The Shapley value for fuzzy games on vague sets

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Abstract: - In this paper, a general expression of the Shapley value for fuzzy games on vague sets is proposed, where the player participation levels are vague sets. The existence and uniqueness of the given Shapley value are showed by establishing axiomatic system. When the fuzzy games on vague sets are convex, the given Shapley value is a vague population monotonic allocation function (VPMAF) and an element in the core. Furthermore, we study a special kind of this class of fuzzy games, which can be seen as an extension of fuzzy games with multilinear extension form. An application of the proposed model in joint production problem is provided.

Key-Words: - fuzzy game; vague set; Shapley value; core; multilinear extension

1 Introduction

Cooperative fuzzy games [1] describe the situations that some players do not fully participate in a coalition, but to a certain degree. In this situation, a coalition is called a fuzzy coalition, which is formed by some players with partial participation. A special kind of fuzzy games, which is called fuzzy games with multilinear extension form [9], was discussed. In [2], the author defined a class of fuzzy games with proportional values, and gave the expression of the Shapley function on this limited class of games. Later, reference [12] pointed the fuzzy games given in [2] are neither monotone nondecreasing nor continuous with regard to rates of players' participation, and defined a kind of fuzzy games with Choquet integral forms. The Shapley function defined on this class of games is given. Recently, reference [3] expanded the fuzzy games with proportional values to fuzzy games with weighted function. And the corresponding Shapley function is also given. The fuzzy games with multilinear extension form and Choquet integral form are both monotone nondecreasing and continuous with respect to rates of players' participation.

As a well-known solution concept in cooperative game theory, the Shapley value for fuzzy games [2, 3, 6-8, 12] has been studied by many researchers. Besides the Shapley value, the fuzzy core of fuzzy games [13, 14] and the lexicographical solution for fuzzy games [10] are also discussed. When every player's participation level is 1, a fuzzy game reduces to be a traditional game. Namely, the traditional game is a special case of the fuzzy game [1]. Furthermore, the Shapley value for fuzzy games with fuzzy payoffs [4, 15] is also considered.

All above mentioned researches only consider the situation where the player participation is determined. There are many uncertain factors during the process of negotiation and coalition forming. In order to reduce risk and get more payoffs, when the players take part in cooperation, sometime they only know the determining participation levels and the participation levels that they do not participate. At this situation, fuzzy games can not be applied. But the vague sets [5] can well describe the participation levels of the players. Based on above analysis, we shall study fuzzy games on vague sets, where the player participation levels are vague sets.

This paper is organized as follows. In the next section, we recall some notations and basic definitions, which will be used in the following. In section 3, the expression of the Shapley value is given and an axiomatic definition is offered. Some properties are researched. In section 4, we pay a special attention to discuss a special kind of fuzzy games on vague sets and give and investigate the explicit forms of the Shapley value for this kind of fuzzy games.

2 Preliminaries

2.1 The concept of vague sets

Let X be an initial universe set, $X=\{x_1, x_2,\ldots,x_n\}$. A vague set over X is characterized by a ruth-membership function t_v and a false-membership function f_v, $t_v: X\to[0, 1]$, $f_v: X\to[0, 1]$satisfying $t_v + f_v\le 1$, where $t_v(x_i)$ is a lower bound on the grade of membership of x_i derived from the evidence for x_i, and $f_v(x_i)$ is a lower bound on the negation of x_i derived from the evidence against x_i. The grade of membership of x_i in the vague set is bounded to a subinterval $[t_v(x_i), 1-f_v(x_i)]$ of [0, 1]. The vague value $[t_v(x_i), 1-f_v(x_i)]$ indicates that the exact grade of membership $\mu_v(x_i)$of x_i may be unknown, but it is bounded by $t_v(x_i)\le \mu_v(x_i)\le 1-f_v(x_i)$, where $t_v+f_v\le 1$.

When the universe X is discrete, a vague set A can be written as

$$A=\sum_{1\le i\le n}[t_A(x_i),1-f_A(x_i)]/x_i,\quad x_i\in X .$$

Let the vague sets $x=[t(x), 1-f(x)]$ and $y=[t(y), 1-f(y)]$, where $0\le t(x)+f(x)\le 1$ and $0\le t(y)+f(y)\le 1$, then

(1) $x\wedge y=[t(x)\wedge t(y), 1-(f(x)\vee f(y))]$;

(2) $x\vee y=[t(x)\vee t(y), 1-(f(x)\wedge f(y))]$;

(3) $x\le y\Leftrightarrow t(x)\le t(y), f(x)\ge f(y)$;

(4) $x=y\Leftrightarrow t(x)=t(y), f(x)=f(y)$.

2.2 Some basic concepts for fuzzy games on vague sets

Let $N=\{1,2,\ldots,n\}$ be the player set, and $P(N)$ be the set of all crisp subsets in N. The coalitions in $P(N)$ are denoted by S_0, T_0,…. For $S_0\in P(N)$, the cardinality of S_0 is denoted by the corresponding lower case s. A function $v_0: P(N)\to\mathbb{R}_+$, satisfying $v_0(\varnothing)=0$, is called a crisp game. Let $G_0(N)$ denote the set of all crisp games in N.

A vague set S in N is denoted by

$$S=\sum_{i\in \mathrm{Supp}S}[t_S(i),1-f_S(i)]/i=[t_S, e^{\mathrm{Supp}S}-f_S],$$

where $\mathrm{Supp}S=\{i\in N|\ t_S(i)>0\}$denotes the support of t_S, and $e^{\mathrm{Supp}S}-f_S=\{(1-f_S(i))_{i\in\mathrm{Supp}S}\}$. $0\le t_S(i)+f_S(i)\le 1$ for any $i\in \mathrm{Supp}S$. $t_S(i)$ indicates the true membership grade of the player i in vague set S, and $f_S(i)$ denotes the false membership grade of the player i in vague set S. $e^{\mathrm{Supp}S}$ denotes a n-dimension vector, where $e^{\mathrm{Supp}S}(i)=1$ for any $i\in \mathrm{Supp}S$, otherwise, $e^{\mathrm{Supp}S}(i)=0$. The set of all vague sets in N is denoted by $TF(N)$. Let $S\in TF(N)$, the cardinality of t_S is denoted by $|\mathrm{Supp}S|$ and $|\mathrm{Supp}(e^{\mathrm{Supp}S}-f_S)|$ indicates the cardinality of $\mathrm{Supp}(e^{\mathrm{Supp}S}-f_S)=\{i\in \mathrm{Supp}U\,|\,1-f_S(i)\ge 0\}$. For any $S, K\in TF(N)$, we use the notation $S\subseteq K$ if and only if $t_S(i)=t_K(i)$ and $f_S(i)=f_K(i)$ or $t_S(i)=f_S(i)=0$ for any $i\in N$. A function $v: TF(N)\to\mathbb{R}_+$, satisfying $v(\varnothing)=0$, is called a fuzzy game on vague sets. Let $G_V(N)$ denote the set of all fuzzy games on $TF(N)$. For any $S\in TF(N)$, $v(t_S)$ is said to be true value for S, and $v(e^{\mathrm{Supp}S}-f_S)$ is said to be upper value for S.

Let $S, K\in TF(N)$, we have

$$(t_S\vee t_K)(i)=t_S(i)\vee t_K(i),$$

$$(t_S\wedge t_K)(i)=t_S(i)\wedge t_K(i),$$

$$(f_S\vee f_K)(i)=f_S(i)\vee f_K(i)$$

and

$$(f_S\wedge f_K)(i)=f_S(i)\wedge f_K(i)$$

for any $i\in N$.

Definition 1 Let $v\in G_V(N)$ and $U\in TF(N)$, v is said to be convex in vague set U if

$$v(t_S\wedge t_K)+v(t_S\vee t_K)\ge v(t_S)+v(t_K)$$

and

$$v\left(e^{\mathrm{Supp}S\cup\mathrm{Supp}K}-(f_S\vee f_K)\right)+v\left(e^{\mathrm{Supp}S\cap\mathrm{Supp}K}-(f_S\wedge f_K)\right)\ge v(e^{\mathrm{Supp}S}-f_S)+v(e^{\mathrm{Supp}K}-f_K)$$

for any $S, K\subseteq U$.

Definition 2 Let $v \in G_V(N)$ and $U \in TF(N)$, the vector $x = \left\{\left\langle x_1^t, x_1^f \right\rangle, \left\langle x_2^t, x_2^f \right\rangle, \ldots, \left\langle x_n^t, x_n^f \right\rangle\right\}$ is said to be an imputation for *v* in *U* if

(1) $x_i^t \geq v(t_U(i)), x_i^f \geq v(1 - f_U(i)) \quad \forall i \in \text{Supp}U$;

(2) $\sum_{i \in \text{Supp}U} x_i^t = v(t_U), \sum_{i \in \text{Supp}U} x_i^f = v(e^{\text{Supp}U} - f_U)$.

Definition 3 Let $v \in G_V(N)$ and $U \in TF(N)$, the core $C_V(U, v)$ for *v* in *U* is defined by

$$C_V(U,v) = \left\{ x = \left\{\left\langle x_i^t, x_i^f \right\rangle_{i \in \text{Supp}U}\right\} \mid \sum_{i \in \text{Supp}U} x_i^t = v(t_U), \sum_{i \in \text{Supp}U} x_i^f = v(e^{\text{Supp}U} - f_U), \sum_{i \in \text{Supp}S} x_i^t \geq v(t_S), \sum_{i \in \text{Supp}S} x_i^f \geq v(e^{\text{Supp}S} - f_S), \forall S \subseteq U \right\}.$$

Definition 4 Let $v \in G_V(N)$ and $U \in TF(N)$, the vague set $S \subseteq U$ is said to be a carrier for *v* in U , if we have

$$v(t_S \wedge t_K) = v(t_K)$$

and

$$v\left(e^{\text{Supp}S \cap \text{Supp}K} - (f_S \wedge f_K)\right) = v\left(e^{\text{Supp}K} - f_K\right)$$

for any $K \subseteq U$.

From Definition 4, we know the vague set $S \subseteq U$ is a carrier for *v* in *U* if and only if t_S and $e^{\text{Supp}S} - f_S$ is a carrier for *v* in t_U and $e^{\text{Supp}U} - f_U$, respectively.

Similar to the definition of population monotonic allocation function (PMAF) [11], we give the definition of VPMAF as follows:

Definition 5 Let $v \in G_V(N)$ and $U \in TF(N)$, the vector $y = \left\{\left\langle y_i^t, y_i^f \right\rangle_{i \in \text{Supp}U}\right\}$ is said to be a VPMAF for *v* in U , if *y* satisfies

1) $\sum_{i \in \text{Supp}S} y_i^t = v(t_S)$, $\sum_{i \in \text{Supp}S} y_i^f = v(e^{\text{Supp}S} - f_S)$

$\forall S \subseteq U$;

2) $y_i^t(t_K) \leq y_i^t(t_S)$, $y_i^f(e^{\text{Supp}K} - f_K) \leq y_i^f(e^{\text{Supp}S} - f_S)$

$\forall i \in \text{Supp}K$, $\forall S, K \subseteq U$ $s.t.$ $K \subseteq S$.

3 The Shapley value for fuzzy games on vague sets

Similar to the definition of the Shapley value for traditional games and fuzzy games, we give the definition of the Shapley value for fuzzy games on vague sets as follows:

Definition 6 Let $v \in G_V(N)$ and $U \in TF(N)$. A function $\phi : G_V(N) \to \mathbb{R}_+$ is said to be the Shapley value for *v* in *U*, if it satisfies the following axioms:

Axiom 1: If *S* is a carrier for *v* in *U*, then we have

$$v(t_S) = \sum_{i \in \text{Supp}S} \phi_i(t_U, v)$$

and

$$v(e^{\text{Supp}S} - f_S) = \sum_{i \in \text{Supp}S} \phi_i(e^{\text{Supp}U} - f_U, v);$$

Axiom 2: For $i, j \in \text{Supp}U$, if we have

$$v(t_K \vee t_U(i)) = v(t_K \vee t_U(j))$$

and

$$v(e^{\text{Supp}K \cup i} - (f_K \vee f_U(i))) = v(e^{\text{Supp}K \cup j} - (f_K \vee f_U(j)))$$

for any $K \subseteq U$ with $i, j \notin \text{Supp}K$, then we get

$$\phi_i(t_U, v) = \phi_j(t_U, v)$$

and

$$\phi_i(e^{\text{Supp}U} - f_U, v) = \phi_j(e^{\text{Supp}U} - f_U, v);$$

Axiom 3: Let $v, w \in G_V(N)$, we have

$$\phi(t_U, v + w) = \phi(t_U, v) + \phi(t_U, w)$$

and

$$\phi(e^{\text{Supp}U} - t_U, v + w)$$
$$= \phi(e^{\text{Supp}U} - t_U, v) + \phi(e^{\text{Supp}U} - t_U, w).$$

Theorem 1 Let $v \in G_V(N)$, $U \in TF(N)$ and the function $\left\langle \varphi(t_U, v), \varphi(e^{\text{Supp}U} - f_U, v) \right\rangle : G_V(N) \to \mathbb{R}_+$ defined by

$$\varphi_i(t_U,v)=\sum_{t_K\subseteq t_U,i\notin \mathrm{Supp}K}\beta_U^K(v(t_K\vee t_U(i))-v(t_K)) \quad (1)$$

and

$$\varphi_i(e^{\mathrm{Supp}U}-f_U,v)=\sum_{\substack{f_K\subseteq f_U,\\ i\notin \mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)}}\beta_{e^{\mathrm{Supp}U}-f_U}^{e^{\mathrm{Supp}K}-f_K}$$

$$\times(v(e^{\mathrm{Supp}K\cup i}-(f_K\vee f_U(i)))-v(e^{\mathrm{Supp}K}-f_K)) \quad (2)$$

for any $i\in \mathrm{Supp}U$, where

$$\beta_U^K=|\mathrm{Supp}K|!(|\mathrm{Supp}U|-|\mathrm{Supp}K|-1)!/|\mathrm{Supp}U|!$$

and

$$\beta_{e^{\mathrm{Supp}U}-f_U}^{e^{\mathrm{Supp}K}-f_K}=|\mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)|!$$

$$\times\frac{(|\mathrm{Supp}(e^{\mathrm{Supp}U}-f_U)|-|\mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)|-1)!}{|\mathrm{Supp}(e^{\mathrm{Supp}U}-f_U)|!}.$$

$t_K\subseteq t_U$ if and only if $t_K(i)=t_U(i)$ or $t_K(i)=0$ for any $i\in \mathrm{Supp}U$, and $f_K\subseteq f_U$ if and only if $f_K(i)=f_U(i)$ or $f_K(i)=0$ for any $i\in \mathrm{Supp}U$.

Then $\langle\varphi(t_U,v),\varphi(e^{\mathrm{Supp}U}-f_U,v)\rangle$ is the unique Shapley value for *v* in *U*.

Remark 1 $\varphi(t_U,v)$ is said to be the player true Shapley values and $\varphi(e^{\mathrm{Supp}U}-f_U,v)$ is said to be the player upper Shapley values. When the given fuzzy games are convex, the interval number $(\varphi(t_U,v),\ \varphi(e^{\mathrm{Supp}U}-f_U,v))$ is called the player possible payoffs with respect to the Shapley function.

Proof. Axiom 1: For any $i\in \mathrm{Supp}U\setminus \mathrm{Supp}S$, from Definition 4, we have

$$v(t_K\vee t_U(i))=v(t_S\wedge(t_K\vee t_U(i)))$$

$$=v((t_S\wedge t_K)\vee(t_S\wedge t_U(i)))$$

$$=v(t_S\wedge t_K)$$

$$=v(t_K)$$

for any $K\subseteq U$ with $i\notin \mathrm{Supp}K$.

From (1), we get

$$v(t_S)=v(t_S\wedge t_U)$$

$$=v(t_U)$$

$$=\sum_{i\in \mathrm{Supp}U}\varphi_i(t_U,v)$$

$$=\sum_{i\in \mathrm{Supp}U}\sum_{t_K\subseteq t_U,i\notin \mathrm{Supp}K}\beta_U^K(v(t_K\vee t_U(i))-v(t_K))$$

$$=\sum_{i\in \mathrm{Supp}S}\sum_{t_K\subseteq t_U,i\notin \mathrm{Supp}K}\beta_U^K(v(t_K\vee t_U(i))-v(t_K))$$

$$=\sum_{i\in \mathrm{Supp}S}\varphi_i(t_U,v);$$

Similarly, we have

$$v(e^{\mathrm{Supp}S}-f_S)=\sum_{i\in \mathrm{Supp}S}\varphi_i(e^{\mathrm{Supp}U}-f_U,v).$$

Axiom 2: From (1), we get

$$\varphi_i(t_U,v)$$

$$=\sum_{t_K\subseteq t_U,i\notin \mathrm{Supp}K}\beta_U^K(v(t_K\vee t_U(i))-v(t_K))$$

$$=\sum_{\substack{t_K\subseteq t_U,\\ i,j\notin \mathrm{Supp}K}}\beta_U^K(v(t_K\vee t_U(i))-v(t_K))+$$

$$\sum_{\substack{t_K\subseteq t_U,\\ i,j\notin \mathrm{Supp}K}}\beta_U^{K\cup t_U(j)}(v(t_K\vee t_U(i)\vee t_U(j))-v(t_K\vee t_U(j)))$$

$$=\sum_{\substack{t_K\subseteq t_U,\\ i,j\notin \mathrm{Supp}K}}\beta_U^K(v(t_K\vee t_U(j))-v(t_K))+$$

$$\sum_{\substack{t_K\subseteq t_U,\\ i,j\notin \mathrm{Supp}K}}\beta_U^{K\cup t_U(i)}(v(t_K\vee t_U(i)\vee t_U(j))-v(t_K\vee t_U(i)))$$

$$=\sum_{t_K\subseteq t_U,j\notin \mathrm{Supp}K}\beta_U^K(v(t_K\vee t_U(j))-v(t_K))$$

$$=\varphi_j(t_U,v);$$

Similarly, we obtain

$$\varphi_i(e^{\mathrm{Supp}U}-f_U,v)=\varphi_j(e^{\mathrm{Supp}U}-f_U,v).$$

From (1) and (2), we can easily get Axiom 3.

Uniqueness: For any $v\in G_V(N)$ and $U\in TF(N)$, since *v* restricted in *U* can be uniquely expressed by

$$v=\begin{cases}\sum_{t_K\subseteq t_U,t_K\neq\varnothing}\alpha_{t_K}u_{t_K}\\ \sum_{f_K\subseteq f_U,e^{\mathrm{Supp}K}-f_K\neq\varnothing}\alpha_{e^{\mathrm{Supp}K}-f_K}u_{e^{\mathrm{Supp}K}-f_K}\end{cases},$$

where

$$\alpha_K = \sum_{t_S \subseteq t_K} (-1)^{|\mathrm{Supp}K|-|\mathrm{Supp}S|} v(t_S),$$

$$\alpha_{e^{\mathrm{Supp}K}-f_K} = \sum_{f_S \subseteq f_K} (-1)^{|e^{\mathrm{Supp}K}-f_K|-|e^{\mathrm{Supp}S}-f_S|} v(e^{\mathrm{Supp}S}-f_S),$$

$$u_{t_K}(t_S) = \begin{cases} 1 & t_K \subseteq t_S \subseteq t_U \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_{e^{\mathrm{Supp}K}-f_K}(e^{\mathrm{Supp}S}-f_S) = \begin{cases} 1 & f_K \subseteq f_S \subseteq f_U \\ 0 & \text{otherwise} \end{cases}.$$

From Axiom 3, we only need to show the uniqueness of φ on unanimity game u_{t_K} and $u_{e^{\mathrm{Supp}K}-f_K}$, where $t_K \neq \varnothing$.

Since t_K is a carrier for u_{t_K}, from Axiom 1 and Axiom 2, we get

$$\varphi_i(t_U, u_{t_K}) = \begin{cases} \dfrac{1}{|\mathrm{Supp}K|} & i \in \mathrm{Supp}K \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, we have

$$\varphi_i(e^{\mathrm{Supp}U}-f_U, u_{e^{\mathrm{Supp}K}-f_K})$$

$$= \begin{cases} \dfrac{1}{|\mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)|} & i \in \mathrm{Supp}(e^{\mathrm{Supp}K}-f_K) \\ 0 & \text{otherwise} \end{cases}.$$

The proof is finished.

Theorem 2 Let $v \in G_V(N)$ and $U \in TF(N)$, if v is convex in U, then

$$\left\{ \left\langle \varphi_i(t_U, v), \varphi_i(e^{\mathrm{Supp}U}-f_U, v) \right\rangle_{i \in \mathrm{Supp}U} \right\}$$

is a VPMAF for v in U.

Proof. From (1) and (2), we can easily get

$$\sum_{i \in \mathrm{Supp}S} \varphi_i(t_S, v) = v(t_S)$$

and

$$\sum_{i \in \mathrm{Supp}S} \varphi_i(e^{\mathrm{Supp}S}-f_S, v) = v(e^{\mathrm{Supp}S}-f_S).$$

In the following, we shall show the second condition in Definition 5 holds.

From Definition 1, we have

$$v(t_S \vee t_U(i)) - v(t_S) \geq v(t_K \vee t_U(i)) - v(t_K)$$

and

$$v\left(e^{\mathrm{Supp}S \cup i} - (f_S \vee f_U(i))\right) - v\left(e^{\mathrm{Supp}S} - f_S\right)$$

$$\geq v\left(e^{\mathrm{Supp}K \cup i} - (f_K \vee f_U(i))\right) - v(e^{\mathrm{Supp}K} - f_K)$$

for any $K \subseteq S \subseteq U$, where $i \notin \mathrm{Supp}S$.

When $|\mathrm{Supp}K|+1 = |\mathrm{Supp}S|$. For any $W \subseteq S$, we have

$$\beta_K^W = \sum_{H \subseteq S \setminus K} \beta_S^{W \vee H},$$

where

$$\beta_K^W = |\mathrm{Supp}W|!(|\mathrm{Supp}K| - |\mathrm{Supp}W| - 1)! / |\mathrm{Supp}K|!$$

and

$$\beta_S^{W \vee H} = \frac{|\mathrm{Supp}(W \vee H)|!(|\mathrm{Supp}S| - |\mathrm{Supp}(W \vee H)| - 1)!}{|\mathrm{Supp}S|!}.$$

From (1), we get

$$\varphi_i(t_K, v)$$

$$= \sum_{t_W \subseteq t_K, i \notin \mathrm{Supp}W} \beta_K^W (v(t_W \vee t_U(i)) - v(t_W))$$

$$\leq \sum_{t_W \subseteq t_K, i \notin \mathrm{Supp}W} \beta_K^W (v(t_W \vee t_H \vee t_U(i)) - v(t_W \vee t_H))$$

$$= \sum_{\substack{t_W \subseteq t_K, \\ i \notin \mathrm{Supp}W}} \sum_{H \subseteq S \setminus K} \beta_S^{W \vee H} (v(t_W \vee t_H \vee t_U(i)) - v(t_W \vee t_H))$$

$$= \sum_{t_W \subseteq t_S, i \notin \mathrm{Supp}W} \beta_S^W (v(t_W \vee t_U(i)) - v(t_W))$$

$$= \varphi_i(t_S, v)$$

for any $i \in \mathrm{Supp}K$.

By induction, we have $\varphi_i(t_K, v) \leq \varphi_i(t_S, v)$ for any $K \subseteq S \subseteq U$ and any $i \in \mathrm{Supp}K$.

Similarly, we have

$$\varphi_i(e^{\mathrm{Supp}K} - f_K, v) \leq \varphi_i(e^{\mathrm{Supp}S} - f_S, v)$$

for any $K \subseteq S \subseteq U$ and any $i \in \mathrm{Supp}K$.

Thus, we obtain $\left\{ \left\langle \varphi_i(t_U, v), \varphi_i(e^{\mathrm{Supp}U}-f_U, v) \right\rangle_{i \in \mathrm{Supp}U} \right\}$ is a VPMAF for v in U.

Theorem 3 Let $v \in G_V(N)$ and $U \in TF(N)$. If v is convex in U, then

$$\left\{\left\langle \varphi_i(t_U,v),\varphi_i(e^{\mathrm{Supp}U}-f_U,v)\right\rangle_{i\in\mathrm{Supp}U}\right\}\in C_V(U,v).$$

Proof. From Theorem 1, we obtain

$$\sum_{i\in\mathrm{Supp}U}\varphi_i(t_U,v)=v(t_U)$$

and

$$\sum_{i\in\mathrm{Supp}U}\varphi_i(e^{\mathrm{Supp}U}-f_U,v)=v(e^{\mathrm{Supp}U}-f_U).$$

From Theorem 2, we have

$$v(t_S)=\sum_{i\in\mathrm{Supp}S}\varphi_i(t_S,v)\le\sum_{i\in\mathrm{Supp}S}\varphi_i(t_U,v)$$

and

$$v(e^{\mathrm{SuppS}}-f_S)=\sum_{i\in\mathrm{Supp}S}\varphi_i(e^{\mathrm{SuppS}}-f_S,v)$$
$$\le\sum_{i\in\mathrm{Supp}S}\varphi_i(e^{\mathrm{Supp}U}-f_U,v).$$

Namely,

$$\left\{\left\langle \varphi_i(t_U,v),\varphi_i(e^{\mathrm{Supp}U}-f_U,v)\right\rangle_{i\in\mathrm{Supp}U}\right\}\in C_V(U,v).$$

Corollary 1 Let $v\in G_V(N)$ and $U\in TF(N)$. Suppose v is convex in U, then

$$\left\{\left\langle \varphi_i(t_U,v),\varphi_i(e^{\mathrm{Supp}U}-f_U,v)\right\rangle_{i\in\mathrm{Supp}U}\right\}$$

is an imputation for v in U.

Proposition 1 Let $v\in G_V(N)$ and $U\in TF(N)$. Suppose we have

$$v(t_S\vee t_U(i))-v(t_S)=v(t_U(i))$$

and

$$v(e^{\mathrm{SuppS}\cup i}-(f_S\vee f_U(i)))-v(e^{\mathrm{Supp}S}-f_S)=v(1-f_U(i))$$

for any $S\subseteq U$ with $i\notin\mathrm{Supp}S$.
Then we have

$$\varphi_i(t_U,v)=v(t_U(i))$$

and

$$\varphi_i(e^{\mathrm{Supp}U}-f_U,v)=v(1-f_U(i)).$$

Corollary 2 Let $v\in G_V(N)$ and $U\in TF(N)$. Suppose we have

$$v(t_S\vee t_U(i))=v(t_S)$$

and

$$v(e^{\mathrm{SuppS}\cup i}-(f_S\vee f_U(i)))=v(e^{\mathrm{SuppS}}-f_S)$$

for any $S\subseteq U$ with $i\notin\mathrm{Supp}S$.
Then we have $\varphi_i(t_U,v)=\varphi_i(e^{\mathrm{Supp}U}-f_U,v)=0$.

Proposition 2 Let $v,w\in G_V(N)$ and $U\in TF(N)$. Suppose we have

$$v(t_S\vee t_U(i))-v(t_S)\le w(t_S\vee t_U(i))-w(t_S)$$

and

$$v(e^{\mathrm{SuppS}\cup i}-(f_S\vee f_U(i)))-v(e^{\mathrm{SuppS}}-f_S)$$
$$\le w(e^{\mathrm{SuppS}\cup i}-(f_S\vee f_U(i)))-w(e^{\mathrm{SuppS}}-f_S)$$

for any $i\in\mathrm{Supp}U$ and any $S\subseteq U$ with $i\notin\mathrm{Supp}S$.
Then we have

$$\varphi_i(t_U,v)\le\varphi_i(t_U,w)$$

and

$$\varphi_i(e^{\mathrm{Supp}U}-f_U,v)\le\varphi_i(e^{\mathrm{Supp}U}-f_U,w)$$

for any $i\in\mathrm{Supp}U$.

4 The Shapley value for a special kind of fuzzy games on vague sets

In this section, we will discuss a special kind of fuzzy games, which is named as fuzzy games with multilinear extension. The fuzzy coalition value for this class of fuzzy games is written as in [9]:

$$v_O(R)=\sum_{T_0\subseteq\mathrm{Supp}R}\{\Pi_{i\in T_0}R(i)\Pi_{i\in\mathrm{Supp}R\setminus T_0}(1-R(i))\}v_0(T_0),\tag{3}$$

where R is a fuzzy coalition as usual.

Let $G_V^O(N)$ denote the set of all fuzzy games on vague sets with multilinear extension form. For any $S\in TF(N)$, we have

$$v_O(t_S)=\sum_{H_0\subseteq\mathrm{Supp}S}\{\Pi_{i\in H_0}t_S(i)\Pi_{i\in\mathrm{Supp}S\setminus H_0}(1-t_S(i))\}v_0(H_0)\tag{4}$$

and

$$v_O(e^{\mathrm{Supp}S}-f_S)=\sum_{H_0\subseteq \mathrm{Supp}(e^{\mathrm{Supp}S}-f_S)}\Big\{\Pi_{i\in H_0}(1-f_S(i))$$

$$\times\Pi_{i\in \mathrm{Supp}(e^{\mathrm{Supp}S}-f_S)\setminus H_0} f_S(i)\Big\}v_0(H_0)\ . \quad (5)$$

When we restrict the domain of $G_V(N)$ in the setting of $G_V^O(N)$, from definitions of VPMAF and imputation given in section 3, we can get the definitions of VPMAF and imputation for v_O in *U*. Here, we omit them.

Definition 6 $v_0\in G_0(N)$ is said to be convex if

$$v_0(T_0)+v_0(S_0)\le v_0(S_0\cup T_0)+v_0(S_0\cap T_0)$$

for all $S_0, T_0\in P(N)$.

Definition 7 For $v_O\in G_V^O(N)$ and $U\in TF(N)$, the core $C_V^O(U,v_O)$ for v_O in *U* is defined by

$$C_V^O(U,v_O)=\Big\{x=\Big\{\big\langle x_i^t,x_i^f\big\rangle_{i\in \mathrm{Supp}U}\Big\}\mid \sum_{i\in \mathrm{Supp}U}x_i^t=v_O(t_U),$$

$$\sum_{i\in \mathrm{Supp}U}x_i^f=v_O(e^{\mathrm{Supp}U}-f_U),\ \sum_{i\in \mathrm{Supp}S}x_i^t\ge v_O(t_S),$$

$$\sum_{i\in \mathrm{Supp}S}x_i^f\ge v_O(e^{\mathrm{Supp}S}-f_S),\forall S\subseteq U\Big\}.$$

Theorem 4 Let $v_O\in G_V^O(N)$, $U\in TF(N)$ and the function $\big\langle\varphi^O(t_U,v_O),\varphi^O(e^{\mathrm{Supp}U}-f_U,v_O)\big\rangle$: $G_V^O(N)\to \mathbb{R}_+$ defined by

$$\varphi_i^O(t_U,v_O)=\sum_{t_K\subseteq t_U,i\notin \mathrm{Supp}K}\beta_U^K\sum_{H_0\subseteq \mathrm{Supp}K}\Big\{t_U(i)\Pi_{j\in H_0}t_U(j)$$

$$\times\Pi_{j\in \mathrm{Supp}S\setminus H_0}(1-t_U(j))\Big\}(v_0(H_0\cup i)-v_0(H_0))\quad (6)$$

and

$$\varphi_i^O(e^{\mathrm{Supp}U}-f_U,v_O)$$

$$=\sum_{\substack{f_K\subseteq f_U,\\ i\notin \mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)}}\beta_{e^{\mathrm{Supp}U}-f_U}^{e^{\mathrm{Supp}K}-f_K}\sum_{H_0\subseteq \mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)}\Big\{(1-f_U(i))$$

$$\times\Pi_{j\in H_0}(1-f_U(j))\Pi_{j\in \mathrm{Supp}(e^{\mathrm{Supp}K}-f_K)\setminus H_0}f_U(j)\Big\}$$

$$\times(v_0(H_0\cup i)-v_0(H_0))\quad (7)$$

for any $i\in \mathrm{Supp}U$, where β_U^K, $\beta_{e^{\mathrm{Supp}U}-f_U}^{e^{\mathrm{Supp}K}-f_K}$, $t_K\subseteq t_U$ and $f_K\subseteq f_U$ are like in Theorem 1.
Then $\big\langle\varphi^O(t_U,v_O),\varphi^O(e^{\mathrm{Supp}U}-f_U,v_O)\big\rangle$ is the unique Shapley value for v_O in *U*.

Proof. From (4) and (5), we have

$$v_O(t_S\vee t_U(i))-v_O(t_S)$$

$$=\sum_{H_0\subseteq \mathrm{Supp}S}\Big\{t_U(i)\Pi_{i\in H_0}t_S(i)\Pi_{i\in \mathrm{Supp}S\setminus H_0}(1-t_S(i))\Big\}$$

$$\times(v_0(H_0\cup i)-v_0(H_0))$$

and

$$v_O(e^{\mathrm{Supp}S\cup i}-(f_S\vee t_U(i)))-v_O(e^{\mathrm{Supp}S}-f_S)$$

$$=\sum_{H_0\subseteq \mathrm{Supp}(e^{\mathrm{Supp}S}-f_S)}\Big\{(1-f_U(i))\Pi_{j\in H_0}(1-f_U(j))$$

$$\times\Pi_{j\in \mathrm{Supp}(e^{\mathrm{Supp}S}-f_S)\setminus H_0}f_U(j)\Big\}(v_0(H_0\cup i)-v_0(H_0))\ .$$

From Theorem 1, (6) and (7), we know the existence holds.

In the following, we shall show the uniqueness holds.

Since $v_O\in G_V^O(N)$, the restricted in *U* can be uniquely expressed by

$$v_O=\begin{cases}\sum_{t_K\subseteq t_U,t_K\ne\varnothing}\alpha_{t_K}u_{t_K}\\ \sum_{f_K\subseteq f_U,e^{\mathrm{Supp}K}-f_K\ne\varnothing}\alpha_{e^{\mathrm{Supp}K}-f_K}u_{e^{\mathrm{Supp}K}-f_K}\end{cases},$$

where

$$\alpha_K=\sum_{t_S\subseteq t_K}(-1)^{|\mathrm{Supp}K|-|\mathrm{Supp}S|}v_O(t_S)$$

and

$$\alpha_{e^{\mathrm{Supp}K}-f_K}=\sum_{f_S\subseteq f_K}(-1)^{|e^{\mathrm{Supp}K}-f_K|-|e^{\mathrm{Supp}S}-f_S|}v_O(e^{\mathrm{Supp}S}-f_S)\ .$$

u_{t_K} and $u_{e^{\mathrm{Supp}K}-f_K}$ as given in Theorem 1.
Thus, we get

$$\varphi_i^O(t_U,u_{t_K})=\begin{cases}\dfrac{1}{|\mathrm{Supp}K|} & i\in \mathrm{Supp}K\\ 0 & \text{otherwise}\end{cases}.$$

and

$$\varphi_i^O(e^{\mathrm{Supp}U} - f_U, u_{e^{\mathrm{Supp}K} - f_K})$$

$$= \begin{cases} \dfrac{1}{|\mathrm{Supp}(e^{\mathrm{Supp}K} - f_K)|} & i \in \mathrm{Supp}(e^{\mathrm{Supp}K} - f_K) \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 5 Let $v_O \in G_V^O(N)$ and $U \in TF(N)$. If the associated crisp game $v_0 \in G_0(N)$ of v_O is convex, then $\left\{\left\langle \varphi_i^O(t_U, v_O), \varphi_i^O(e^{\mathrm{Supp}U} - f_U, v_O)\right\rangle_{i\in \mathrm{Supp}U}\right\}$ is a VPMAF for v_O in U.

Proof. From (6) and (7), we can easily get

$$\sum_{i\in \mathrm{Supp}S} \varphi_i^O(t_S, v_O) = v_O(t_S)$$

and

$$\sum_{i\in \mathrm{Supp}S} \varphi_i^O(e^{\mathrm{Supp}S} - f_S, v_O) = v_O(e^{\mathrm{Supp}S} - f_S).$$

Next, we shall show the second condition holds. From the convexity of $v_0 \in G_0(N)$, (4) and (5), we get $v_O \in G_V^O(N)$ restricted in U is convex. From Definition 1, we get

$$v_O(t_S \vee t_U(i)) - v_O(t_S) \geq v_O(t_K \vee t_U(i)) - v_O(t_K)$$

and

$$v_O\left(e^{\mathrm{Supp}S\cup i} - (f_S \vee f_U(i))\right) - v_O\left(e^{\mathrm{Supp}S} - f_S\right)$$

$$\geq v_O\left(e^{\mathrm{Supp}K\cup i} - (f_K \vee f_U(i))\right) - v_O(e^{\mathrm{Supp}K} - f_K)$$

for any $K \subseteq S \subseteq U$, where $i \notin \mathrm{Supp}S$.
From (6), (7) and Theorem 2 , we obtain

$$\varphi_i^O(t_K, v_O) \leq \varphi_i^O(t_S, v_O)$$

and

$$\varphi_i^O(e^{\mathrm{Supp}K} - f_K, v_O) \leq \varphi_i^O(e^{\mathrm{Supp}S} - f_S, v_O)$$

for any $K \subseteq S \subseteq U$ and any $i \in \mathrm{Supp}K$.
Namely, $\left\{\left\langle \varphi_i^O(t_U, v_O), \varphi_i^O(e^{\mathrm{Supp}U} - f_U, v_O)\right\rangle_{i\in \mathrm{Supp}U}\right\}$ is a VPMAF for v_O in U.

Theorem 6 Let $v_O \in G_V^O(N)$ and $U \in TF(N)$. If the associated crisp game $v_0 \in G_0(N)$ of v_O is convex, then

$$\left\{\left\langle \varphi_i^O(t_U, v_O), \varphi_i^O(e^{\mathrm{Supp}U} - f_U, v_O)\right\rangle_{i\in \mathrm{Supp}U}\right\} \in C_V^O(U, v_O).$$

Proof. The proof of Theorem 6 is similar to that of Theorem 3.

Corollary 3 Let $v_O \in G_V^O(N)$ and $U \in TF(N)$. If the associated crisp game $v_0 \in G_0(N)$ of v_O is convex, then $\left\{\left\langle \varphi_i^O(t_U, v_O), \varphi_i^O(e^{\mathrm{Supp}U} - f_U, v_O)\right\rangle_{i\in \mathrm{Supp}U}\right\}$ is an imputation for v_O in U.

Theorem 7 Let $v_O \in G_V^O(N)$ and $U \in TF(N)$. If the associated crisp game $v_0 \in G_0(N)$ of v_O is convex, then the core $C_V^O(U, v_O) \neq \varnothing$ and it can be expressed by

$$C_V^O(U, v_O) = \left\{ x = \left\{\left\langle x_i^t, x_i^f \right\rangle_{i\in \mathrm{Supp}U}\right\} \mid \sum_{i\in \mathrm{Supp}U} x_i^t = \right.$$

$$\sum_{H_0 \subseteq \mathrm{Supp}U} \left\{\Pi_{i\in H_0} t_U(i) \Pi_{i\in \mathrm{Supp}U\setminus H_0}(1 - t_U(i))\right\} y^{H_0},$$

$$\sum_{i\in \mathrm{Supp}U} x_i^f = \sum_{R_0 \subseteq \mathrm{Supp}(e^{\mathrm{Supp}U} - f_U)} \left\{\Pi_{i\in R_0}(1 - f_U(i))\right.$$

$$\left.\Pi_{i\in \mathrm{Supp}(e^{\mathrm{Supp}U} - f_U)\setminus R_0} f_U(i)\right\} y^{R_0}, \forall H_0 \subseteq \mathrm{Supp}U,$$

$$\forall y^{T_0} \in C(H_0, v_0), \forall R_0 \subseteq \mathrm{Supp}(e^{\mathrm{Supp}U} - f_U),$$

$$\left. \forall y^{R_0} \in C(R_0, v_0) \right\},$$

where $C(H_0, v_0)$ denotes the core in H_0 for v_0 and $C(R_0, v_0)$ denotes the core in R_0 for v_0.

Proof. The proof of Theorem 7 is similar to that of Proposition 4.1 given in [15].

Since the fuzzy games in $G_V^O(N)$ establish the specific relationship between the fuzzy coalition values and that of their associated crisp coalitions. The properties for this class of fuzzy games can be obtained by researching their associated crisp games.

5 Numerical example

There are three companies, named 1, 2 and 3, that decide to cooperate with their resources. They can combine freely. For example $S_0 = \{1, 2\}$ denotes the cooperation between company 1 and 2. Since there are many uncertainty factors during the cooperation, each player is not willing to offer all its resources to this specific cooperation. In another word, they only supply part of their resources. In order to reduce risk and get more payoffs, when the players take part in this cooperation, they only know the determining participation levels and the participation levels that they do not participate. For example, the company 1 has 10000 units of resources, the determining participation level is 3000 units, and 2000 units are not devoted to cooperation. Namely, the true membership grade of the player 1 is 0.3 = 3000/ 10000, and the false membership grade of the player 1 is 0.2 = 2000/10000. In such a way, a vague set is interpreted. Consider a vague coalition U defined by

$$U = \sum_{i\in\{1,2,3\}} [t_U(i), 1 - f_U(i)]/i$$

$$= [0.3, 0.6]/1 + [0.2, 0.3]/2 + [0.6, 0.8]/3.$$

If the crisp coalition values are given by table 1

Table 1. The fuzzy payoffs of the crisp coalitions (millions of dollars)

S_0	$v_0(S_0)$	S_0	$v_0(S_0)$
{1}	1	{1,3}	3
{2}	2	{2,3}	5
{3}	1	{1,2,3}	10
{1,2}	6		

From table 1, we know that when the company 1 and 2 cooperate with all their resources, then their payoff is 6 millions of dollars.

When the relationship between the values of the fuzzy coalitions and that of their associated crisp coalitions as given in (4) and (5). Namely, this fuzzy game belongs to $G_V^O(N)$. From (6), we get the player true Shapley values are

$$\varphi_1^O(t_U, v_O) = 0.42\ ,\ \varphi_2^O(t_U, v_O) = 0.64\ ,$$

$$\varphi_3^O(t_U, v_O) = 0.84\ .$$

From (7), we get the player upper Shapley values are

$$\varphi_1^O(e^{\mathrm{Supp}U} - f_U, v_O) = \varphi_2^O(e^{\mathrm{Supp}U} - f_U, v_O) = 1.11$$

$$\varphi_3^O(e^{\mathrm{Supp}U} - f_U, v_O) = 1.28\ .$$

Since the associated crisp game is convex, we know that the player Shapley values is a VMPAF and an element in its core.

Furthermore, the player possible payoffs with respect to the Shapley function are [0.42, 1.11], [0.64, 1.11] and [0.84, 1.28].

6 Conclusion

In some cooperative games, the players only know the determination participation levels and the levels that they do not participate. The fuzzy games on vague sets can well solve this situation. For this purpose, we research the fuzzy games on vague sets and discuss the Shapley value for fuzzy games on vague sets. When the given fuzzy games on vague sets are convex, some properties are investigated. Furthermore, we study a special kind of fuzzy games on vague sets. The Shapley value and the core for this kind of fuzzy games are studied.

However, we only study the Shapley value for fuzzy games on vague sets and it will be interesting to study other payoff indices.

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