in that case, stability of the surface temperature history reconstruction is proved.

## 5 Method of paleotemperature reconstruction

The boundary condition at the surface has to be written in form of a finite segment of the Fourier series to build the unique and stable reconstruction of the past surface temperature.

We will use the dominant periods of temperature variations that are contained in the high-resolution proxy climate indicators of the studied region. For example, we can use these data based on the tree rings, corals and sclerosponges, speleothems and lake varved sediments. For glacier boreholes the periods can be derived from the oxygen or hydrogen isotopic ratios.

To find out the periods in the proxy climate indicators we apply the wavelet analysis [18], [19]. The wavelet analysis allows us to determine both the periods of the climate changes and the time intervals in the past when these changes took place.

Thus, the following algorithm of the paleotemperature reconstructions from the borehole data can be formulated.

1. It is needed to search the high-resolution proxy climate indicators near the borehole region.

2. The relationships between the proxy indicators and instrumental climate data at the nearest meteorological stations have to be derived.

3. The dominant periods of the proxy data for the studied region have to be found out by the wavelet analysis.

4. The unknown past surface temperature is looking for in form of the finite segment of the Fourier series consisted of the set of the dominant periods.

In fact this algorithm combines two different proxy climate indicators that contain both long-term and short-term trends of the temperature changes kept in the borehole temperature-depth profile and in the high-resolution proxy climate indicators, respectively. Indeed, information in the temperature-depth profile is reflection of factual changes of temperature at the surface. Unfortunately, this information on the past temperatures becomes worse when we consider relatively big time intervals. However, this

deficiency of information is compensated by data containing in the high-resolution proxy climate indicators.

## 6 Reconstruction of past temperatures

Let us consider some examples of the past surface temperature reconstructions. We use the input data about the borehole temperature according to NCDC database [20]. Consider region of Bogatyrevo located in Kazakhstan (Longitude:  $84.33^{\circ}$ , Latitude:  $49.83^{\circ}$ ). The depth of the borehole RU-Bogatyrevo3096 is 500 m. The measurements were done in 1977. The measured temperature-depth profile is shown in Fig. 1. The steady-state temperature profile is determined by Eq. (3) and also shown in Fig. 1. Then we can determine parameters  $U_s$  and  $q/k$  taking into account that the bottom part of the measured temperature-depth profile is in the steady-state condition;  $U_s=6.46^{\circ}\text{C}$ ;  $q/k=0.0165^{\circ}\text{C}\cdot\text{m}^{-1}$ .

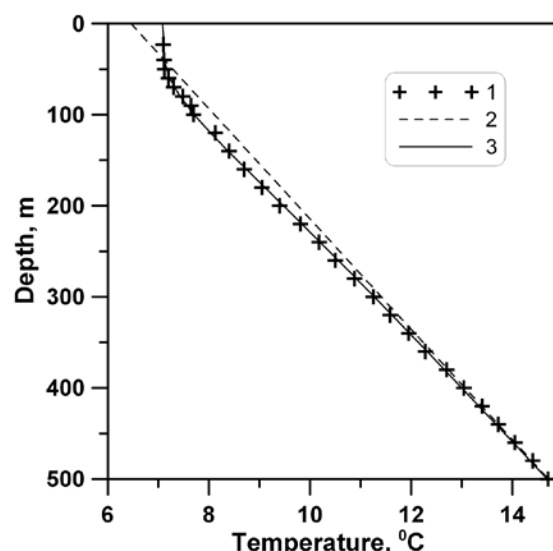


Fig. 1. Borehole temperature profiles (1 is measured temperature profile, 2 is steady state temperature profile, 3 is reconstructed temperature profile)

Many authors reconstruct the past surface temperatures by search of temperature in the form of the step-wise-function [14], [15]. Here we show that such reconstructions do not possess property of uniqueness. To demonstrate it we apply the Monte-Carlo method. This method tests randomly chosen variations in the surface temperature using them as the input data for the direct problem (4) and taking into account the consistency degree between the calculated and measured temperature profiles. Set of possible reconstructions and the most likely one are shown in Fig. 2. One can see the significant

difference in the possible reconstructed temperatures.

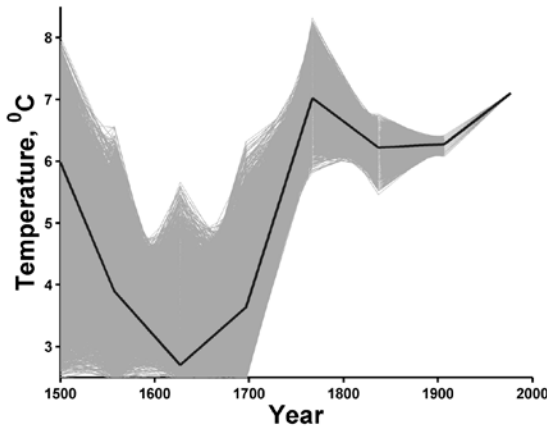


Fig. 2. Set of the possible past surface temperature reconstructions and the most likely one (solid curve)

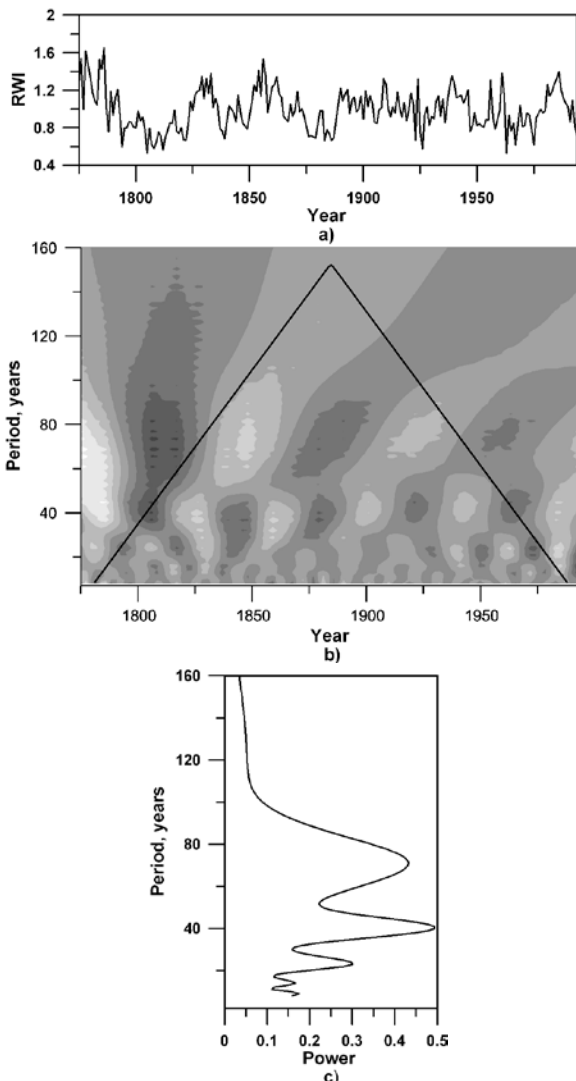


Fig. 3. Tree-ring chronology (a); wavelet power spectrum (b) and global wavelet (c)

Let us apply the developed algorithm of the paleotemperature reconstructions for the borehole data. The nearest tree-ring chronologies are used as the high-resolution proxy climate indicator [20]. The wavelet analysis allows us to derive that the dominant periods of 70, 40, 23 and 14 years are contained in the tree-ring chronologies for the studied region. We also take into account the dominant period of 200 years that observed in the long-term chronologies. The results of the wavelet analysis are shown in Fig. 3. The dominant periods of 23 and 14 years are observed at the nearest meteorostation Kokpekty [21], and correlate with wavelet analysis results. Three other dominant periods could not be observed at this station due to short observation time.

The Tikhonov method is applied to determine the past surface temperatures [22]. The Tikhonov regularization method is the determination of the boundary temperature  $\mu(t)$  minimizing a smoothing functional consisting of the difference and stabilizer:

$$\Psi = \frac{1}{2} \int_0^H [R\{\mu(t)\} - \theta(z)]^2 dz + \alpha \Omega\{\mu(t)\} \quad (15)$$

where  $R\{\mu(t)\}$  is the solution of the direct problem (4) represented in the operator form,  $\alpha$  is the regularization parameter matched with the accuracy of the input data. The functional  $\Omega\{\mu(t)\}$  is called the stabilizing functional or stabilizer:

$$\Omega\{\mu(t)\} = \int_0^{t_f} \sum_{j=0}^r q_j \left( \frac{d^j \mu}{dt^j} \right)^2 dt \quad (16)$$

where  $r$  is the stabilizer order,  $q_j \geq 0$ , and  $q_r > 0$ .

The procedure of the minimization of the smoothing functional  $\Psi$  can be performed by means of the gradient method and is an iteration procedure. The iteration procedure is carried out until the functional  $\Psi$  reaches a minimum with a given accuracy, which corresponds to the optimal solution of the inverse problem.

Let us write the surface temperature in the form of finite set of the Fourier series:

$$\mu(t) = \frac{a_0}{2} + \sum_{m=1}^5 a_m \cos(2\pi t / T_m) + b_m \sin(2\pi t / T_m) \quad (17)$$

The minimization procedure for functional  $\Psi$  can be carried out by the iteration process. The

initial Fourier coefficients are given in the first iteration step while the next  $n$ -th iterations are determined by the following equations:

$$\begin{aligned} a_0^{n+1} &= a_0^n - \gamma^n \frac{\partial \Psi^n}{\partial a_0^n}, \\ a_m^{n+1} &= a_m^n - \gamma^n \frac{\partial \Psi^n}{\partial a_m^n}, \\ b_m^{n+1} &= b_m^n - \gamma^n \frac{\partial \Psi^n}{\partial b_m^n}, \quad m = 1, 2, \dots, 5, \end{aligned} \quad (18)$$

where  $\gamma^n > 0$  is the gradient step. The derivatives of the functional in Eqs. (18) with respect to the corresponding Fourier coefficients are given by the expressions:

$$\begin{aligned} \frac{\partial \Psi^n}{\partial a_0^n} &= \int_0^H W_{a_0}(z) [R\{\mu^n(t)\} - \theta(z)] dz + \alpha \frac{\partial \Omega^n}{\partial a_0^n}, \\ \frac{\partial \Psi^n}{\partial a_m^n} &= \int_0^H W_{a_m}(z) [R\{\mu^n(t)\} - \theta(z)] dz + \alpha \frac{\partial \Omega^n}{\partial a_m^n}, \\ \frac{\partial \Psi^n}{\partial b_m^n} &= \int_0^H W_{b_m}(z) [R\{\mu^n(t)\} - \theta(z)] dz + \alpha \frac{\partial \Omega^n}{\partial b_m^n}, \end{aligned} \quad (19)$$

$m = 1, 2, \dots, 5.$

Here, the profiles  $W_{a_0}(z)$ ,  $W_{a_m}(z)$ , and  $W_{b_m}(z)$  are the solutions of the problem specified by Eqs. (4) with the boundary conditions on the surface  $\mu(t) = 1/2$ ,  $\mu(t) = \cos(2\pi t / T_m)$ , and  $\mu(t) = \sin(2\pi t / T_m)$ , respectively. It is easy to show that the term with the stabilizer in Eq. (15) when the boundary condition on the surface  $\mu(t)$  has the form of the segment of the trigonometric Fourier series has the form:  $\alpha \Omega = \alpha_0 \frac{a_0^2}{2} + \sum_{m=1}^5 (a_m^2 + b_m^2) \xi_m$ , where  $\xi_m = \alpha_0 + \alpha_1 (2\pi / T_m)^2 + \dots + \alpha_r (2\pi / T_m)^{2r}$ ,  $\alpha_i = \alpha q_i t_f / 2$ ,  $i = 0, 1, \dots, r$ . In this case

$$\begin{aligned} \alpha \frac{\partial \Omega^n}{\partial a_0^n} &= \alpha_0 a_0^n, \\ \alpha \frac{\partial \Omega^n}{\partial a_m^n} &= 2a_m^n \xi_m, \\ \alpha \frac{\partial \Omega^n}{\partial b_m^n} &= 2b_m^n \xi_m, \quad m = 1, 2, \dots, 5. \end{aligned} \quad (18)$$

Thus, to determine the Fourier coefficients of the boundary condition given by Eq. (17), the iteration

procedure specified by Eqs. (18) is performed with the use of Eqs. (19) and (20).

There construction is shown in Fig. 4. The reconstructed temperature correlates with the annual temperature of the North Hemisphere registered in 1850-1977. The correlation coefficient equals to 0.75. The calculated temperature-depth profile is shown in Fig. 1 (curve 3).

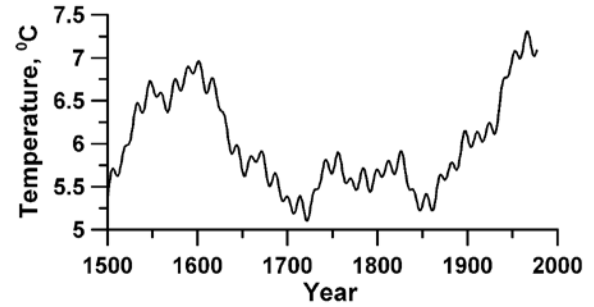


Fig. 4. Past surface temperature reconstruction

## 7 Conclusion

It is found out the conditions when the past surface temperature reconstructions are based on solutions possessing the uniqueness and stability properties with mathematical point of view. The unknown past surface temperature has to be presented as a finite segment of Fourier series with dominant periods. These periods can be found out in the high-resolution proxy climate indicators such as the tree rings, corals, sclerosponges, speleothems, and lake varved sediments. In practices, the measured borehole temperature contains continuous set of harmonics. It is due to both errors of measurements and unknown nature of climatic changes. It means that the problem of the past surface temperature reconstruction based on the measured borehole temperature has not the uniqueness and stability properties.

In fact all previous reconstructions of the past surface temperatures implicitly assume that the retrieval surface temperatures can be presented by the finite set of harmonics. In these cases the amplitudes of the harmonics can be found and the solutions are unique and stable. Moreover, the approach based on the additional information can improve an accuracy of the past surface temperature reconstructions.

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