Riesz Basis and Stability Analysis of the Feedback Controlled Networks of 1-D Wave equations

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Abstract: In this paper, using graph theory and functional analysis approach, we study the Riesz basis property and the stabilization of general networks of 1-D wave equations. Firstly, we derive the vector form of the model under consideration and then discuss the controllers design. We prove that the controlled network is a Riesz system under certain conditions and hence the spectrum determined growth assumption holds. Further we give some necessary and sufficient conditions for the asymptotic stability and the non-stability of the controlled network by spectral conditions. Finally, we apply the obtained results to two networks of special shapes, and analyze their stability by the "irrational dependence".

Key–Words: wave equation, partial differential network, geometrically continuous type network, Riesz basis, stability.

1 Introduction

In the latest decades, the partial differential equations on metric graph have always been an attractive topic in the field of engineering control and mathematical control, involving controllability, observability and stabilization ([1, 2, 3, 4, 5, 6, 7, 8, 9] and [10] as well as the references therein). Also, S. Nicaise and J. Valein in [11] investigated the stabilization of the network of 1-D the wave equations with time delay, in which the wave network is assumed that the set of Dirichlet boundary points, $\mathcal{D}$, is not empty. J. Valein and E. Zuazua in [12] considered the stabilization of a planar network of strings with a damping term on one boundary vertex v_1 and the other boundary vertices are fixed. They use observability estimate to analyze the stability of the damped network. We observed that the networks discussed are simple ones and the method used in the papers mentioned above seems invalid for the complex networks that have parallel edges and self-loop or for the wave equation with variable coefficients. Besides the approach used in [11] and [12], the Riesz basis approach, as one of the powerful tool in the control theory of distributed parameter system, was also used successfully in study of vibration control of the flexible system ([13, 14, 15, 16, 17, 18, 19, 20] and the references therein). According to [14], if a system is Riesz one, then the system satisfies the spectrum determined growth condition, and hence one can get stability of the system via the spectral distribution. However, for most practice models, after small perturbation, they do not have a Riesz basis property. So G. Q. Xu, Z. J. Han and S. P. Yung in [17] and [20] extended the Riesz basis property to the Riesz basis with parentheses and proved that the system satisfies the spectrum determined growth assumption under certain assumption on spectral distribution. The advantage of Riesz basis approach is that it can be used in analysis of any system although its verification is very difficult in practice model.

In this paper, we mainly study the stabilization problem of general complex network of 1-d wave equations, which might contain circuits and parallel edges. The wave network under consideration is assumed of continuous type. Our strategy is the velocity feedback controllers on nodes and boundary of the network. With suitable choice of the feedback controllers our aim is to stabilize the network. However, from earlier research for example [4], [5], [11], [12], [13], [18], we found that stability analysis of the closed loop system is a difficult task. Therefore, the second aim of our research is to find a manner that can be coded by computer language and then use computer to analyze stability of the closed loop system including design of controllers.

The main result of the present paper is that the 1-D wave network system with appropriate vertex controllers is a Riesz system under certain conditions, here we mainly give these conditions (see formula (26)), so it satisfies the spectrum determined growth assumption (Theorem 8). Thus the asymptotical stability is equivalent to the non-existence of purely imaginary eigenvalues. Consequently, the criterions for the asymptotical stability of the networks, Theorem 16, Corollary 17, Theorem 21, and the criterion for the non-stability of the networks, Theorem 22 are derived. Note that our aim is to find a microprogrammable analysis process, so the results are given by analytic forms.

The rest is organized as follows. In section 2, we use the graph theory to formulate the connected network and write the 1-D wave network of continuous type into an abstract differential equation in $\mathbb{C}^n$. Based on the energy function, we design the feedback controllers for the system so that its energy decays, inhere we discuss two modes of controllers according to different type networks. In section 3, we discuss Riesz basis property of the system by asymptotical analysis, in which the conditions ensuring the closed loop system is Riesz one are explicitly given. In section 4, we analyze the stability of the controlled network, and give some sufficient and necessary conditions for the asymptotical stability. In section 5, as application we study two examples. Finally, we give some complementary results about the graph theory in the appendix.

2 The general wave network and the controller design

Usually, a network can be regarded as a graph, we shall identify the graph with the network in this paper. To formulate the wave network, we at first list some fundamental concepts. Some related results are presented in appendix. For more details, we refer to [21].

2.1 Basic notions in graph theory

Let $G = (V, E)$ be a connected planar graph with the vertex set $V = \{p_1, p_2, \dots, p_m\}$ and the edge set $E = \{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$. Suppose that the length of edge ϵ_i is ℓ_i and ϵ_i is incident with vertices p_{j_i} and p_{k_i}. A continuous function $\pi_i : [0, \ell_i] \mapsto \epsilon_i$ satisfying $\pi_i(s) \in \epsilon_i$, $s \in [0, \ell_i]$ with $\pi_i(0) = p_{j_i}$ and $\pi_i(\ell_i) = p_{k_i}$, is called the parameterized mapping of ϵ_i. Obviously, if every edge of G has finite length ℓ_j, then it can be parameterized, and hence G becomes a metric graph. If every edge of G is signed a direction that coincides with the parameter increasing, G become a digraph. Thus, the vertices p_{j_i} and p_{k_i} are called the tail and the head of ϵ_i respectively. An edge with the head and the tail at the same vertex is called a loop (or self-loop), and an edge with distinct ends is called a link. Two or more links with the same pair of ends are called parallel edges. A simple graph is one without loops and parallel edges. A sequence consisting of the different vertices and edges of G alternatively,

$$p_1, \epsilon_1, p_2, \epsilon_2, \cdots, p_i, \epsilon_i, p_{i+1}, \cdots, p_k,$$

is denoted by $P(p_1, p_k)$, if the ends of edge ϵ_i are the vertices p_i and p_{i+1} ($1 \le i \le k-1$), then $P(p_1, p_k)$ is called a path from p_1 to p_k. If p_1 and p_k are the same vertex, then $P(p_1, p_k)$ is called a circuit.

In this paper, we always suppose that $G = (V, E)$ is a connected planar metric digraph without loops. Set $\mathcal{I}_E = \{1, 2, \dots, n\}$ and $\mathcal{I}_V = \{1, 2, \dots, m\}$. The following three sets

$$\mathcal{I}_E(p_j) = \{k \in \mathcal{I}_E | \epsilon_k \text{ is incident with } p_j, \epsilon_k \in E\},$$

$$\mathcal{I}_E^+(p_j) = \left\{ k \in \mathcal{I}_E \;\middle|\; \begin{array}{l} p_j \text{ is the starting point (tail)} \\ \text{of the edge } \epsilon_k, \epsilon_k \in E \end{array} \right\}$$

and

$$\mathcal{I}_E^-(p_j) = \left\{ k \in \mathcal{I}_E \;\middle|\; \begin{array}{l} p_j \text{ is the final point (head)} \\ \text{of the edge } \epsilon_k, \epsilon_k \in E \end{array} \right\}$$

are called the incident index set, the outgoing incident index set and the incoming incident index set of the vertex p_j, respectively. The notations $\deg(p_j) = \#\mathcal{I}_E(p_j)$, $\deg^+(p_j) = \#\mathcal{I}_E^+(p_j)$ and $\deg^-(p_j) = \#\mathcal{I}_E^-(p_j)$ represent the degree, out-degree and in-degree of a vertex p_j of G, respectively, where $\#\Gamma$ represents the number of elements in Γ. The set $Int(G) = \{p_j \in V| \ \deg(p_j) > 1\}$ is called the interior vertex (or node) set of G and the set $\partial G = \{p_j \in V| \ \deg(p_j) = 1\}$ is called the boundary of G.

Definition 1. *The matrices $\Psi^+ = (\psi_{i,j}^+)_{m\times n}$ and $\Psi^- = (\psi_{i,j}^-)_{m\times n}$, defined by*

$$\psi_{i,j}^+ = \begin{cases} 1, & \text{if} \quad \pi_j(0) = p_i, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\psi_{i,j}^- = \begin{cases} 1, & \text{if} \quad \pi_j(\ell_j) = p_i, \\ 0, & \text{otherwise}, \end{cases}$$

are called the outgoing incidence matrix and the incoming incidence matrix, respectively. The incidence matrix is defined by $\Psi = \Psi^+ - \Psi^-$.

Note that Ψ^+ and Ψ^- have exactly one nonzero entry in each column and Ψ has exactly two nonzero entries in each column, -1 and 1, this is because each edge has exactly one tail and head. Thus,

$$\Psi^+ = \left(e_{j_1^+}, \dots, e_{j_k^+}, \dots, e_{j_n^+}\right)_{m\times n} \tag{1}$$

and

$$\Psi^- = \left(e_{j_1^-}, \dots, e_{j_k^-}, \dots, e_{j_n^-}\right)_{m\times n}, \tag{2}$$

where $j_k^+, j_k^- \in \{1, 2, \dots, m\}$, $k = 1, 2, \dots, n$, and $e_{j_k^+}$ and $e_{j_k^-}$ are the j_k^+-th and the j_k^--th column vector of the identity matrix of order m respectively. The sets

$$\mathcal{J}_\Psi^+ = \{j_1^+, \dots, j_n^+\} \text{ and } \mathcal{J}_\Psi^- = \{j_1^-, \dots, j_n^-\}, \tag{3}$$

are called the outgoing incident index set and incoming incident index set respectively.

2.2 Description of 1-D wave network

For convenience, we denote $G = \left(\bigcup_{j=1}^n \epsilon_j\right) \bigcup V$. Let $\mathrm{y}(z,t)$ be a function defined on $G \times [0, +\infty)$ ([22]), where $z \in G$ stands for position, $t \in [0, \infty)$ is time variable. We denote by $y_j(x,t)$ the parametrization realization of $\mathrm{y}(z,t)$ on j-th edge, that is, $y_j(x,t) = \mathrm{y}(z,t)|_{z\in\epsilon_j} = \mathrm{y}(\pi_j(x), t)$. If for each $j \in \mathcal{I}_E$, $y_j(x,t)$ satisfies the wave equation

$$T_j y_{j,xx}(x,t) = \rho_j y_{j,tt}(x,t), x \in (0, \ell_j), \tag{4}$$

where $T_j > 0$ and $\rho_j > 0$, then we say that $\mathrm{y}(z,t)$ satisfies the wave equation on G, briefly, G is an 1-D wave network.

Remark 2. *If $\ell_j \neq 1$, we let the change of variable $x = \widetilde{x}\ell_j$ and denote $y_j(\widetilde{x}\ell_j, t)$ by $\widetilde{y}_j(\widetilde{x}, t)$, thus,*

$$\widetilde{T}_j \widetilde{y}_{j,\widetilde{x}\widetilde{x}}(\widetilde{x}, t) = \widetilde{\rho}_j \widetilde{y}_{j,tt}(\widetilde{x}, t),\ 0 < \widetilde{x} < 1, t > 0,$$

where $\widetilde{\rho}_j = \ell_j\rho_j$ and $\widetilde{T}_j = T_j/\ell_j$. So, when we consider the wave networks, without loss of generality we can assume that every edge of G has length 1. In this case, we call G is a normalized wave network.

Let $\mathcal{D} = \{p \in V | \mathrm{y}(p,t) = 0\}$ and $\partial G_D = \partial G \cap \mathcal{D}$. Sets $\mathcal{D}$ and ∂G_D are called the Dirichlet set and the Dirichlet boundary set respectively. Thus, we say $\mathrm{y}(z,t)$ satisfies Dirichlet conditions on $\mathcal{D}$, sometimes also say the network is fixed on $\mathcal{D}$. The set $\mathcal{J}_D = \{j \in \mathcal{I}_V | p_j \in \mathcal{D}\}$ is called the Dirichlet index set. A subset ∂G_N of ∂G is called the Neumann boundary if the parameterized function $y_j(x,t)$ satisfy

$$\partial G_N = \left\{p \in \partial G \,\middle|\, \frac{\partial y_j(p,t)}{\partial \nu} = f(p,t),\ j \in \mathcal{I}_E(p)\right\}$$

where $f(p,t)$ is called an exterior force acting on p, $\frac{\partial y_j(p,t)}{\partial \nu}$ is outward normal derivative defined by

$$\frac{\partial y_j(p,t)}{\partial \nu} = \begin{cases} \frac{\partial y_j(1,t)}{\partial x}, & \text{if } j \in \mathcal{I}_E^-(p), \\ -\frac{\partial y_j(0,t)}{\partial x}, & \text{if } j \in \mathcal{I}_E^+(p). \end{cases}$$

If for some $p \in \partial G$, $\frac{\partial y_j(p,t)}{\partial \nu} = 0$, then p is called a free boundary point. The set

$$\mathcal{N} = \left\{p \in \partial G \,\middle|\, \frac{\partial y_j(p,t)}{\partial \nu} = 0, j \in \mathcal{I}_E(p)\right\}$$

is called Neumann free boundary set, or free boundary for short.

Definition 3. *Let $\mathrm{y}(z,t)$ be a function on $G \times [0, +\infty)$. For each $j \in \mathcal{I}_E$, $y_j(x,t)$, satisfies the wave equation*

$$\rho_j y_{j,tt}(x,t) = T_j y_{j,xx}(x,t), x \in (0,1). \tag{5}$$

If $\mathrm{y}(z,t)$ and $y_j(x,t)$ satisfy the following conditions:

1 *Boundary conditions: $\forall p \in \mathcal{D}$,*

$$\begin{cases} y_j(1,t) = 0, & \textit{if } j \in \mathcal{I}_E^-(p) \\ y_i(0,t) = 0, & \textit{if } i \in \mathcal{I}_E^+(p), \end{cases} \tag{6}$$

and $\forall p \in \partial G_N$,

$$\begin{cases} y_{j,x}(1,t) = f(p,t), & \textit{if } j \in \mathcal{I}_E^-(p), \\ y_{j,x}(0,t) = f(p,t), & \textit{if } j \in \mathcal{I}_E^+(p); \end{cases} \tag{7}$$

2 *Geometrical continuity conditions at nodes: $\forall p \in Int(G) \setminus \mathcal{D}$,*

$$y_j(1,t) = \mathrm{y}(p,t) = y_i(0,t), \tag{8}$$

where $j \in \mathcal{I}_E^-(p), i \in \mathcal{I}_E^+(p)$;

3 *Dynamic condition at vertices: $\forall p \in Int(G) \setminus \mathcal{D}$,*

$$\sum_{j\in\mathcal{I}_E^-(p)} T_j y_{j,x}(1,t) - \sum_{k\in\mathcal{I}_E^+(p)} T_k y_{k,x}(0,t) = f(p,t), \tag{9}$$

where $f(p,t)$ represents an exterior force acting on p, which takes the different function with varying of p.

Then G is said to be a continuous-type wave network, $\mathrm{y}(z,t)$ is said to satisfy the mixed (Dirichlet-Neumann, or, D-N) boundary conditions.

From the above definition, if $p \in \mathcal{D}$, the network is fixed at p, which is called the Dirichlet point; if $p \in \mathcal{N}$, the network is free at p; if $p \in \partial G_N \setminus \mathcal{N}$, there is a damping term located on the vertex p. In (9), if for some $p \in Int(G) \setminus \mathcal{D}$, $f(p,t) = 0$, then the network G satisfies the Kirchhoff flow continuous condition at p. Therefore, the motion of the network G with mixed boundary conditions is completely governed by (5)–(9).

For convenience, we reformulate the equations on the network G in the vector form. For this, we define a subspace of $L^2_{loc}([0,+\infty);\mathbb{C}^m)$ by

$$L^2_D([0,+\infty);\mathbb{C}^m) = \left\{ (w_1(t),\ldots,w_m(t))^{\mathrm{T}} \,\middle|\, \begin{array}{l} w_j \in L^2_{loc}([0,+\infty);\mathbb{C}), \\ \text{and } w_j(t)=0, \text{ if } p_j \in \mathcal{D} \end{array} \right\}$$

and a subspace of $\mathbb{C}^m$ by

$$\mathbb{D} = \{ (\xi_1,\xi_2,\ldots,\xi_m)^{\mathrm{T}} \in \mathbb{C}^m \,|\, \xi_j = 0, \text{ if } p_j \in \mathcal{D} \}.$$

Let $r = \dim(\mathbb{D})$, then $\mathbb{D} \cong \mathbb{C}^r$. Let $P_{\mathbb{D}}$ be the orthogonal projection from $\mathbb{C}^m$ to $\mathbb{C}^r$, whose matrix representation is also denoted by $P_{\mathbb{D}}$, that is,

$$P_{\mathbb{D}} = (e_{j_1}, e_{j_2}, \ldots, e_{j_r})^{\mathrm{T}}, \; p_{j_k} \notin \mathcal{D}, \tag{10}$$

where the vector e_{j_k} is the j_k-th column of the identity matrix I_m, $j_k \in \{1,2,\ldots,m\}$, $k = 1,2,\ldots,r$. We call the set $\mathcal{J}_P = \{j_1,j_2,\ldots,j_r\}$ the projection index set, or, the non-Dirichlet index set.

Let

$$Y(x,t) = \begin{pmatrix} y_1(x,t) \\ \vdots \\ y_n(x,t) \end{pmatrix} \text{ and } \boldsymbol{y}(\boldsymbol{v},t) = \begin{pmatrix} \mathbb{y}(p_1,t) \\ \vdots \\ \mathbb{y}(p_m,t) \end{pmatrix},$$

where $\boldsymbol{v} = (p_1,p_2,\ldots,p_m)^{\mathrm{T}}$, $Y(x,t)$ and $\boldsymbol{y}(\boldsymbol{v},t)$ are said to be the vectorization of $\mathbb{y}(z,t)$. Thus, equations (5)–(9) can be rewritten into the following vector form

$$\begin{cases} TY_{xx}(x,t) = MY_{tt}(x,t), \; x \in (0,\,1), \; t > 0, \\ Y(0,t) = (\Psi^+)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},t), Y(1,t) = (\Psi^-)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},t), \\ P_{\mathbb{D}}\left[\Psi^- TY_x(1,t) - \Psi^+ TY_x(0,t)\right] = F(t), \\ Y(x,0) = Y_0(x), \; Y_t(x,0) = Y_1(x), \end{cases} \tag{11}$$

where $\boldsymbol{y}(\boldsymbol{v},t) \in L^2_D([0,+\infty);\mathbb{C}^m)$ for vertex vector $\boldsymbol{v}$; $F(t) = (f(p_{j_1},t),\ldots,f(p_{j_r},t))^{\mathrm{T}}$ is the vertex force function; Y_0 and Y_1 are two initial state vector functions satisfy conditions $Y_0(0) = (\Psi^+)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},0)$ and $Y_0(1) = (\Psi^-)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},0)$.

2.3 Feedback controller design

In this subsection we consider the design problem of controllers. Here we mainly choose the feedback control law to stabilize the network according the following two cases.

2.3.1 Case I: $\mathcal{D} \neq \emptyset$.

We define the energy function of the network by

$$\begin{aligned} \mathscr{E}(t) &= \frac{1}{2}\int_0^1 \langle MY_t(x,\,t),\,Y_t(x,\,t)\rangle_{\mathbb{C}^n} dx \\ &\quad + \frac{1}{2}\int_0^1 \langle TY_x(x,\,t),\,Y_x(x,\,t)\rangle_{\mathbb{C}^n} dx, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ represents the Euclidean inner product in $\mathbb{C}^n$. Without causing confusion, the subscript $\mathbb{C}^n$ will be omitted below. Using the differential equation in (11) and integration by parts arrives at

$$\begin{aligned} \frac{d\mathscr{E}(t)}{dt} &= \frac{1}{2}\langle TY_x(1,\,t),\,Y_t(1,\,t)\rangle \\ &\quad - \frac{1}{2}\langle TY_x(0,\,t),\,Y_t(0,\,t)\rangle \\ &\quad + \frac{1}{2}(\langle Y_t(1,\,t),\,TY_x(1,\,t)\rangle \\ &\quad - \frac{1}{2}\langle Y_t(0,\,t),\,TY_x(0,\,t)\rangle. \end{aligned}$$

So, from the continuity condition

$$Y(0,t) = (\Psi^+)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},t), Y(1,t) = (\Psi^-)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},t) \tag{12}$$

and the dynamical condition in (11), it is reduced that

$$\frac{d\mathscr{E}(t)}{dt} = \Re\left(\langle F(t),\,P_{\mathbb{D}}\boldsymbol{y}_t(\boldsymbol{v},t)\,\rangle_{\mathbb{C}^r}\right)$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^r}$ represents the Euclidean inner product in $\mathbb{C}^r$. Thus, we take the feedback control law as follows:

$$F(t) = -\beta P_{\mathbb{D}}\boldsymbol{y}_t(\boldsymbol{v},t) \tag{13}$$

where $\beta = \mathrm{diag}\{\beta_1,\ldots,\beta_r\}$ with $\beta_k \geq 0$, $k = 1,2,\ldots,r$. If for some k, $\beta_k = 0$, it means that there is no control at the corresponding vertex. Under this feedback control law, we have

$$\frac{d\mathscr{E}(t)}{dt} = -\langle \beta P_{\mathbb{D}}\boldsymbol{y}_t(\boldsymbol{v},t),\,P_{\mathbb{D}}\boldsymbol{y}_t(\boldsymbol{v},t)\rangle \leq 0. \tag{14}$$

2.3.2 Case II: $\mathcal{D} = \emptyset$.

If the network has no Dirichlet points, then $\mathcal{D} = \emptyset$, $r = m$ and $P_{\mathbb{D}} = I_m$. We define the energy function of the network by

$$\begin{aligned} \mathscr{E}(t) &= \frac{1}{2}\int_0^1 \langle MY_t(x,\,t),\,Y_t(x,\,t)\rangle_{\mathbb{C}^n} dx \\ &\quad + \frac{1}{2}\int_0^1 \langle TY_x(x,\,t),\,Y_x(x,\,t)\rangle_{\mathbb{C}^n} dx \\ &\quad + \frac{1}{2}\langle \gamma\boldsymbol{y}(\boldsymbol{v},t),\,\boldsymbol{y}(\boldsymbol{v},t)\rangle, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product in $\mathbb{C}^n$ or $\mathbb{C}^m$, $\gamma = \mathrm{diag}\{\gamma_1,\dots,\gamma_m\}$ with $\gamma_k \geq 0$ and $\sum_{k=1}^m \gamma_k > 0$. Such a choice of γ ensures that $\mathscr{E}(t)$ vanishing implies that $Y(x,t) = 0$.

Similar to Case I, we obtain

$$\frac{d\mathscr{E}(t)}{dt} = \Re\left(\langle F(t) + \gamma \boldsymbol{y}(\boldsymbol{v},t),\, \boldsymbol{y}_t(\boldsymbol{v},t)\,\rangle\right).$$

Take the feedback control law

$$F(t) = -\beta \boldsymbol{y}_t(\boldsymbol{v},t) - \gamma \boldsymbol{y}(\boldsymbol{v},t)$$

where $\beta = \mathrm{diag}\{\beta_1,\dots,\beta_m\}$ with $\beta_k \geq 0$, and if $\beta_k = 0$, $\gamma_k = 0$, then (14) also holds.

All in all, the feedback control law can be formulated uniformly by

$$F(t) = -\beta P_{\mathbb{D}} \boldsymbol{y}_t(\boldsymbol{v},t) - \gamma P_{\mathbb{D}} \boldsymbol{y}(\boldsymbol{v},t), \qquad (15)$$

where $\beta \geq 0$ and $\gamma \geq 0$. One takes $\gamma = 0$ if $\mathcal{D} \neq \emptyset$. Thus, the equations (11) together with the feedback control law (15) forms a closed loop system.

Finally we define two sets

$$\mathcal{I}_\beta = \{k \in \mathcal{I}_V \mid \beta_k > 0\} \text{ and } \mathcal{C} = \{p_k \in V \mid k \in \mathcal{I}_\beta\}. (16)$$

$\mathcal{I}_\beta$ is called the controlled vertex index set, $\mathcal{C}$ is called the controlled vertex set. If $\mathcal{C} = V \setminus \mathcal{D}$ (or equivalently, $\mathcal{I}_\beta = \mathcal{I}_V \setminus \mathcal{J}_D = \mathcal{J}_P$), i.e., the controllers are located on all vertices of the network G but the fixed ones, then the network G is called a completely controlled network; otherwise, it is call a non-completely (or partially) controlled one. If $\mathcal{C} = \partial G \setminus \mathcal{D}$, namely, the controllers are located on all boundary ∂G but the fixed ones, the network G is called a completely boundary controlled one. If $\mathcal{C}$ is a proper subset of $\partial G \setminus \mathcal{D}$, then G is called a non-completely (or partially) boundary controlled one. If $\mathcal{C} \subset Int(G)$, the network G is called a internally controlled one. If $\mathcal{C} = Int(G) \setminus \mathcal{D}$, the network G is called a completely internally controlled one. If $\mathcal{C}$ is a proper subset of $Int(G) \setminus \mathcal{D}$, the network G is called a non-completely (or partially) internally controlled one.

2.4 The evolution equation of the controlled network

To study the Riesz basis property and stability of the controlled network, we need to introduce an appropriate state space. Let $\mathbb{f}$ be a function on G with its parametrization on ϵ_j, $f_j(x)$. Denote $(f_1(x),\dots,f_n(x))^{\mathrm{T}}$ by $f(x)$, and $(\mathbb{f}(p_1), \mathbb{f}(p_2), \dots, \mathbb{f}(p_m))^{\mathrm{T}}$ by $\boldsymbol{f}(\boldsymbol{v})$, where $\boldsymbol{v} = (p_1, p_2, \dots, p_m)^{\mathrm{T}}$, $f(x)$ and $\boldsymbol{f}(\boldsymbol{v})$ are called the vectorization of $\mathbb{f}$.

Define the function spaces $L^2(E)$ and $H^k(E)$ by

$$L^2(E) = \{f(x) \mid f_j \in L^2(0,1), j \in \mathcal{I}_E\}$$

and

$$H^k(E) = \{f \in L^2(E) \mid f_j \in H^k(0,1), j \in \mathcal{I}_E\},$$

where $H^k(0,1)$ ($k = 1,2$) are the usual Sobolev spaces and $L^2(0,1)$ is the usual Hilbert space. Let

$$V_E^k(0,1) = \left\{ f \in H^k(E) \;\middle|\; \begin{array}{l} f(0) = (\Psi^+)^{\mathrm{T}} \boldsymbol{f}(\boldsymbol{v}), \\ f(1) = (\Psi^-)^{\mathrm{T}} \boldsymbol{f}(\boldsymbol{v}), \\ \boldsymbol{f}(\boldsymbol{v}) \in \mathbb{D} \end{array} \right\}$$

with $k = 1,2$, where $\boldsymbol{f}(\boldsymbol{v}) \in \mathbb{D}$ means that

$$\boldsymbol{f}(\boldsymbol{v}) = (\mathbb{f}(p_1), \mathbb{f}(p_2), \dots, \mathbb{f}(p_m))^{\mathrm{T}} \qquad (17)$$

and $\mathbb{f}(p_j) = 0$ as $p_j \in \mathcal{D}$. Take the state space

$$\mathcal{H} = V_E^1(0,1) \times L^2(E). \qquad (18)$$

In space $\mathcal{H}$, the inner product is defined by $\forall (f,g)^{\mathrm{T}}, (\hat{f},\hat{g})^{\mathrm{T}} \in \mathcal{H}$,

$$\begin{aligned} &\langle \begin{pmatrix} f \\ g \end{pmatrix} \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} \rangle_{\mathcal{H}} \\ &= \int_0^1 \Big(\langle Tf'(x), \hat{f}'(x) \rangle + \langle Mg(x), \hat{g}(x) \rangle \Big)\, dx \\ &\quad + \langle \gamma P_{\mathbb{D}} \boldsymbol{f}(\boldsymbol{v}),\, P_{\mathbb{D}} \hat{\boldsymbol{f}}(\boldsymbol{v}) \rangle_{\mathbb{C}^m} \end{aligned} \qquad (19)$$

and the norm is given by

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathcal{H}} = \langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle_{\mathcal{H}}^{\frac{1}{2}}. \qquad (20)$$

Obviously, $\mathcal{H}$ is a Hilbert space. Defined an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g(x) \\ M^{-1} T f''(x) \end{pmatrix} \qquad (21)$$

with the domain

$$\mathrm{dom}(\mathcal{A}) = \qquad (22)$$
$$\left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H} \;\middle|\; \begin{array}{l} f \in V_E^2(0,1), g \in V_E^1(0,1), \\ P_{\mathbb{D}} \left[(\Psi^-) T f'(1) - (\Psi^+) T f'(0) \right] \\ \quad = -\beta P_{\mathbb{D}} \boldsymbol{g}(\boldsymbol{v}) - \gamma P_{\mathbb{D}} \boldsymbol{f}(\boldsymbol{v}) \end{array} \right\}.$$

Note that $g \in V_E^1(0,1)$ means that $\boldsymbol{g}(\boldsymbol{v}) \in \mathbb{D}$,

$$g(1) = (\Psi^-)^{\mathrm{T}} \boldsymbol{g}(\boldsymbol{v}) \text{ and } g(0) = (\Psi^+)^{\mathrm{T}} \boldsymbol{g}(\boldsymbol{v}). \quad (23)$$

Thus, the closed loop system (11) with (15) can be rewritten into an evolution equation in $\mathcal{H}$

$$\begin{cases} \dfrac{dZ}{dt} = \mathcal{A} Z, \\ Z(t_0) = Z_0, \end{cases} \qquad (24)$$

where $Z = (Y, Y_t)^{\mathrm{T}}$ and $Z_0 = (Y_0, Y_1)^{\mathrm{T}}$.

3 Spectrum of $\mathcal{A}$ and Riesz basis

In the present paper we suppose that the network has no self-loop. In what follows, we introduce the main result about spectrum of A and Riesz basis, whose proofs will be given later.

Theorem 4. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined by* (18) *and* (21) *respectively. Then*

(1) *$\mathcal{A}$ is dissipative, i.e.*

$$\Re\langle \mathcal{A}(f,g)^{\mathrm{T}}, (f,g)^{\mathrm{T}}\rangle = -\langle\, \beta P_{\mathbb{D}}\mathbf{g}(\mathbf{v}),\, P_{\mathbb{D}}\mathbf{g}(\mathbf{v})\,\rangle_{\mathbb{C}^r} \le 0, \quad (25)$$

and $\mathcal{A}^{-1}$ is compact in $\mathcal{H}$,

(2) *the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of finite multiplicity, i.e. $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$,*

(3) *$\mathcal{A}$ generates a C_0 semigroup of contraction $S(t)$ on $\mathcal{H}$.*

Theorem 4 shows that system (24) (i.e., (11) with (15)) is well posed. The next theorem shows that the spectrum of $\mathcal{A}$ lie in a strip parallel to the imaginary axis under certain conditions.

Theorem 5. *Let the state space $\mathcal{H}$ and the operator $\mathcal{A}$ be defined by* (18) *and* (21)*, respectively. Then when β satisfies*

$$\beta_i \neq \sum_{k\in\mathcal{I}_E(p_{j_i})} \sqrt{\rho_k T_k},\ j_i \in \mathcal{J}_P\ (\text{ i.e., } p_{j_i} \notin \mathcal{D}), \quad (26)$$

$\sigma(\mathcal{A})$ is a finite union of separable sets (taking the multiplicity into account), i.e.,

$$\sigma(\mathcal{A}) = \cup_{k=1}^{N}\Lambda_k$$

where Λ_k are separable set consisting of eigenvalues of $\mathcal{A}$, N is the uniform bound of the multiplicities of eigenvalues in $\mathcal{A}$. Moreover, there exists a positive constant δ such that

$$\sigma(\mathcal{A}) \subset \{\, \lambda \in \mathbb{C}\,|\, -\delta \le \Re\lambda \le 0\,\}. \quad (27)$$

Remark 6. *A set $\Lambda \subset \mathbb{C}$ is said to be separable if* $\forall\lambda, \mu \in \Lambda,\ \inf_{\lambda\neq\mu} |\lambda - \mu| > 0.$

The following theorem indicates the completeness of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ and its conditions.

Theorem 7. *Let the state space $\mathcal{H}$ and the operator $\mathcal{A}$ be defined by* (18) *and* (21) *respectively. If β satisfies* (26) *, then the system of eigenvectors and generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

The following Theorem gives the Riesz basis property of eigenvector and generalized eigenvectors of $\mathcal{A}$, including the spectrum determined growth assumption of the system (24).

Theorem 8. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined by* (18) *and* (21) *respectively. If β satisfies* (26)*, then there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$. So, the C_0 semigroup $S(t)$ generated by $\mathcal{A}$ satisfies the spectrum determined growth assumption.*

Remark 9. *The condition* (26) *is a necessary condition for the Riesz basis property. If* (26) *does not hold, the controlled network system may not be a Riesz one, see the special examples in [23] and [24]. In addition, according to Remark 2, the results of Theorem 8 are independent of the length of edge.*

3.1 The proof of Theorem 4

Proof According to (18), (19), (21), (22), (23) and integration by parts, it can be obtained that

$$\begin{aligned}
& \Re\langle \mathcal{A}(f,g)^{\mathrm{T}}, (f,g)^{\mathrm{T}}\rangle \\
=\ & \frac{1}{2}[\langle \mathcal{A}(f,g)^{\mathrm{T}}, (f,g)^{\mathrm{T}}\rangle + \langle (f,g)^{\mathrm{T}}, \mathcal{A}(f,g)^{\mathrm{T}}\rangle] \\
=\ & \Re\left\{\langle\, Tf'(1),\, g(1)\,\rangle - \langle\, Tf'(0),\, g(0)\,\rangle \right. \\
& \left. +\ \langle \gamma P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}),\, P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v})\,\rangle\right\} \\
=\ & \Re\left\{\langle\, P_{\mathbb{D}}\left[(\Psi^-)Tf'(1) - (\Psi^+)Tf'(0)\right]\right. \\
& \left. +\gamma P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}),\, P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v})\,\rangle\right\},
\end{aligned}$$

which implies that (25) holds.

Next,we verify $0 \in \rho(\mathcal{A})$, i.e., $\mathcal{A}^{-1}$ exists and is bounded. Consider the following resolvent equation

$$\mathcal{A}(f,g)^{\mathrm{T}} = (\zeta,\nu)^{\mathrm{T}},\ (f,g)^{\mathrm{T}} \in \mathrm{dom}(\mathcal{A})$$

with $(\zeta,\nu)^{\mathrm{T}} \in \mathcal{H}$, namely,

$$\begin{cases} g(x) = \zeta(x), \\ Tf''(x) = M\nu(x). \end{cases} \quad (28)$$

Integrating the above second equation from x to 1 leads to

$$Tf'(1) = Tf'(0) + \varrho(0) \quad (29)$$

and

$$\begin{aligned} Tf(x) &= Tf(1) - (1-x)Tf'(1) + \int_x^1 \varrho(s)ds \\ &= T(\Psi^-)^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}) - (1-x)Tf'(1) + \kappa(x), \end{aligned} \quad (30)$$

where $\kappa(x) = \int_x^1 \varrho(s)ds$, $\varrho(x) = \int_x^1 M\nu(s)ds$. In the above equality, taking $x = 0$ yields

$$T\Psi^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}) + Tf'(1) = \kappa(0). \quad (31)$$

By the domain of $\mathcal{A}$, (22) and (29), we have

$$-\gamma P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v})+P_{\mathbb{D}}\Psi Tf'(1)=\beta P_{\mathbb{D}}\boldsymbol{\zeta}(\boldsymbol{v})+P_{\mathbb{D}}\Psi^{+}\varrho(0). \tag{32}$$

Since $\boldsymbol{f}(\boldsymbol{v})\in\mathbb{D}$, there exists $d\in\mathbb{C}^r$ such that $\boldsymbol{f}(\boldsymbol{v})=P_{\mathbb{D}}^{\mathrm{T}}d$. Thus, it follows from (31) and (32) that d and $Tf'(1)$ satisfy algebraic equations

$$\begin{cases}\gamma d-P_{\mathbb{D}}\Psi Tf'(1)=-\beta P_{\mathbb{D}}\boldsymbol{\zeta}(\boldsymbol{v})-P_{\mathbb{D}}\Psi^{+}\varrho(0),\\ T\Psi^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}}d+Tf'(1)=\kappa(0).\end{cases} \tag{33}$$

The determinant of the coefficient matrix of above equations is

$$\det\begin{pmatrix}\gamma & -P_{\mathbb{D}}\Psi\\ T(P_{\mathbb{D}}\Psi)^{\mathrm{T}} & I\end{pmatrix}=\det\left(\gamma+P_{\mathbb{D}}\Psi T(P_{\mathbb{D}}\Psi)^{\mathrm{T}}\right).$$

The Lemma 28 in the Appendix asserts that $\gamma+P_{\mathbb{D}}\Psi T(P_{\mathbb{D}}\Psi)^{\mathrm{T}}$ is a symmetrical and positive definite matrix, so, d and $Tf'(1)$ can be determined by (33) uniquely. From $\boldsymbol{f}(\boldsymbol{v})=P_{\mathbb{D}}^{\mathrm{T}}d$, (30) and (28), f,g can be determined by ζ,ν uniquely, which implies that $0\in\rho(\mathcal{A})$ i.e., $\mathcal{A}^{-1}$ exists. The Sobolev's Embedding Theorem shows that $\mathcal{A}^{-1}$ is compact. So the assertion (2) holds. Finally, according to Lumer-Phillips Theorem ([25]), $\mathcal{A}$ generates a C_0 semigroup of contraction. $\square$

3.2 The proof of Theorem 5

Let $\lambda\in\sigma(\mathcal{A})$, $\lambda\neq 0$ and $(f,g)^{\mathrm{T}}\in\mathrm{dom}(\mathcal{A})$ be the corresponding eigenvector. Then the eigenvalue problem of $\mathcal{A}$ is given by

$$Tf''(x)=\lambda^2 Mf(x),\ g(x)=\lambda f(x), \tag{34}$$

with the boundary conditions

$$\begin{cases} f(0)=(\Psi^{+})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}),\\ f(1)=(\Psi^{-})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}),\boldsymbol{f}(\boldsymbol{v})\in\mathbb{D},\\ P_{\mathbb{D}}\left[(\Psi^{-})Tf'(1)-(\Psi^{+})Tf'(0)\right]\\ \quad=-(\lambda\beta+\gamma)P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}).\end{cases} \tag{35}$$

Set $\eta(x)=(f(x),\lambda^{-1}Tf'(x))^{\mathrm{T}}$, then $\eta(x)$ satisfies equation

$$\frac{d\eta}{dx}=\lambda\begin{bmatrix}0 & T^{-1}\\ M & 0\end{bmatrix}\eta. \tag{36}$$

The fundamental solution of (36) is given by

$$W(x,\lambda)=Q\begin{pmatrix}e^{-\lambda xA} & 0\\ 0 & e^{\lambda xA}\end{pmatrix}Q^{-1}, \tag{37}$$

where

$$Q=\begin{pmatrix}-(\sqrt{MT})^{-1} & (\sqrt{MT})^{-1}\\ I & I\end{pmatrix}$$

and

$$Q^{-1}=\frac{1}{2}\begin{pmatrix}-\sqrt{MT} & I\\ \sqrt{MT} & I\end{pmatrix},$$

$\sqrt{MT}=\mathrm{diag}\{q_1,\dots,q_n\}$, $A=M^{1/2}T^{-1/2}=\mathrm{diag}\{a_1,\dots,a_n\}$, $a_k=\rho_k^{1/2}T_k^{-1/2}$, $q_k=\sqrt{\rho_k T_k}$, $k=1,\dots,n$. Thus, the general solution of (36) is

$$\eta(x)=W(x,\lambda)\eta(0). \tag{38}$$

Since $\boldsymbol{f}(\boldsymbol{v})\in\mathbb{D}$, there exists a $d\in\mathbb{C}^r$ such that $d=P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v})$ and $\boldsymbol{f}(\boldsymbol{v})=P_{\mathbb{D}}^{\mathrm{T}}d$. Thus,

$$\eta(0)=\begin{pmatrix}(\Psi^{+})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v})\\ \lambda^{-1}Tf'(0)\end{pmatrix}=\begin{pmatrix}(P_{\mathbb{D}}\Psi^{+})^{\mathrm{T}}d\\ \lambda^{-1}Tf'(0)\end{pmatrix}$$

and

$$\eta(1)=\begin{pmatrix}(\Psi^{-})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v})\\ \lambda^{-1}Tf'(1)\end{pmatrix}=\begin{pmatrix}(P_{\mathbb{D}}\Psi^{-})^{\mathrm{T}}d\\ \lambda^{-1}Tf'(1)\end{pmatrix}.$$

From (38) and the last condition in (35), it can be derived that

$$D(\lambda)\begin{pmatrix}d\\ \lambda^{-1}Tf'(0)\end{pmatrix}=0 \tag{39}$$

where

$$\begin{aligned}&D(\lambda)\\ =&\begin{pmatrix}I & 0\\ 0 & P_{\mathbb{D}}\Psi^{-}\end{pmatrix}W(1,\lambda)\begin{pmatrix}(P_{\mathbb{D}}\Psi^{+})^{\mathrm{T}} & 0\\ 0 & I\end{pmatrix}\\ &-\begin{pmatrix}(P_{\mathbb{D}}\Psi^{-})^{\mathrm{T}} & 0\\ -(\boldsymbol{\beta}+\lambda^{-1}\boldsymbol{\gamma}) & P_{\mathbb{D}}\Psi^{+}\end{pmatrix},\end{aligned}$$

namely,

$$D(\lambda)=\begin{pmatrix}(\sqrt{MT})^{-1}\cosh(\lambda A)\sqrt{MT}(P_{\mathbb{D}}\Psi^{+})^{\mathrm{T}}-(P_{\mathbb{D}}\Psi^{-})^{\mathrm{T}} & (\sqrt{MT})^{-1}\sinh(\lambda A)\\ P_{\mathbb{D}}(\Psi^{-})\sinh(\lambda A)\sqrt{MT}(P_{\mathbb{D}}\Psi^{+})^{\mathrm{T}}+(\boldsymbol{\beta}+\lambda^{-1}\boldsymbol{\gamma}) & P_{\mathbb{D}}(\Psi^{-})\cosh(\lambda A)-P_{\mathbb{D}}(\Psi^{+})\end{pmatrix}. \tag{40}$$

Obviously, the above algebraic equation (39) has nonzero solution if and only if the determinant of the coefficients matrix vanishes at $\lambda\neq 0$, i.e.,

$$\Delta(\lambda)=\det(D(\lambda))=0. \tag{41}$$

Therefore, according to (38), (39) and the identity $\boldsymbol{f}(\boldsymbol{v})=P_{\mathbb{D}}^{\mathrm{T}}d$, we have the following result.

Lemma 10. *Let $\mathcal{A}$ be defined by* (21). *Then we have*

$$\sigma(\mathcal{A})=\{\,\lambda\in\mathbb{C}\setminus\{0\}\,|\Delta(\lambda)=0\,\},$$

where $\Delta(\lambda)$ is given by (41).

We call the matrix $D(\lambda)$ characteristic matrix of $\mathcal{A}$ (or, the controlled network), $\Delta(\lambda)$ characteristic determinant of $\mathcal{A}$ (or, the controlled network).

The proof of Theorem 5: From (40), it is yielded that

$$\begin{aligned}
&D(\lambda)\begin{pmatrix} I & 0\\ \sqrt{MT}(\Psi^+)^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}} & I\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & e^{\lambda A}\end{pmatrix}\\
=&\begin{pmatrix} \sqrt{MT}^{-1}e^{\lambda A}\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{\sqrt{MT}^{-1}e^{2\lambda A}}{2}\\ P_{\mathbb{D}}(\Psi^-)e^{\lambda A}\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{P_{\mathbb{D}}(\Psi^-)e^{2\lambda A}}{2}\end{pmatrix}\\
&+\begin{pmatrix} 0 & 0\\ \lambda^{-1}\gamma & -P_{\mathbb{D}}(\Psi^+)e^{\lambda A}\end{pmatrix} \qquad (42)\\
&+\begin{pmatrix} (\sqrt{MT})^{-1} & 0\\ -P_{\mathbb{D}}(\Psi^-) & I\end{pmatrix}\begin{pmatrix} -\sqrt{MT}(P_{\mathbb{D}}\Psi^-)^{\mathrm{T}} & -\dfrac{1}{2}I\\ D_{\beta-} & 0\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
D_{\beta-} \;=\; &\beta - P_{\mathbb{D}}(\Psi^+)\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}}\\
&-P_{\mathbb{D}}(\Psi^-)\sqrt{MT}(P_{\mathbb{D}}\Psi^-)^{\mathrm{T}}.
\end{aligned}$$

So,

$$\begin{aligned}
&\lim_{\Re\lambda\to-\infty}\Delta(\lambda)\det(e^{\lambda A})\\
=&\det(\sqrt{MT})^{-1}\det\begin{pmatrix} -\sqrt{MT}(P_{\mathbb{D}}\Psi^-)^{\mathrm{T}} & -\dfrac{I}{2}\\ D_{\beta-} & 0\end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta_- &= \lim_{\Re\lambda\to-\infty}\frac{\Delta(\lambda)}{\det(e^{-\lambda A})}\\
&= \frac{(-1)^{n+\frac{(n+r)(n+r+1)}{2}}}{2^n\det\left(\sqrt{MT}\right)}\det\left(D_{\beta-}\right).
\end{aligned}$$

Similarly, it follows from (40) that

$$\begin{aligned}
&D(\lambda)\begin{pmatrix} I & 0\\ -\sqrt{MT}(\Psi^+)^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}} & I\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & e^{-\lambda A}\end{pmatrix}\\
=&\begin{pmatrix} \sqrt{MT}^{-1}e^{-\lambda A}\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{\sqrt{MT}^{-1}e^{-2\lambda A}}{-2}\\ -P_{\mathbb{D}}(\Psi^-)e^{-\lambda A}\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{P_{\mathbb{D}}(\Psi^-)e^{-2\lambda A}}{2}\end{pmatrix}\\
&+\begin{pmatrix} 0 & 0\\ \lambda^{-1}\gamma & P_{\mathbb{D}}(\Psi^+)e^{-\lambda A}\end{pmatrix}\\
&+\begin{pmatrix} \sqrt{MT}^{-1} & 0\\ P_{\mathbb{D}}(\Psi^-) & I\end{pmatrix}\begin{pmatrix} -\sqrt{MT}(P_{\mathbb{D}}\Psi^-)^{\mathrm{T}} & \dfrac{I}{2}\\ D_{\beta+} & 0\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
D_{\beta+} \;=\; &\beta + P_{\mathbb{D}}(\Psi^+)\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}}\\
&+P_{\mathbb{D}}(\Psi^-)\sqrt{MT}(P_{\mathbb{D}}\Psi^-)^{\mathrm{T}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta_+ &= \lim_{\Re\lambda\to+\infty}\frac{\Delta(\lambda)}{\det(e^{\lambda A})}\\
&= \frac{\det\begin{pmatrix} -\sqrt{MT}(P_{\mathbb{D}}\Psi^-)^{\mathrm{T}} & \dfrac{I}{2}\\ D_{\beta+} & 0\end{pmatrix}}{\det(\sqrt{MT})}\\
&= \frac{(-1)^{\frac{(n+r)(n+r+1)}{2}}}{2^n\det(\sqrt{MT})}\det\left(D_{\beta+}\right).
\end{aligned}$$

It follows from Lemma 29 in the Appendix that

$$\begin{aligned}
\Delta_- &= \frac{(-1)^{n+\frac{(n+r)(n+r+1)}{2}}}{2^n\prod_{j=1}^n\sqrt{\rho_jT_j}}\\
&\quad\cdot\prod_{i=1}^r\left(\beta_i-\sum_{k\in\mathcal{I}_E(p_{j_i})}\sqrt{\rho_kT_k}\right)
\end{aligned}$$

and

$$\begin{aligned}
\Delta_+ &= \frac{(-1)^{\frac{(n+r)(n+r+1)}{2}}}{2^n\prod_{j=1}^n\sqrt{\rho_jT_j}}\\
&\quad\cdot\prod_{i=1}^r\left(\beta_i+\sum_{k\in\mathcal{I}_E(p_{j_i})}\sqrt{\rho_kT_k}\right).
\end{aligned}$$

Therefore, when $\Delta_-\neq 0$, i.e., β satisfies (26), there exist positive constants $\widetilde{c}_1$, $\widetilde{c}_2$ and δ such that for $|\Re\lambda|>\delta$,

$$\widetilde{c}_1e^{\Re(\lambda)tr(A)}\le|\Delta(\lambda)|\le\widetilde{c}_2e^{\Re(\lambda)tr(A)}, \qquad (43)$$

which implies that the zeros set of $\Delta(\lambda)$ is contained in the region $\{\lambda\in\mathbb{C}\big|\ |\Re(\lambda)|\le\delta\}$. By Theorem 4, the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis, i.e., (27) holds. In addition, (43) shows that $\Delta(i\lambda)$ is a sine type function on $\mathbb{C}$ (e.g., see [26, Definition II, 1.27, p61]). Levin theorem (e.g., see[26, Proposition II, 1.28, p61], also [15, Proposition 3.5]) asserts that the set of zeros of $\Delta(\lambda)$ is a finite union of separable sets. So is $\sigma(\mathcal{A})$, and for each $\lambda_k\in\sigma(\mathcal{A})$, its algebraic multiplicity is uniformly upper bounded. □

3.3 The proof of Theorem 7

To prove Theorem 7, we first prove two lemmas.

Lemma 11. *Let $\mathcal{H}$ be defined by* (18). *Define operator $\mathcal{A}_0$ in $\mathcal{H}$ by*

$$\mathcal{A}_0\begin{pmatrix} f\\ g\end{pmatrix}=\begin{pmatrix} g(x)\\ M^{-1}Tf''(x)\end{pmatrix}, \qquad (44)$$

with

$$\mathrm{dom}(\mathcal{A}_0) = \tag{45}$$
$$\left\{\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H} \,\middle|\, \begin{array}{r} f \in V_E^2(0,1), g \in V_E^1(0,1), \\ P_{\mathbb{D}}\left[(\Psi^-)Tf'(1) - (\Psi^+)Tf'(0)\right] \\ -\gamma P_{\mathbb{D}} \boldsymbol{f}(\boldsymbol{v}) = 0 \end{array}\right\}.$$

Then $\mathcal{A}_0$ is a skew adjoint operator in $\mathcal{H}$, and $\forall(\zeta,\varpi)^{\mathrm{T}} \in \mathcal{H}$, $\lambda \in \mathbb{R}$, the solution of the resolvent equation

$$(\lambda I - \mathcal{A}_0)\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \zeta \\ \varpi \end{pmatrix} \tag{46}$$

satisfies

$$\|\boldsymbol{g}(\boldsymbol{v})\| = \|P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v})\| \le c\|(\zeta,\varpi)^{\mathrm{T}}\|, \tag{47}$$

where c is a positive constant.

Proof $\forall\,(\zeta,\varpi)^{\mathrm{T}}$, $(f,\,g)^{\mathrm{T}} \in D(\mathcal{A}_0)$, using identities

$$f(0) = (\Psi^+)^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}), f(1) = (\Psi^-)^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}),$$
$$\zeta(0) = (\Psi^+)^{\mathrm{T}}\boldsymbol{\zeta}(\boldsymbol{v}) \text{ and } \zeta(1) = (\Psi^-)^{\mathrm{T}}\boldsymbol{\zeta}(\boldsymbol{v}),$$

and (44),(45), we derive

$$\begin{aligned} &\langle \mathcal{A}_0(f,g)^{\mathrm{T}}, (\zeta,\varpi)^{\mathrm{T}}\rangle + \langle (f,g)^{\mathrm{T}}, \mathcal{A}_0(\zeta,\varpi)^{\mathrm{T}}\rangle = \\ &\quad \langle P_{\mathbb{D}}[(\Psi^-)Tf'(1) - (\Psi^+)Tf'(0)], P_{\mathbb{D}}\boldsymbol{\varpi}(\boldsymbol{v})\,\rangle \\ &\quad +\langle \gamma P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}), P_{\mathbb{D}}\boldsymbol{\varpi}(\boldsymbol{v})\,\rangle \\ &\quad +\langle\, P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v}),\, P_{\mathbb{D}}[(\Psi^-)T\zeta'(1) - (\Psi^+)T\zeta'(0)]\rangle \\ &\quad +\langle\, P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v}), \gamma P_{\mathbb{D}}\boldsymbol{\zeta}(\boldsymbol{v})\,\rangle = 0. \end{aligned}$$

So, $\mathcal{A}_0$ is a skew adjoint operator in $\mathcal{H}$, which implies that

$$\|\lambda R(\lambda,\mathcal{A}_0)\| \le 1,\ \forall \lambda \in \mathbb{R}. \tag{48}$$

From (46), we have $g(x) = \lambda f(x) - \zeta(x)$, which implies that $\boldsymbol{g}(\boldsymbol{v}) = \lambda\boldsymbol{f}(\boldsymbol{v}) - \boldsymbol{\zeta}(\boldsymbol{v})$ and $g'(x) = \lambda f'(x) - \zeta'(x)$. From the equalities $g(0) = (\Psi^+)^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{v})$ and $g(1) = (\Psi^-)^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{v})$, it follows that

$$\begin{aligned} g(1) &= g(0) + \int_0^1 g'(x)dx \\ &= g(0) + \int_0^1 (\lambda f'(x) - \zeta'(x))dx, \end{aligned}$$

and so,

$$\Psi^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{v}) = -\int_0^1 (\lambda f'(x) - \zeta'(x))dx.$$

When $\mathcal{D} = \emptyset$, then $P_{\mathbb{D}} = I$, $\gamma \ge 0$ and $\sum_{k=1}^m \gamma_k^2 \ne 0$. So,

$$\begin{aligned} &\left(\gamma + \Psi\Psi^{\mathrm{T}}\right)\boldsymbol{g}(\boldsymbol{v}) = \\ &\quad -\Psi\int_0^1 (\lambda f'(x) - \zeta'(x))dx + \gamma(\lambda\,\boldsymbol{f}(\boldsymbol{v}) - \boldsymbol{\zeta}(\boldsymbol{v})). \end{aligned}$$

From Lemma 28 in the Appendix, it follows that $\gamma + \Psi\Psi^{\mathrm{T}}$ is nonsingular. Therefore,

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{v}) &= \left(\gamma + \Psi\Psi^{\mathrm{T}}\right)^{-1}\left[\Psi\int_0^1 \zeta'(x)dx + \gamma\boldsymbol{\zeta}(\boldsymbol{v})\right] \\ &\quad -\lambda\left(\gamma + \Psi\Psi^{\mathrm{T}}\right)^{-1}\left[\Psi\int_0^1 f'(x)dx + \gamma\boldsymbol{f}(\boldsymbol{v})\right]. \end{aligned}$$

Thus, from (48) and the following two inequalities:

$$\left\|\int_0^1 T^{1/2}f'(x)dx\right\|^2 \le \int_0^1 \langle\, Tf'(x), f'(x)\,\rangle dx \tag{49}$$

and

$$\begin{aligned} &\int_0^1 \langle Tf'(x),\ f'(x)\rangle dx + \|\gamma^{1/2}\boldsymbol{f}(\boldsymbol{v})\|^2 \\ &\quad \le\ \|(f,\ g)^{\mathrm{T}}\|^2 = \|R(\lambda,\mathcal{A}_0)(\zeta,\varpi)^{\mathrm{T}}\|^2. \end{aligned} \tag{50}$$

From above estimates we get

$$\begin{aligned} &\|\boldsymbol{g}(\boldsymbol{v})\|^2 \\ \le\ & 4c_0|\lambda|\left\|\Psi T^{-1/2}\right\|^2\left\|\int_0^1 T^{1/2}f'(x)dx\right\|^2 \\ & +4c_0|\lambda|\left\|\gamma^{1/2}\right\|^2\left\|\gamma^{1/2}\boldsymbol{f}(\boldsymbol{v})\right\|^2 \\ & +4c_0\left\|\Psi T^{-1/2}\right\|^2\left\|\int_0^1 T^{1/2}\zeta'(x)dx\right\|^2 \\ & +4c_0\left\|\gamma^{1/2}\right\|^2\|\gamma^{1/2}\boldsymbol{\zeta}(\boldsymbol{v})\|^2 \\ \le\ & \frac{|c\lambda|^2}{2}\left[\int_0^1 \langle Tf'(x), f'(x)\rangle dx + \left\|\gamma^{1/2}\boldsymbol{f}(\boldsymbol{v})\right\|^2\right] \\ & +\frac{c^2}{2}\left[\int_0^1 \langle\, T\zeta'(x), \zeta'(x)\,\rangle dx + \|\gamma^{1/2}\boldsymbol{\zeta}(\boldsymbol{v})\|^2\right] \\ \le\ & \frac{c^2}{2}\left(|\lambda|^2\|R(\lambda,\mathcal{A}_0)(\zeta,\varpi)^{\mathrm{T}}\|^2 + \|(\zeta,\varpi)^{\mathrm{T}}\|^2\right) \\ \le\ & c^2\|(\zeta,\varpi)^{\mathrm{T}}\|^2 \end{aligned}$$

where $c^2 = 8c_0\max\{\left\|\Psi T^{-1/2}\right\|^2, \left\|\gamma^{1/2}\right\|^2\}$, $c_0 = \left\|\left(\gamma + \Psi\Psi^{\mathrm{T}}\right)^{-1}\right\|^2$.

When $\mathcal{D} \ne \emptyset$, $P_{\mathbb{D}} \ne I$ and $\gamma = 0$, there exists a $d \in \mathbb{D}$ such that $\boldsymbol{g}(\boldsymbol{v}) = P_{\mathbb{D}}^{\mathrm{T}}d$ and

$$\Psi^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}}d = \Psi^{\mathrm{T}}\boldsymbol{g}(\boldsymbol{v}) = -\int_0^1 (\lambda f'(x) - \zeta'(x))dx.$$

Thus,

$$P_{\mathbb{D}}\Psi\Psi^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}}d = -P_{\mathbb{D}}\Psi\int_0^1 (\lambda f'(x) - \zeta'(x))dx.$$

Lemma 28 in the Appendix asserts that $P_{\mathbb{D}}\Psi\Psi^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}}$ is nonsingular, and hence

$$\begin{aligned} d &= -(P_{\mathbb{D}}\Psi\Psi^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}})^{-1}P_{\mathbb{D}}\Psi T^{-1/2} \\ &\quad \cdot\left(\lambda\int_0^1 T^{1/2}f'(x)dx - \int_0^1 T^{1/2}\zeta'(x)dx\right). \end{aligned}$$

From (48),(49) and (50) with $\gamma = 0$, it is derived that

$$\begin{aligned} &\|P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v})\|^2 = \|d\|^2 \\ &\le \frac{c^2}{2}|\lambda|^2\left\|\int_0^1 T^{1/2}f'(x)dx\right\|^2 \\ &\quad +\frac{c^2}{2}\left\|\int_0^1 T^{1/2}\zeta'(x)dx\right\|^2 \\ &\le \frac{c^2}{2}\left[|\lambda|^2\|R(\lambda,\mathcal{A}_0)(\zeta,\varpi)^{\mathrm{T}}\|^2 + \|(\zeta,\varpi)^{\mathrm{T}}\|^2\right] \\ &\le c^2\|(\zeta,\varpi)^{\mathrm{T}}\|^2, \end{aligned}$$

where $c^2 = 4\|(P_{\mathbb{D}}\Psi\Psi^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}})^{-1}P_{\mathbb{D}}\Psi T^{-1/2}\|^2$. □

Lemma 12. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined by* (18) *and* (21)*, respectively. Then, $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$, is given by*

$$\mathcal{A}^*\begin{pmatrix}\zeta\\ \varpi\end{pmatrix} = \begin{pmatrix}-\varpi(x)\\ -M^{-1}T\zeta''(x)\end{pmatrix} \tag{51}$$

with the domain

$$\begin{aligned} &\mathrm{dom}(\mathcal{A}^*) = \\ &\left\{\begin{pmatrix}\zeta\\ \varpi\end{pmatrix}\in\mathcal{H}\left|\begin{array}{c}\zeta\in V_E^2(0,1),\varpi\in V_E^1(0,1),\\ P_{\mathbb{D}}\left[(\Psi^-)T\zeta'(1)-(\Psi^+)T\zeta'(0)\right]\\ =\beta P_{\mathbb{D}}\boldsymbol{\varpi}(\boldsymbol{v})-\gamma P_{\mathbb{D}}\boldsymbol{\zeta}(\boldsymbol{v})\end{array}\right.\right\}, \end{aligned} \tag{52}$$

and $R(\lambda,\mathcal{A}^*)(f,g)^{\mathrm{T}} = (\zeta,\varpi)^{\mathrm{T}}$ ($\lambda \neq 0$) with

$$\begin{aligned} \zeta(x) &= \cosh(\lambda x A)\eta_{0,1} + (\sqrt{MT})^{-1}\sinh(\lambda x A)\eta_{0,2} \\ &\quad -\lambda^{-1}\int_0^x A\sinh(\lambda(x-s)A)(g(s)-\lambda f(s))ds \end{aligned} \tag{53}$$

and

$$\varpi(x) = f(x) - \lambda\zeta(x), \tag{54}$$

where $\eta_{0,1}$ and $\eta_{0,2}$ are two vectors.

Proof A direct verification shows that $\mathcal{A}^*$ is of the form (51) with the domain (52). In what follows, we mainly consider the resolvent of $\mathcal{A}^*$.

Set $\lambda \in \rho(\mathcal{A}^*)$ and $(f,g)^T \in \mathcal{H}$, consider the resolvent problem

$$(\lambda I - \mathcal{A}^*)(\zeta,\varpi)^{\mathrm{T}} = (f,g)^{\mathrm{T}},$$

which is equivalent to the differential equations $\varpi(x) = f(x) - \lambda\zeta(x)$ and

$$\begin{cases} T\zeta''(x) - \lambda^2 M\zeta(x) = M(g(x)-\lambda f(x)), \\ \zeta(0) = (\Psi^+)^{\mathrm{T}}\boldsymbol{\zeta}(\boldsymbol{v}), \zeta(1) = (\Psi^-)^{\mathrm{T}}\boldsymbol{\zeta}(\boldsymbol{v}), \\ P_{\mathbb{D}}\left[(\Psi^-)T\zeta'(1) - (\Psi^+)T\zeta'(0)\right] \\ \qquad = \beta P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}) - (\lambda\beta+\gamma)P_{\mathbb{D}}\boldsymbol{\zeta}(\boldsymbol{v}) \end{cases} \tag{55}$$

with $\boldsymbol{\zeta}(\boldsymbol{v}) \in \mathbb{D}$.

Set $\eta(x) = (\zeta(x), \lambda^{-1}T\zeta'(x))^{\mathrm{T}}$, similar to (36), $\eta(x)$ satisfies nonhomogenous equations

$$\frac{d\eta}{dx} = \lambda\begin{pmatrix}0 & T^{-1}\\ M & 0\end{pmatrix}\eta - \begin{pmatrix}0\\ \lambda^{-1}M(g-\lambda f)\end{pmatrix}.$$

Thus,

$$\begin{aligned} \eta(x) &= W(x,\lambda)\eta_0 \\ &\quad -\int_0^x W(x-s,\lambda)\begin{pmatrix}0\\ \lambda^{-1}M(g(s)-\lambda f(s))\end{pmatrix}, \end{aligned}$$

where

$$\eta_0 = \begin{pmatrix}\zeta(0)\\ \lambda^{-1}T\zeta'(0)\end{pmatrix} = \begin{pmatrix}(\Psi^+)^{\mathrm{T}}\boldsymbol{\zeta}(\boldsymbol{v})\\ \lambda^{-1}T\zeta'(0)\end{pmatrix}. \tag{56}$$

From (56) and the boundary conditions in (55), it follows that

$$\begin{aligned} &\begin{pmatrix}I & 0\\ 0 & P_{\mathbb{D}}\Psi^-\end{pmatrix}\eta(1) = \\ &\begin{pmatrix}I & 0\\ 0 & P_{\mathbb{D}}\Psi^-\end{pmatrix}W(1,\lambda)\begin{pmatrix}(\Psi^+)^{\mathrm{T}} & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\boldsymbol{\zeta}(\boldsymbol{v})\\ \lambda^{-1}T\zeta'(0)\end{pmatrix} \\ &-\int_0^1\begin{pmatrix}\lambda^{-1}A\sinh(\lambda(1-s)A)(g(s)-\lambda f(s))\\ P_{\mathbb{D}}(\Psi^-)\lambda^{-1}M\cosh(\lambda(1-s)A)(g(s)-\lambda f(s))\end{pmatrix}ds \end{aligned}$$

and

$$\begin{aligned} &\begin{pmatrix}I & 0\\ 0 & P_{\mathbb{D}}\Psi^-\end{pmatrix}\eta(1) \\ &= \begin{pmatrix}(\Psi^-)^{\mathrm{T}} & 0\\ -(\beta+\lambda^{-1}\gamma)P_{\mathbb{D}} & P_{\mathbb{D}}\Psi^+\end{pmatrix}\begin{pmatrix}\boldsymbol{\zeta}(\boldsymbol{v})\\ \lambda^{-1}T\zeta'(0)\end{pmatrix} \\ &\quad + \begin{pmatrix}0\\ \lambda^{-1}\beta P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v})\end{pmatrix}. \end{aligned}$$

Since $\boldsymbol{\zeta}(\boldsymbol{v}) \in \mathbb{D}$, there exists a $d \in \mathbb{C}^r$ such that $d = P_{\mathbb{D}}\boldsymbol{\zeta}(\boldsymbol{v})$. Thus, similar to (39), it is deduced that

$$D(\lambda)\begin{pmatrix}d\\ \lambda^{-1}T\zeta'(0)\end{pmatrix} = \widehat{K}, \tag{57}$$

where $D(\lambda)$ is given by (40) and

$$\begin{aligned} &\widehat{K} = \begin{pmatrix}0\\ \lambda^{-1}\beta P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v})\end{pmatrix} + \\ &\lambda^{-1}\int_0^1\begin{pmatrix}A\sinh(\lambda(1-s)A)(g(s)-\lambda f(s))\\ P_{\mathbb{D}}(\Psi^-)M\cosh(\lambda(1-s)A)(g(s)-\lambda f(s))\end{pmatrix}ds. \end{aligned}$$

Note that $\lambda \in \rho(\mathcal{A}^*) = \rho(\mathcal{A})$, so $\det(D(\lambda)) \neq 0$, which implies that the equations (57) has a unique solution $(d, \lambda^{-1}T\zeta'(0))^{\mathrm{T}} \in \mathbb{C}^{r+n}$. Hence, $\eta_0 = (\eta_{0,1}, \eta_{0,2})^{\mathrm{T}}$ can be solved by (56) and $\boldsymbol{\zeta}(\boldsymbol{v}) = P_{\mathbb{D}}^{\mathrm{T}}d$. Thus, (53) is obtained by (55) and (56). □

Remark 13. *When $\lambda = 0$, it follows from* (55) *that* $\zeta(x) = \zeta(1) - \zeta'(1)(1-x) + T^{-1}M\int_x^1\int_r^1 g(s)ds\,dr$. *Thus,*

$$R(0, \mathcal{A}^*)\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \zeta(1) - \zeta'(1)(1-x) + T^{-1}M\int_x^1\int_r^1 g(s)ds\,dr \\ f(x) \end{pmatrix},$$

where $\zeta(1)$ and $\zeta'(1)$ are two vectors, which can be determined uniquely, similar to $f(1)$ and $f'(1)$ in (33).

Proof of Theorem 7: Denote by

$$Sp(\mathcal{A}) = \overline{\operatorname{span}\left\{\sum_{k=1}^{\widetilde{m}} y_k \,\middle|\, \begin{array}{l} y_k \in E(\lambda_k, \mathcal{A})\mathcal{H}, \\ \forall \lambda_k \in \sigma(\mathcal{A}), \widetilde{m} \in \mathbb{N} \end{array}\right\}}, \tag{58}$$

where $E(\lambda_k, \mathcal{A})$ is the Riesz projector corresponding to λ_k. It is easy to prove that

$$Sp(\mathcal{A})^{\perp} = \{z \in \mathcal{H} | E(\overline{\lambda}_k, \mathcal{A}^*)z = 0, \forall \lambda_k \in \sigma(\mathcal{A})\}. \tag{59}$$

To prove $Sp(\mathcal{A}) = \mathcal{H}$, we need only to prove $Sp(\mathcal{A})^{\perp} = \{0\}$, according to (59). In what follows, we will prove $Sp(\mathcal{A})^{\perp} = \{0\}$.

Let $(\widetilde{\zeta}, \widetilde{\nu})^{\mathrm{T}} \in \mathcal{H}$ and $(\widetilde{\zeta}, \widetilde{\nu})^{\mathrm{T}} \perp Sp(\mathcal{A})$. It follows from [27, Lemma 6,pp.2296] that $R^*(\lambda, \mathcal{A})(\widetilde{\zeta}, \widetilde{\nu})^{\mathrm{T}}$ is an entire function on $\mathbb{C}$ valued in $\mathcal{H}$. Thus, $\forall (\zeta, \nu)^{\mathrm{T}} \in \mathcal{H}$, we define a function on complex plane $\mathbb{C}$ by

$$F(\lambda) = \langle (\zeta, \nu)^{\mathrm{T}}, R^*(\lambda, \mathcal{A})(\widetilde{\zeta}, \widetilde{\nu})^{\mathrm{T}} \rangle_{\mathcal{H}}. \tag{60}$$

It follows from Lemma 12 that $F(\lambda)$ is an entire function of finite exponential type. In addition, Theorem 4 and Hille-Yosida Theorem asserts that

$$|F(\lambda)| \leq (\Re\lambda)^{-1} \|(\zeta, \nu)\| \left\|(\widetilde{\zeta}, \widetilde{\nu})\right\|, \text{ for } \Re\lambda > 0.$$

Hence $\lim\limits_{\Re\lambda \to +\infty} |F(\lambda)| = 0$.

Now, let us consider the solution of the following equations

$$\begin{cases} (\lambda I - \mathcal{A})(f, g)^{\mathrm{T}} = (\zeta, \nu)^{\mathrm{T}}, \\ (\lambda I - \mathcal{A}_0)\left(\hat{f}, \hat{g}\right)^{\mathrm{T}} = (\zeta, \nu)^{\mathrm{T}}, \end{cases} \tag{61}$$

where $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0)$ and $\lambda < 0$. Let $\widetilde{f}(x) = f(x) - \widehat{f}(x)$, $\widetilde{g}(x) = g(x) - \widehat{g}(x)$. Then

$$R(\lambda, \mathcal{A})(\zeta, \nu)^{\mathrm{T}} = R(\lambda, \mathcal{A}_0)(\zeta, \nu)^{\mathrm{T}} + \left(\widetilde{f}, \widetilde{g}\right)^{\mathrm{T}} \tag{62}$$

and $\left(\widetilde{f}, \widetilde{g}\right)^{\mathrm{T}}$ satisfies $\widetilde{g}(x) = \lambda\widetilde{f}(x)$ and

$$\begin{cases} T\widetilde{f}''(x) = \lambda^2 M\widetilde{f}(x), \\ \widetilde{f}(0) = (\Psi^+)^{\mathrm{T}}\widetilde{\boldsymbol{f}}(\boldsymbol{v}), \widetilde{f}(1) = (\Psi^-)^{\mathrm{T}}\widetilde{\boldsymbol{f}}(\boldsymbol{v}), \\ P_{\mathbb{D}}\left[(\Psi^-)T\widetilde{f}'(1) - (\Psi^+)T\widetilde{f}'(0)\right] \\ \quad = -(\lambda\beta + \gamma)P_{\mathbb{D}}\widetilde{\boldsymbol{f}}(\boldsymbol{v}) - \beta P_{\mathbb{D}}\widehat{\boldsymbol{g}}(\boldsymbol{v}) \end{cases} \tag{63}$$

with $\widetilde{\boldsymbol{f}}(\boldsymbol{v}) \in \mathbb{D}$.

Set $\eta(x) = (\widetilde{f}(x), \lambda^{-1}T\widetilde{f}'(x))^{\mathrm{T}}$. Similar to (34) and (35), the solution of (63), $\eta(x)$ satisfies

$$\eta = W(x, \lambda)\eta_0 \text{ with } \eta_0 = \begin{pmatrix} (\Psi^+)^{\mathrm{T}}\widetilde{\boldsymbol{f}}(\boldsymbol{v}) \\ \lambda^{-1}T\widetilde{f}'(0) \end{pmatrix}, \tag{64}$$

where $W(x, \lambda)$ is defined by (37). Since $\widetilde{\boldsymbol{f}}(\boldsymbol{v}) \in \mathbb{D}$, there is a $d \in \mathbb{C}^r$ such that $\widetilde{\boldsymbol{f}}(\boldsymbol{v}) = P_{\mathbb{D}}^{\mathrm{T}}d$. Let $\eta_d = (d, \lambda^{-1}T\widetilde{f}'(0))^{\mathrm{T}} \in \mathbb{C}^{r+n}$, then

$$\begin{aligned} \eta_0 &= \begin{pmatrix} (P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} d \\ \lambda^{-1}T\widetilde{f}'(0) \end{pmatrix} \\ &= \begin{pmatrix} (P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & 0 \\ 0 & I \end{pmatrix}\eta_d. \end{aligned} \tag{65}$$

Thus, similar to (39), η_d satisfies

$$D(\lambda)\eta_d = -\lambda^{-1}\begin{pmatrix} 0 \\ \beta P_{\mathbb{D}}\widehat{\boldsymbol{g}}(\boldsymbol{v}) \end{pmatrix}, \tag{66}$$

where $D(\lambda)$ is defined by (40).

Set

$$\begin{aligned} D_{(\lambda,\gamma)} &= \begin{pmatrix} 0 & 0 \\ \lambda^{-1}\gamma & -P_{\mathbb{D}}\Psi^+ \end{pmatrix} \\ &+ \begin{pmatrix} (\sqrt{MT})^{-1}e^{\lambda A}\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{(\sqrt{MT})^{-1}e^{\lambda A}}{2} \\ P_{\mathbb{D}}(\Psi^-)e^{\lambda A}\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{P_{\mathbb{D}}(\Psi^-)e^{\lambda A}}{2} \end{pmatrix} \end{aligned} \tag{67}$$

and

$$D_c = \begin{pmatrix} -(\Psi^-)^{\mathrm{T}}P_{\mathbb{D}}^{\mathrm{T}} & -\dfrac{\sqrt{MT}^{-1}}{2} \\ \beta - P_{\mathbb{D}}\Psi^+\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & \dfrac{P_{\mathbb{D}}\Psi^-}{2} \end{pmatrix}, \tag{68}$$

then it follows from (40) that

$$D(\lambda) = \left[D_{(\lambda,\gamma)} + D_c\begin{pmatrix} I & 0 \\ 0 & e^{-\lambda A} \end{pmatrix}\right]\begin{pmatrix} I & 0 \\ -\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & I \end{pmatrix}.$$

Let

$$\widetilde{\eta}_d = \begin{pmatrix} I & 0 \\ 0 & e^{-\lambda A} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} & I \end{pmatrix} \eta_d$$

i.e.,

$$\widetilde{\eta}_d = \begin{pmatrix} d \\ e^{-\lambda A}\Big(\lambda^{-1}T\widetilde{f}'(0) - \sqrt{MT}(P_{\mathbb{D}}\Psi^+)^{\mathrm{T}} d\Big) \end{pmatrix},$$

thus, (66) implies that

$$\widetilde{\eta}_d = -\lambda^{-1}\left[D_{(\lambda,\gamma)}\begin{pmatrix} I & 0 \\ 0 & e^{\lambda A} \end{pmatrix} + D_c\right]^{-1} \begin{pmatrix} 0 \\ \beta P_{\mathbb{D}}\widehat{\boldsymbol{g}}(\boldsymbol{v}) \end{pmatrix},$$

where $D_{(\lambda,\gamma)}$ and D_c are defined by (67) and (68), respectively. Therefore, $\|d\| = O(\lambda^{-1})\|\beta P_{\mathbb{D}}\widehat{\boldsymbol{g}}(\boldsymbol{v})\|$, for sufficiently large $-\lambda > 0$.

According to Definition (20), the direct calculation shows that

$$\begin{aligned} \left\|(\widetilde{f}, \widetilde{g})^{\mathrm{T}}\right\|^2 &= -\lambda\langle P_{\mathbb{D}}\widetilde{\boldsymbol{f}}(\boldsymbol{v}),\, \beta P_{\mathbb{D}}\widetilde{\boldsymbol{f}}(\boldsymbol{v})\,\rangle \\ &\quad -\langle P_{\mathbb{D}}\widetilde{\boldsymbol{f}}(\boldsymbol{v}),\, \beta P_{\mathbb{D}}\widehat{\boldsymbol{g}}(\boldsymbol{v})\,\rangle. \end{aligned}$$

By $d = P_{\mathbb{D}}\widetilde{\boldsymbol{f}}(\boldsymbol{v})$ and (47) in Lemma 11, it can be derived that

$$\left\|(\widetilde{f}, \widetilde{g})^{\mathrm{T}}\right\| = O(\sqrt{|\lambda|})\left\|(\zeta, \nu)^{\mathrm{T}}\right\|.$$

Thus, for sufficiently large $-\lambda > 0$, it follows from (60) and (62) that

$$\begin{aligned} |F(\lambda)| &\le |\lambda^{-1}|\left\|(\zeta,\nu)^{\mathrm{T}}\right\|\left\|(\widetilde{\zeta},\widetilde{\nu})^{\mathrm{T}}\right\| \\ &\quad + \left\|\left(\widetilde{f}, \widetilde{g}\right)^{\mathrm{T}}\right\|\left\|(\widetilde{\zeta},\widetilde{\nu})^{\mathrm{T}}\right\| \\ &= O(|\lambda|^{-1/2})\left\|(\zeta,\nu)^{\mathrm{T}}\right\|\left\|(\widetilde{\zeta},\widetilde{\nu})^{\mathrm{T}}\right\|. \end{aligned}$$

Since $F(\lambda)$ is an entire function of finite exponential type, the Phrangmen-Lindelöf theorem in [30] and the above inequality show that $F(\lambda) \equiv 0$. Thus, it follows from (60) that $R^*(\lambda, \mathcal{A})(\widetilde{\zeta},\widetilde{\nu})^{\mathrm{T}} \equiv 0$, which leads to $(\widetilde{\zeta},\widetilde{\nu})^{\mathrm{T}} \equiv 0$. So, $Sp\,(\mathcal{A}) = \mathcal{H}$. $\square$

3.4 The proof of Theorem 8

Firstly, we introduce the definition of the subspaces Riesz basis ([28]) and the Riesz basis with parentheses ([29]).

Definition 14. *A subspace sequence of $\{\mathcal{H}_j\}_{j=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a Riesz basis of subspaces for space $\mathcal{H}$, if for every $\phi \in \mathcal{H}$ there is a unique $\phi_j \in \mathcal{H}_j$, $j = 1, 2, \ldots$ such that $\phi = \sum_{j=1}^{\infty}\phi_j$, and there exists positive constants c_1 and c_2, such that $\forall\phi \in \mathcal{H}$,*

$$c_1\sum_{j=1}^{\infty}\|\phi_j\|^2 \le \|\phi\|^2 \le c_2\sum_{j=1}^{\infty}\|\phi_j\|^2.$$

A sequence $\{\phi_j\}_{j=1}^{\infty}$ is called a Riesz basis with parentheses if there is a sequence of integers $n_0 = 1 \le n_1 \le \cdots \le n_k \le \cdots$ such that $\{\mathcal{H}_k\}_{k=1}^{\infty}$ with $\mathcal{H}_k = \operatorname{span}\{\phi_j | j = n_{k-1}+1, \ldots, n_k\}$ forms a Riesz basis of subspaces.

The proof of Theorem 8: Denote by $S(t)$ the semigroup generated by $\mathcal{A}$. Let $\sigma_1(\mathcal{A}) = \{-\infty\}$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Theorem 4 and 5 shows that all the conditions of [19, Lemma 4.5] are fulfilled. So, there exist two $S(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ corresponding to $\sigma_1(\mathcal{A})$ and $\sigma_2(\mathcal{A})$, and there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}_2$. Theorem 7 claims that the sequence is complete in $\mathcal{H}$, which implies that $\mathcal{H}_2 = \mathcal{H}$. Therefore the sequence is also a Riesz basis with parentheses for $\mathcal{H}$. The Riesz basis property together with spectral distribution and the uniform boundedness of the multiplicities of eigenvalues of $\mathcal{A}$ implies that $S(t)$ satisfies the spectrum determined growth assumption. $\square$

4 Stability of the closed loop system

As a consequence of the Riesz basis with parentheses and Theorem 5, we asserts the system satisfies the spectrum determined growth assumption. Therefore, asymptotical stability of both the semigroup $S(t)$ generated by $\mathcal{A}$ and the closed loop system ((11) and (15)) are equivalent to the non-existence of purely imaginary eigenvalues of the operator $\mathcal{A}$. In what follows, by the spectral analysis, we present some sufficient and necessary conditions for the asymptotical stability of the closed loop system. For this, we first introduce the following definition [31, 32]

Definition 15. *Let $\mathcal{A}$ be defined by* (21)*, which is an infinitesimal generator of a C_0 semigroup $S(t)$ on the Hilbert space $\mathcal{H}$ defined by* (18)*. $S(t)$ is asymptotical stable (strongly stable) if*

$$\lim_{t\to\infty} S(t)x = 0, \forall x \in \mathcal{H}.$$

The corresponding Cauchy problem (24) *is also called asymptotical stable.*

Next, we verify whether or not there exist the spectral points on the imaginary axis. Let $\lambda = i\theta \in$

$\sigma_p(\mathcal{A})$ ($\theta \in \mathbb{R}$ and $\theta \neq 0$) and $(f,g)^{\mathrm{T}} \in \mathrm{dom}(\mathcal{A})$ be the corresponding eigenvector, then (25) implies that

$$\begin{aligned} & \Re(\lambda)\langle (f,g)^{\mathrm{T}}, (f,g)^{\mathrm{T}}\rangle \\ = \;& \Re(\mathcal{A}(f,g)^{\mathrm{T}}, (f,g)^{\mathrm{T}}) \\ = \;& -\langle \beta P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v}),\, P_{\mathbb{D}}\boldsymbol{g}(\boldsymbol{v})\rangle = 0. \end{aligned} \tag{69}$$

According to (34) and (35), the eigenvalue problem is equivalent to existence of the nonzero solution of the following differential equations

$$Tf''(x) = -\theta^2 Mf(x), \quad g(x) = i\theta f(x), \tag{70}$$

with the boundary conditions

$$\begin{cases} f(0) = (\Psi^+)^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}), \\ f(1) = (\Psi^-)^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{f}(\boldsymbol{v}) \in \mathbb{D} \\ P_{\mathbb{D}}\left[(\Psi^-)Tf'(1) - (\Psi^+)Tf'(0)\right] = -\gamma P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}). \end{cases} \tag{71}$$

Since we suppose that $\lambda \in i\mathbb{R}$, the spectral points is on the imaginary axis, the eigenvalue problem (70)-(71) is called the purely imaginary spectral analysis or spectral analysis on the imaginary axis.

Case 1: $\mathcal{D} = \emptyset$ and β is a diagonal positive definite matrix.

In this case, $P_{\mathbb{D}} = I$, $\mathbb{D} \cong \mathbb{C}^m$ and it is required that $\sum_{k=1}^m \gamma_k > 0, \gamma_k \geq 0\,(k \in \mathcal{I}_V)$. The wave network is controlled completely. It follows from (69) that $\boldsymbol{g}(\boldsymbol{v}) = 0$, which implies that $\boldsymbol{f}(\boldsymbol{v}) = 0$. So, the eigenvalue problem (70)-(71) becomes

$$Tf''(x) = -\theta^2 Mf(x) \tag{72}$$

with the boundary conditions

$$\begin{cases} f(0) = 0,\ f(1) = 0 \\ (\Psi^-)Tf'(1) - (\Psi^+)Tf'(0) = 0 \end{cases} \tag{73}$$

and $g(x) = i\theta f(x)$. From the first boundary condition of (73) it follows that solution of the equation (72) is of the form

$$f(x) = T^{-1/2}\sin(\theta Ax)\xi, \tag{74}$$

where $A = \sqrt{T^{-1}M} = \mathrm{diag}\{a_1, a_2, \cdots, a_n\}$ and the $\xi \in \mathbb{C}^n$ is an arbitrary constant vector. From the second and third boundary conditions in (73) it follows that

$$\begin{cases} \sin(\theta A)\xi = 0, \\ (\Psi^-)\cos(\theta A)M^{1/2}\xi - (\Psi^+)M^{1/2}\xi = 0. \end{cases} \tag{75}$$

According to (1) and (2), it can be derived that

$$\begin{aligned} &(\Psi^-)\cos(\theta A)M^{1/2}\xi \\ &\quad = \Big(e_{j_1^-}\cos(\theta a_1)\sqrt{\rho_1}, \\ &\qquad \ldots, e_{j_k^-}\cos(\theta a_k)\sqrt{\rho_k}, \ldots, \\ &\qquad\qquad e_{j_n^-}\cos(\theta a_n)\sqrt{\rho_n}\Big)\,\xi \end{aligned}$$

and

$$\begin{aligned} &(\Psi^+)M^{1/2}\xi = \\ &\quad \Big(e_{j_1^+}\sqrt{\rho_1}, \ldots, e_{j_k^+}\sqrt{\rho_k}, \ldots, e_{j_n^+}\sqrt{\rho_n}\Big)\,\xi. \end{aligned}$$

So

$$\sum_{k\in\mathcal{I}_E^-(p_i)} \cos(\theta a_k)\sqrt{\rho_k}\xi_k = \sum_{k\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_k}\xi_k, i \in \mathcal{I}_V. \tag{76}$$

From (75) and (76), it is yielded that

$$\begin{cases} \sin(\theta a_k)\sqrt{\rho_k}\xi_k = 0, k \in \mathcal{I}_E, \\ \sum_{k\in\mathcal{I}_E^-(p_i)}\cos(\theta a_k)\sqrt{\rho_k}\xi_k \\ \qquad = \sum_{k\in\mathcal{I}_E^+(p_i)}\sqrt{\rho_k}\xi_k, i \in \mathcal{I}_V. \end{cases} \tag{77}$$

Hence, there are not spectral points on imaginary axis, if and only if (77) implies that $\xi \equiv 0$.

Case 2: $\mathcal{D} = \emptyset$ and β is a diagonal and semi-positive definite matrix.

In this case, $P_{\mathbb{D}} = I$, $\mathbb{D} \cong \mathbb{C}^m$ and $\mathcal{I}_\beta \subset \mathcal{J}_P = \mathcal{I}_V$, which means that the network G is a non-completely controlled one. When $\mathcal{N} \neq \emptyset$, the controlled network has free boundary vertices. If $p_k \in Int(G)$ and $k \in \mathcal{J}_P \setminus \mathcal{I}_\beta$, which means there is no controller at p_k, and hence the controlled network satisfies the Kirchhoff condition (dynamic condition) at node p_k.

According to the definition of $\mathcal{C}$, the controller vertex set, for each $p_k \in \mathcal{C}$, it is derived from (69) that $\mathrm{g}(p_k) = 0\ (k \in \mathcal{I}_\beta)$, and hence $\mathbb{f}(p_k) = 0\ (k \in \mathcal{I}_\beta)$. Define subspace $\mathbb{D}_\beta$ of $\mathbb{C}^m$ by

$$\mathbb{D}_\beta = \{(x_1, x_2, \cdots, x_m) \in \mathbb{C}^m \;\big|\; x_k = 0, k \in I_\beta\}$$

Obviously, $\boldsymbol{f}(\boldsymbol{v}) \in \mathbb{D}_\beta$. Therefore, the eigenvalue problem (70)-(71) becomes $g(x) = i\theta f(x)$ and

$$Tf''(x) = -\theta^2 Mf(x) \tag{78}$$

with the boundary conditions

$$\begin{cases} \boldsymbol{f}(\boldsymbol{v}) \in \mathbb{D}_\beta, \\ f(0) = (\Psi^+)^T\boldsymbol{f}(\boldsymbol{v}), \\ f(1) = (\Psi^-)^T\boldsymbol{f}(\boldsymbol{v}), \\ (\Psi^-)Tf'(1) - (\Psi^+)Tf'(0) = 0. \end{cases}$$

From (1), (2), (17) and (71), it follows that $f_k(0) = e^{\mathrm{T}}_{j_k^+}\boldsymbol{f}(\boldsymbol{v}) = \mathbb{f}(p_{j_k^+})$ and $f_k(1) = e^{\mathrm{T}}_{j_k^-}\boldsymbol{f}(\boldsymbol{v}) = \mathbb{f}(p_{j_k^-})$. So, the above boundary conditions are equivalent to

$$\begin{cases} f_k(0) = 0, j_k^+ \in \mathcal{I}_\beta, \\ f_k(0) = \mathbb{f}(p_{j_k^+}), j_k^+ \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ f_k(1) = 0, j_k^- \in \mathcal{I}_\beta, \\ f_k(1) = \mathbb{f}(p_{j_k^-}), j_k^- \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ \qquad j_k^-, j_k^+ \in \mathcal{I}_V, k \in \mathcal{I}_E, \\ (\Psi^-)Tf'(1) - (\Psi^+)Tf'(0) = 0. \end{cases} \tag{79}$$

The solution of the equation (78) is of the from

$$f(x) = T^{-1/2}\sin(\theta Ax)\xi + T^{-1/2}\cos(\theta Ax)\omega,$$

where $\xi, \omega \in \mathbb{C}^n$ are constant vectors. The boundary conditions (79) lead to ξ and ω satisfy

$$\begin{cases} \omega_k = 0, j_k^+ \in \mathcal{I}_\beta, \\ T_k^{-1/2}\omega_k = \mathbb{f}(p_{j_k^+}), j_k^+ \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ T_k^{-1/2}\sin(\theta a_k)\xi_k + T_k^{-1/2}\cos(\theta a_k)\omega_k = 0, j_k^- \in \mathcal{I}_\beta, \\ T_k^{-1/2}\sin(\theta a_k)\xi_k + T_k^{-1/2}\cos(\theta a_k)\omega_k = \mathbb{f}(p_{j_k^-}), \\ \qquad j_k^- \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ ((\Psi^-)T^{1/2}\cos(\theta A) - (\Psi^+)T^{1/2})A\xi \\ \qquad -(\Psi^-)T^{1/2}\sin(\theta A)A\omega = 0. \end{cases}$$

The last equations in the above is equivalent to

$$\sum_{k\in\mathcal{I}_E^-(p_i)} [\cos(\theta a_k)\sqrt{\rho_k}\xi_k - \sin(\theta a_k)\sqrt{\rho_k}\omega_k] = \sum_{k\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_k}\xi_k, i \in \mathcal{I}_V.$$

Therefore, there are not spectral points on imaginary axis if, and only if

$$\begin{cases} \omega_k = 0, j_k^+ \in \mathcal{I}_\beta, \\ \omega_k = T_k^{1/2}\mathbb{f}(p_{j_k^+}), j_k^+ \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ \sin(\theta a_k)\xi_k + \cos(\theta a_k)\omega_k = 0, j_k^- \in \mathcal{I}_\beta, \\ \sin(\theta a_k)\xi_k + \cos(\theta a_k)\omega_k = T_k^{1/2}\mathbb{f}(p_{j_k^-}), \\ \qquad j_k^- \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ \sum_{s\in\mathcal{I}_E^-(p_i)} \left(\cos(\theta a_s)\sqrt{\rho_s}\xi_s - \sin(\theta a_s)\sqrt{\rho_s}\omega_s\right) \\ \qquad = \sum_{s\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_s}\xi_s, \\ j_k^-, j_k^+, i \in \mathcal{I}_V, k \in \mathcal{I}_E, \end{cases} \tag{80}$$

implies that $\xi = \omega \equiv 0$.

Case 3: $\mathcal{D} \neq \emptyset$ and β is a diagonal positive definite matrix.

In this case, $\gamma = 0$, $\mathbb{D} \cong \mathbb{C}^r$, $r < m$, the wave network is controlled completely. It follows from (69) that $P_{\mathbb{D}}\boldsymbol{f}(\boldsymbol{v}) = 0$. So, the eigenvalue problem (70) and (71) is given by $g(x) = i\theta f(x)$ and

$$Tf''(x) = -\theta^2 Mf(x)$$

with the boundary conditions

$$\begin{cases} f(0) = 0,\ f(1) = 0 \\ P_{\mathbb{D}}\left[(\Psi^-)Tf'(1) - (\Psi^+)Tf'(0)\right] = 0. \end{cases}$$

So the solution of the above equation is defined by (74) with ξ satisfying

$$\begin{cases} \sin(\theta A)\xi = 0, \\ P_{\mathbb{D}}\left[(\Psi^-)\cos(\theta A) - (\Psi^+)\right] M^{1/2}\xi = 0. \end{cases} \tag{81}$$

Similar to **Case 1**, the spectral points are not on imaginary axis if and only if that $\xi = 0$ can be derived from the following equations

$$\begin{cases} \sin(\theta a_k)\sqrt{\rho_k}\xi_k = 0, k \in \mathcal{I}_E, \\ \sum_{k\in\mathcal{I}_E^-(p_i)} \cos(\theta a_k)\sqrt{\rho_k}\xi_k = \sum_{k\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_k}\xi_k, \\ \qquad p_i \notin \mathcal{D}, i \in \mathcal{I}_V. \end{cases} \tag{82}$$

Case 4: $\mathcal{D} \neq \emptyset$ and β is a diagonal and semi-positive definite matrix.

In this case, $\gamma = 0, \mathbb{D} \cong \mathbb{C}^r, r < m$, the wave network is a non-completely controlled one. Similar to **Case 2**, it follows from (69) that $\mathrm{g}(p_k) = 0$, for $k \in \mathcal{I}_\beta$, which implies that $\mathbb{f}(p_k) = 0$, for $k \in \mathcal{I}_\beta$. So, the eigenvalue problem (70)–(71) is formulated by

$$Tf''(x) = -\theta^2 Mf(x) \tag{83}$$

with the boundary conditions

$$\begin{cases} f_k(0) = 0, j_k^+ \in \mathcal{I}_\beta \cup \mathcal{J}_D, \\ f_k(0) = \mathbb{f}(p_{j_k^+}), j_k^+ \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ f_k(1) = 0, j_k^- \in \mathcal{I}_\beta \cup \mathcal{J}_D, \\ f_k(1) = \mathbb{f}(p_{j_k^-}), j_k^- \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ \qquad j_k^-, j_k^+ \in \mathcal{I}_V, k \in \mathcal{I}_E, \\ P_{\mathbb{D}}\left[(\Psi^-)Tf'(1) - (\Psi^+)Tf'(0)\right] = 0, \end{cases}$$

and $g(x) = i\theta f(x)$. Therefore, the solution of the equation (83) is

$$f(x) = T^{-1/2}\sin(\theta Ax)\xi + T^{-1/2}\cos(\theta Ax)\omega, \tag{84}$$

where $\xi, \omega \in \mathbb{C}^n$ are two arbitrary constant vectors. Thus, ξ and ω satisfy

$$\begin{cases} \omega_k = 0, j_k^+ \in \mathcal{I}_\beta \cup \mathcal{J}_D, \\ \omega_k = T_k^{1/2}\mathbb{f}(p_{j_k^+}), j_k^+ \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ \sin(\theta a_k)\xi_k + \cos(\theta a_k)\omega_k = 0, j_k^- \in \mathcal{I}_\beta \cup \mathcal{J}_D, \\ \sin(\theta a_k)\xi_k + \cos(\theta a_k)\omega_k = T_k^{1/2}\mathbb{f}(p_{j_k^-}), \\ \quad j_k^- \in \mathcal{J}_P \setminus \mathcal{I}_\beta, \\ \sum\limits_{s\in\mathcal{I}_E^-(p_i)} \big(\cos(\theta a_s)\sqrt{\rho_s}\xi_s - \sin(\theta a_s)\sqrt{\rho_s}\omega_s\big) \\ \quad = \sum\limits_{s\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_s}\xi_s, \\ k \in \mathcal{I}_E, p_i \notin \mathcal{D}, i \in \mathcal{I}_V. \end{cases} \tag{85}$$

Therefore, the spectral points are not on imaginary axis if and only if it can be derived from (85) that $\xi = \omega = 0$.

From the above four cases, we can draw a conclusion as follows.

Theorem 16. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined by* (18) *and* (21)*, respectively, and β satisfies* (26)*. Then the controlled network* ((11) *with* (15)) *is asymptotically stable if and only if one of the following four conditions holds.*

1. *When $\mathcal{D} = \emptyset$ and β is a diagonal positive definite matrix, for all $\theta \in \mathbb{R}$,* (77) *implies that $\xi = 0$.*
2. *When $\mathcal{D} = \emptyset$ and β is a diagonal and semi-positive definite matrix, for all $\theta \in \mathbb{R}$,* (80) *implies that $\xi = \omega = 0$.*
3. *When $\mathcal{D} \neq \emptyset$ and β is a diagonal and semi-positive definite matrix, for all $\theta \in \mathbb{R}$,* (82) *implies that $\xi = 0$.*
4. *When $\mathcal{D} \neq \emptyset$ and β is a diagonal positive definite matrix, $\forall\theta \in \mathbb{R}$,* (85) *implies that $\xi = \omega = 0$.*

Proof We only prove the sufficient and necessary condition (1), the others can be proven similarly.

If the controlled network is asymptotically stable, then according to the spectrum determined growth assumption, there is no spectral point of $\mathcal{A}$ on the imaginary axis. Suppose that for some real number θ_0, a nonzero vector $\xi \in \mathbb{C}^n$ can be found such that (77) holds, then $\lambda = i\theta_0$ is a spectral point of $\mathcal{A}$, which is a contradiction.

Conversely, if $\forall\theta \in \mathbb{R}$, (77) implies that $\xi = 0$, then from above the spectral analysis on the imaginary axis, it follows that the spectral points of $\mathcal{A}$ are not on the imaginary axis. Thus, the spectrum determined growth assumption asserts that the controlled network is asymptotically stable. □

If the network (11) is controlled completely, i.e., β is a diagonal positive definite matrix, from (77) and (82) it can be deduced that $\forall\, i \in \mathcal{J}_P$,

$$\sum_{k\in\mathcal{I}_E^-(p_i)} \delta_k\sqrt{\rho_k}\xi_k = \sum_{k\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_k}\xi_k, \tag{86}$$

where δ_k is $+1$ or -1. Thus, (77) and (82) are equivalent to

$$\sin(\theta A)M^{1/2}\xi = 0,\ P_{\mathbb{D}}(\Psi^+ - \Psi^- S_\delta)M^{1/2}\xi = 0, \tag{87}$$

where

$$S_\delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \text{ with } \delta_k \in \{-1, 1\}. \tag{88}$$

Note that if $\mathcal{D} = \emptyset$, then $\mathcal{J}_P = \mathcal{I}_V$ and $P_{\mathbb{D}} = I_m$. Thus, we have the following result.

Corollary 17. *Let all assumptions of Theorem 16 be fulfilled and the network* ((11) *with* (15)) *be controlled completely. Then the controlled network is asymptotically stable if the kernel of the matrix $P_{\mathbb{D}}(\Psi^+ - \Psi^- S_\delta)$, $\ker(P_{\mathbb{D}}(\Psi^+ - \Psi^- S_\delta))$, is the null space, for any diagonal matrix S_δ defined by* (88).

To characterize the implication relation in Theorem 16 further, we introduce the following definition and lemma.

Definition 18. *For any two different numbers a_k and a_s, if $\dfrac{a_k}{a_s} \neq \dfrac{n_k}{n_s}$ where $n_k, n_s \in \mathbb{Z}$ and $n_s \neq 0$, then we call the pair (a_k, a_s) satisfying the relation "IR$^{(1,1)}$".*

The above definition shows that the ratio of a_k and a_s is an irrational number, so the property is called irrational dependence.

Remark 19. *In fact, this property reflects relationship between spectra and edges of the network. To see this, let us consider the following Sturm–Liouville problem*

$$T_k f_k''(x) = \lambda^2\rho_k f_k(x), x \in (0, 1)$$

with $f_k(0) = f_k(1) = 0$. The general solution is given by

$$f_k(x) = c_1 e^{-a_k\lambda x} + c_2 e^{a_k\lambda x}.$$

From the boundary conditions $f_k(0) = f_k(1) = 0$, it follows that

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 e^{-a_k\lambda} + c_2 e^{a_k\lambda} = 0. \end{cases}$$

So, its point spectrum is

$$\sigma_k^D = \{\lambda \in \mathbb{C} | e^{2a_k\lambda} = 1\} = \{i\pi n_k/a_k | n_k \in \mathbb{Z}\}.$$

We call it Dirichlet spectrum of k-th edge. Hence $\sigma_k^D \cap \sigma_s^D = \emptyset$ if and only if (a_k, a_s) satisfies the relation "IR$^{(1,1)}$".

Lemma 20. *Suppose that for some $p_i \notin \mathcal{D}$, $\forall\theta \in \mathbb{R}$,*

$$\begin{cases} \sin(\theta a_k)\sqrt{\rho_k}\xi_k = 0, k \in \mathcal{I}_E(p_i), \\ \sum_{k\in\mathcal{I}_E^-(p_i)} \cos(\theta a_k)\sqrt{\rho_k}\xi_k = \sum_{k\in\mathcal{I}_E^+(p_i)} \sqrt{\rho_k}\xi_k, \end{cases} \tag{89}$$

where $a_k > 0$. If (a_k, a_s) satisfies the relation "IR$^{(1,1)}$", $\forall k, s \in \mathcal{I}_E(p_i), k \neq s$, then $\xi_k = 0$, $\forall k \in \mathcal{I}_E(p_i)$.

Proof It can be derived from (89) that (86) holds for p_i. For all $k \in \mathcal{I}_E(p_i)$, if $\sin(\theta a_k) \neq 0$, then $\xi_k = 0, \forall k \in \mathcal{I}_E(p_i)$. If $\sin(\theta a_k) = 0$ for some $k \in \mathcal{I}_E(p_i)$, then $\theta a_k = n_k\pi, n_k \in \mathbb{Z}$. Since (a_k, a_s) satisfies the relation "IR$^{(1,1)}$", $\forall s \in \mathcal{I}_E(p_i)$ with $s \neq k$,

$$\theta a_s = \theta a_k \frac{a_s}{a_k} = \frac{a_s}{a_k} n_k\pi \neq n_s\pi, n_s \in \mathbb{Z}.$$

Thus, $\sin(\theta a_s) \neq 0$, which implies that $\xi_s = 0, \forall s \in \mathcal{I}_E(p_i), s \neq k$. Therefore, it follows from (86) that $\xi_k = 0, \forall k \in \mathcal{I}_E(p_i)$. □

Theorem 21. *Let the network ((11) with (15)) be controlled completely. Suppose that all assumptions of Theorem 16 are true and each edge has at most one fixed vertex. For all $i \in \mathcal{J}_P$, $\forall k, s \in \mathcal{I}_E(p_i), k \neq s$, if (a_k, a_s) satisfies the relation "IR$^{(1,1)}$", then the controlled network is asymptotically stable.*

Proof For each $i \in \mathcal{J}_P$, i.e., $p_i \notin \mathcal{D}$, owing to Lemma 20, $\xi_k = 0$, $\forall k \in \mathcal{I}_E(p_i)$. The fact that each edge has at most one fixed vertex shows that k takes over all $\mathcal{I}_E$. So $\xi \equiv 0$. Thus, the controlled network is asymptotically stable. □

According to Remark 19 and under the assumptions of Theorem 21, if all edges jointed with vertex $p_k, \forall k \in \mathcal{I}_V$, have no the same Dirichlet spectra, then the controlled network is asymptotically stable. We also remark that when Theorem 16, Theorem 21 and Lemma 20 are applied to judge the stability of concrete networks, only if the "IR$^{(1,1)}$" relation is true for partial pairs (a_k, a_s), where $k, s \in \mathcal{I}_E(p_i), k \neq s$, $p_i \notin \mathcal{D}$, owing to the edge-edge joint condition. Refer to the examples given in the next section. But when all pairs (a_k, a_s) with $k \neq s$ are rational numbers, the network ((11) with (15)) can not be stabilized.

Theorem 22. *Let all assumptions of Theorem 16 be fulfilled and the network ((11) with (15)) be controlled completely. Assume that all pairs (a_k, a_s) with $k \neq s$ are rational numbers. If there is a diagonal matrix S_δ defined by (88) such that $\ker(P_{\mathbb{D}}(\Psi^+ - \Psi^- S_\delta)) \neq \{0\}$, then the network is not stable. Especially, if $\ker(P_{\mathbb{D}}\Psi) \neq \{0\}$, then it is not stable.*

Proof Since $\ker(P_{\mathbb{D}}(\Psi^+ - \Psi^- S_\delta)) \neq \{0\}$ for some matrix $S_\delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\}$, there exists nonzero vector ξ such that $P_{\mathbb{D}}(\Psi^+ - \Psi^- S_\delta))\xi = 0$. The assumption that all pairs (a_k, a_s) with $k \neq s$ are rational numbers implies that there is nonzero constant $\theta \in \mathbb{R}$ such that $\cos(\theta A) = S_\delta$, i.e., $\cos(a_k\theta) = \delta_k = \pm 1$, $k = 1, 2, \ldots, n$. Thus,

$$\sin(\theta A)(M^{-1/2}\xi) = 0$$

and

$$P_{\mathbb{D}}[\Psi^+ - \Psi^- \cos(\theta A)]M^{1/2}(M^{-1/2}\xi) = 0.$$

According to (74),(75), or (74), (81), we know that $i\theta$ is the purely imaginary spectrum of the network, which implies that the network is not stable. □

5 Application and conclusion

In this section, we shall apply the previous results to several concrete networks. The process of analyzing is as follows.

Step 1 Denote the network by $G = (V, E)$ with the vertex set V and the edge set E. Write down the outgoing and incoming incident matrix Ψ^+ and Ψ^- of G, the Dirichlet set $\mathcal{D}$ and corresponding index sets J_Ψ^+, J_Ψ^-, $\mathcal{I}_E^+(p)$, $\mathcal{I}_E^-(p)$, $\mathcal{I}_\beta$, and so on.

Step 2 By Theorem 16, together with the concrete network, reduce (77), (80), (82) or (85) to a simplified form. Using Lemma 20, present the conditions for the asymptotical stability of the controlled network.

5.1 A wave network with parallel edges

In the subsection, we consider a wave network $G = (V, E)$ with $V = \{p_1, p_2, p_3, p_4\}$ and $E = \{s_1, s_2, \ldots, s_6\}$, whose shap shown as figure 1 with fixed vertex p_4, i.e., the Dirichlet set $\mathcal{D} = \{p_4\}$. Suppose that the length of each edge is one. The outgoing incident matrix Ψ^+ and incoming incident matrix Ψ^- of G are

$$\Psi^+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Psi^- = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

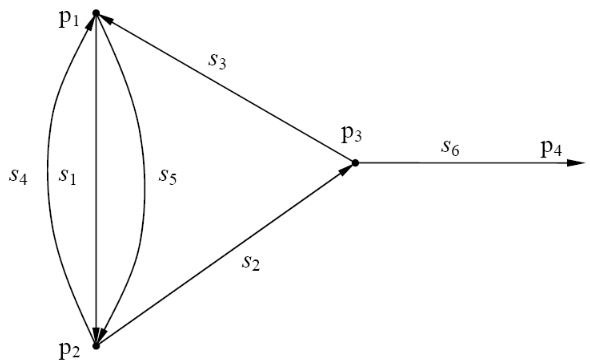

Figure 1: A wave network with parallel edges

respectively, and their corresponding index sets are

$$\mathcal{J}_\Psi^+ = \{j_1^+, j_2^+, j_3^+, j_4^+, j_5^+, j_6^+\} = \{1,2,3,2,1,3\}$$

and

$$\mathcal{J}_\Psi^- = \{j_1^-, j_2^-, j_3^-, j_4^-, j_5^-, j_6^-\} = \{2,3,1,1,2,4\},$$

respectively. The index sets are

$$\begin{array}{lll} \mathcal{I}_E(p_1)=\{1,3,4,5\}, & \mathcal{I}_E^+(p_1)=\{1,5\}, & \mathcal{I}_E^-(p_1)=\{3,4\}; \\ \mathcal{I}_E(p_2)=\{1,2,4,5\}, & \mathcal{I}_E^+(p_2)=\{2,4\}, & \mathcal{I}_E^-(p_2)=\{1,5\}; \\ \mathcal{I}_E(p_3)=\{2,3,6\}, & \mathcal{I}_E^+(p_3)=\{3,6\}, & \mathcal{I}_E^-(p_3)=\{2\}; \\ \mathcal{I}_E(p_4)=\{6\}, & \mathcal{I}_E^-(p_4)=\{6\}, & \mathcal{I}_E^+(p_4)=\emptyset. \end{array}$$

Thus, the dynamic behavior of the controlled network is governed by partial differential equations

$$\begin{cases} TY_{xx}(x,t) = MY_{tt}(x,t),\ x\in(0,\ 1),\ t>0, \\ Y(0,t) = (\Psi^+)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},t), Y(1,t) = (\Psi^-)^{\mathrm{T}}\boldsymbol{y}(\boldsymbol{v},t), \\ P_{\mathbb{D}}[\Psi^- TY_x(1,t) - \Psi^+ TY_x(0,t)] = -\beta P_{\mathbb{D}}\boldsymbol{y}_t(\boldsymbol{v},t), \\ Y(x,0) = Y_0,\ Y_t(x,0) = Y_1, \end{cases} \tag{90}$$

where $M = \{\rho_1,\dots,\rho_6\}$, $T = \{T_1,\dots,T_6\}$, $\beta = \{\beta_1,\beta_2,\beta_3\}$, $\beta_k \geq 0$, $k = 1,2,3$, $P_{\mathbb{D}} = (I_3, 0)$, $v = (p_1,p_2,p_3,p_4)^{\mathrm{T}}$ and $\boldsymbol{y}(\boldsymbol{v},t) = (y(p_1,t), y(p_2,t), y(p_3,t), 0)^{\mathrm{T}}$.

According to Theorem 8 and (26), when β_1, β_2 and β_3 satisfy conditions

$$\begin{cases} \beta_1 \neq \sqrt{\rho_1T_1} + \sqrt{\rho_3T_3} + \sqrt{\rho_4T_4} + \sqrt{\rho_5T_5}, \\ \beta_2 \neq \sqrt{\rho_1T_1} + \sqrt{\rho_2T_2} + \sqrt{\rho_4T_4} + \sqrt{\rho_5T_5}, \\ \beta_3 \neq \sqrt{\rho_2T_2} + \sqrt{\rho_3T_3} + \sqrt{\rho_6T_6}, \end{cases} \tag{91}$$

the controlled network (90) satisfies the Riesz basis property and the spectrum determined growth assumption.

Next, we analyze stability of the network (90). Let the controlled vertex set be $\mathcal{C} = \{p_1,p_2,p_3\}$, namely, $\beta_k > 0$, $k = 1,2,3$. In this case, the network (90) is controlled completely. According to Theorem 21, $a_j = \sqrt{\rho_j/T_j}$, if $\forall k,s \in \mathcal{I}_E(p_1), k\neq s$, $\forall k,s\in\mathcal{I}_E(p_2), k\neq s$ and $\forall k,s\in\mathcal{I}_E(p_3), k\neq s$, (a_k,a_s) satisfies the relation "IR$^{(1,1)}$", that is, the pairs (a_1,a_3), (a_1,a_4), (a_1,a_5), (a_3,a_4), (a_3,a_5), (a_4,a_5), (a_1,a_2), (a_2,a_4), (a_2,a_5), (a_2,a_3), (a_2,a_6) and (a_3,a_6) satisfy the relation "IR$^{(1,1)}$", then the controlled network is asymptotically stable. In fact, by the edge-edge joint conditions, the number of the pairs satisfying the relation "IR$^{(1,1)}$" can be reduced. From (82), it can be derived that

$$\begin{cases} \sin(\theta\sqrt{\rho_k/T_k})\sqrt{\rho_k}\xi_k = 0, k\in\{1,2,3,4,5,6\}, \\ \cos(\theta\sqrt{\rho_3/T_3})\sqrt{\rho_3}\xi_3 + \cos(\theta\sqrt{\rho_4/T_4})\sqrt{\rho_4}\xi_4 \\ \qquad = \sqrt{\rho_1}\xi_1 + \sqrt{\rho_5}\xi_5, \\ \cos(\theta\sqrt{\rho_1/T_1})\sqrt{\rho_1}\xi_1 + \cos(\theta\sqrt{\rho_5/T_5})\sqrt{\rho_5}\xi_5 \\ \qquad = \sqrt{\rho_2}\xi_2 + \sqrt{\rho_4}\xi_4, \\ \cos(\theta\sqrt{\rho_2/T_2})\sqrt{\rho_2}\xi_2 = \sqrt{\rho_3}\xi_3 + \sqrt{\rho_6}\xi_6. \end{cases} \tag{92}$$

So, by Theorem 16, Lemma 20 and (92), we can obtain the following conclusion.

Corollary 23. *Suppose that $\beta_k > 0$ ($k = 1,2,3$), and (91) holds. Let $a_j = \sqrt{\rho_j/T_j}$, $j = 1,2,\dots,6$. If one of the following conditions is fulfilled*

1. *(a_1,a_3), (a_1,a_4), (a_3,a_4), (a_4,a_5), (a_3,a_5) and (a_5,a_1) satisfy the relation "IR$^{(1,1)}$",*

2. *(a_1,a_2), (a_1,a_4), (a_2,a_4), (a_2,a_5), (a_4,a_5) and (a_5,a_1) satisfy the relation "IR$^{(1,1)}$",*

3. *(a_2,a_3), (a_3,a_6) and (a_6,a_2) satisfy the relation "IR$^{(1,1)}$". (a_1,a_5), (a_1,a_4) and (a_4,a_5) satisfy the relation "IR$^{(1,1)}$",*

then (92) implies that $\xi = 0$, so, the controlled network (90) is asymptotically stable.

Proof We only prove the condition (3), because the conditions (1) and (2) can be analyzed similarly. From Lemma 20 and the assumption that (a_2,a_3), (a_3,a_6) and (a_6,a_2) satisfy the relation "IR$^{(1,1)}$", it follows that $\xi_2 = \xi_3 = \xi_6 = 0$. So, (92) can be reduce to

$$\begin{cases} \sin(\theta a_k)\sqrt{\rho_k}\xi_k = 0, k\in\{1,4,5\}, \\ \cos(\theta a_4)\sqrt{\rho_4}\xi_4 = \sqrt{\rho_1}\xi_1 + \sqrt{\rho_5}\xi_5, \\ \cos(\theta a_1)\sqrt{\rho_1}\xi_1 + \cos(\theta a_5)\sqrt{\rho_5}\xi_5 = \sqrt{\rho_4}\xi_4. \end{cases} \tag{93}$$

From Lemma 20 and the assumption that (a_1,a_5), (a_1,a_4) and (a_4,a_5) satisfy the relation "IR$^{(1,1)}$", it follows that $\xi_1 = \xi_4 = \xi_5 = 0$.

Hence, $\xi = 0$ under the condition (3). $\square$

5.2 A network with multiple circuits and without boundary

Let us consider a network of shape shown as figure 2, where p_5 be a fixed point and the boundary $\partial G = \emptyset$. The Dirichlet set $\mathcal{D} = \{p_5\}$, the vertex set $V = \{p_1, \ldots, p_5\}$, the edge set $E = \{s_1, \ldots, s_8\}$ and the length of the edge s_k is ℓ_k, $k = 1, \ldots, 8$. The out-

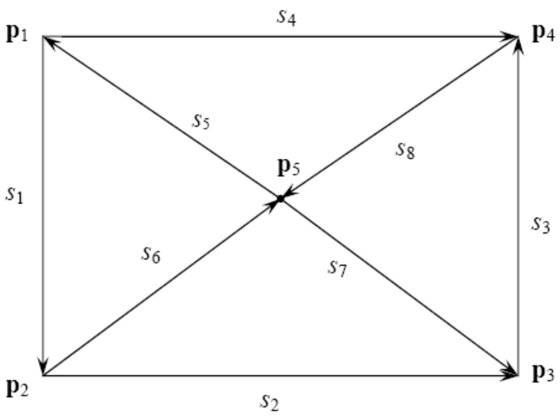

Figure 2: A network with multiple circuits and without boundary

going incidence matrix Ψ^+ and incoming incidence matrix Ψ^- are respectively

$$\Psi^+ = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\Psi^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and corresponding index sets are

$$\begin{aligned} \mathcal{J}^+_\Psi &= \{j^+_1, j^+_2, j^+_3, j^+_4, j^+_5, j^+_6, j^+_7, j^+_8\} \\ &= \{1, 2, 3, 1, 5, 2, 5, 4\} \end{aligned} \tag{94}$$

and

$$\begin{aligned} \mathcal{J}^-_\Psi &= \{j^-_1, j^-_2, j^-_3, j^-_4, j^-_5, j^-_6, j^-_7, j^-_8\} \\ &= \{2, 3, 4, 4, 1, 5, 3, 5\}. \end{aligned} \tag{95}$$

The motion of the controlled network is governed by partial differential equations

$$\begin{cases} y_{k,xx}(x,t) = y_{k,tt}(x,t),\ x \in (0,\ \ell_k), t > 0, \\ y_1(0,\ t) = y_4(0,t) = y_5(\ell_5, t), \\ y_2(0,\ t) = y_6(0,t) = y_1(\ell_1, t), \\ y_3(0,\ t) = y_2(\ell_2, t) = y_7(\ell_7, t), \\ y_8(0,\ t) = y_3(\ell_3, t) = y_4(\ell_4, t), \\ y_5(0,\ t) = y_7(0,\ t) = y_6(\ell_6, t) = y_8(\ell_8, t) = 0, \\ y_{5,x}(\ell_5,\ t) - y_{1,x}(0,\ t) - y_{4,x}(0,\ t) \\ \qquad = -\beta_1 y_t(p_1, t), \\ y_{1,x}(\ell_1,\ t) - y_{2,x}(0,\ t) - y_{6,x}(0,\ t) \\ \qquad = -\beta_2 y_t(p_2, t), \\ y_{2,x}(\ell_2,\ t) + y_{7,x}(\ell_7,\ t) - y_{3,x}(0,\ t) \\ \qquad = -\beta_3 y_t(p_3, t), \\ y_{3,x}(\ell_3,\ t) + y_{4,x}(\ell_4,\ t) - y_{8,x}(0,\ t) \\ \qquad = -\beta_4 y_t(p_4, t), \\ y_k(x,\ 0) = y_{k,0}(x),\ y_{k,t}(x,\ 0) = y_{k,1}(x), \\ k = 1, 2, \ldots, 8. \end{cases} \tag{96}$$

By the variable change $x_k := \ell_k x_k$, $k = 1, 2, \ldots, 8$, the network (96) can be rewritten as (90), where $\boldsymbol{y}(\boldsymbol{v}, t) = (y(p_1,t), y(p_2,t), y(p_3,t), y(p_4,t), 0)^{\mathrm{T}}$, $\boldsymbol{v} = (p_1, p_2, p_3, p_4, p_5)^{\mathrm{T}}$, $T = M^{-1}$, $M = \mathrm{diag}\{\ell_1, \ell_2, \ldots, \ell_8\}$, $\beta = \mathrm{diag}\{\beta_1, \ldots, \beta_4\}$, the Dirichlet projection matrix $P_{\mathbb{D}} = (I_4, 0)$ and I_4 is the identity matrix of order 4.

According to Theorem 8 and assumpation (26), if $\beta_i\,(i = 1, \ldots, 4)$ satisfy

$$\beta_i \geq 0 \text{ and } \beta_i \neq 3, i = 1, 2, 3, 4, \tag{97}$$

then the spectrum determined growth assumption holds.

Next, we discuss the stability of the network (96) under the assumption that $\beta_1, \beta_2, \beta_3 > 0$, $\beta_4 = 0$, and (97) holds. Thus, $\mathcal{J}_P = \{1, 2, 3, 4\}$, $\mathcal{I}_\beta = \{1, 2, 3\}$, $\mathcal{J}_\mathcal{D} = \{5\}$ and the network is non-completely controlled one. From (85), (94) and (95) it follows that

$$\begin{cases} \omega_k = 0, k \in \{1, 2, 3, 4, 5, 6, 7\}, \\ \sin(\theta\ell_k)\xi_k = 0, k \in \{1, 2, 5, 6, 7\}, \\ \sin(\theta\ell_8)\xi_8 + \cos(\theta\ell_8)\omega_8 = 0, \\ T_3^{-1/2}\sin(\theta\ell_3)\xi_3 = T_4^{-1/2}\sin(\theta\ell_4)\xi_4 \\ \qquad = T_8^{-1/2}\omega_8 = f(p_4), \\ \cos(\theta\ell_5)\sqrt{\ell_5}\xi_5 = \sqrt{\ell_1}\xi_1 + \sqrt{\ell_4}\xi_4, \\ \cos(\theta\ell_1)\sqrt{\ell_1}\xi_1 = \sqrt{\ell_2}\xi_2 + \sqrt{\ell_6}\xi_6, \\ \cos(\theta\ell_2)\sqrt{\ell_2}\xi_2 + \cos(\theta\ell_7)\sqrt{\ell_7}\xi_7 = \sqrt{\ell_3}\xi_3, \\ \cos(\theta\ell_3)\sqrt{\ell_3}\xi_3 + \cos(\theta\ell_4)\sqrt{\ell_4}\xi_4 = \sqrt{\ell_8}\xi_8. \end{cases} \tag{98}$$

Thus, Theorem 16 claims that the network (96) is asymptotically stable if and only if (98) implies that $\xi = \omega = 0$. Concretely, we can present a sufficient condition as follows.

Corollary 24. *Let the network be given as* (96). *Assume that* $\mathcal{I}_\beta = \{1, 2, 3\}$ *and* β *satisfies* (97). *If the* *following three conditions are fulfilled, then the controlled network is asymptotically stable.*

(1) *The pairs (ℓ_1, ℓ_2), (ℓ_2, ℓ_6), (ℓ_6, ℓ_1) and (ℓ_5, ℓ_7) satisfy the relation "IR$^{(1,1)}$";*

(2) *one of (ℓ_4, ℓ_5), (ℓ_5, ℓ_8) and (ℓ_8, ℓ_4) satisfies the relation "IR$^{(1,1)}$";*

(3) *one of (ℓ_3, ℓ_7), (ℓ_7, ℓ_8) and (ℓ_8, ℓ_3) also satisfies the relation "IR$^{(1,1)}$".*

Proof Since the pairs (ℓ_1, ℓ_2), (ℓ_2, ℓ_6) and (ℓ_6, ℓ_1) satisfy the relation "IR$^{(1,1)}$", from Lemma 20 and the equalities in the second and sixth lines of (98), it follows that

$$\begin{cases} \omega_k = 0, k \in \{1, 2, \ldots, 7\}, \xi_1 = \xi_2 = \xi_6 = 0, \\ \sin(\theta\ell_k)\xi_k = 0, k \in \{1, 2, 5, 6, 7\}, \\ \ell_5\xi_5^2 = \ell_4\xi_4^2, \ell_7\xi_7^2 = \ell_3\xi_3^2, \\ \sin(\theta\ell_8)\xi_8 + \cos(\theta\ell_8)\omega_8 = 0, \\ \sin(\theta\ell_3)\xi_3 = T_3^{1/2}T_8^{-1/2}\omega_8, \\ \sin(\theta\ell_4)\xi_4 = T_4^{1/2}T_8^{-1/2}\omega_8, \\ \cos(\theta\ell_3)\sqrt{\ell_3}\xi_3 + \cos(\theta\ell_4)\sqrt{\ell_4}\xi_4 = \sqrt{\ell_8}\xi_8. \end{cases} \tag{99}$$

Assume that $\xi_5 \neq 0$, then $\sin(\theta\ell_5) = 0$ as $\sin(\theta\ell_5)\xi_5 = 0$. So, the assumption that (ℓ_5, ℓ_7) satisfies the relation "IR$^{(1,1)}$" asserts that $\sin(\theta\ell_7) \neq 0$, which implies that $\xi_7 = 0$. Therefore, (99) can be reduced to

$$\begin{cases} \omega_k = 0, k \in \{1, 2, \ldots, 8\}, \\ \xi_1 = \xi_2 = \xi_3 = \xi_6 = \xi_7 = 0, \\ \sin(\theta\ell_k)\xi_k = 0, k \in \{1, 2, \ldots, 8\}, \\ \ell_4\xi_4^2 = \ell_5\xi_5^2 = \ell_8\xi_8^2. \end{cases} \tag{100}$$

Thus, the assumption (2) and the equalities in the second and third lines of (100) imply that $\xi_5 = 0$, which is a contradiction. Therefore, $\xi_5 = 0$. From (99), it follows that $\xi_4 = 0$. Thus, (99) can be reduced to

$$\begin{cases} \omega_k = 0, k \in \{1, 2, \ldots, 8\}, \\ \xi_1 = \xi_2 = \xi_5 = \xi_4 = \xi_6 = 0, \\ \sin(\theta\ell_k)\xi_k = 0, k \in \{1, 2, \ldots, 8\}, \\ \ell_3\xi_3^2 = \ell_7\xi_7^2 = \ell_8\xi_8^2. \end{cases} \tag{101}$$

Thus, the assumption (3) and the equalities in the last line of (101) imply that $\xi_3 = \xi_7 = \xi_8 = 0$.

Therefore, the assumptions (1),(2) and (3) claim that $\xi = \omega = 0$, which shows that the controlled network is asymptotically stable. □

Actually, there are other sufficient conditions for the stability of the controlled network. In what follows, we give a nonstable case.

Assume that (ℓ_1, ℓ_2), (ℓ_2, ℓ_6) and (ℓ_6, ℓ_1) satisfy the relation "IR$^{(1,1)}$", $\ell_5 = \ell_7 = \ell_8 = \ell$, $\ell_3 = r_3\ell/c_3$, $\ell_4 = r_4\ell/c_4$, where c_3, r_3, c_4 and r_4 are integers. Let $\omega_8 = 0$, then from (99) it follows that

$$\begin{cases} \omega = 0, \xi_1 = \xi_2 = \xi_6 = 0, \\ \sin(\theta\ell_j)\xi_j = 0, j \in \{3, 4, 5, 7, 8\}, \\ c_4\xi_5^2 = r_4\xi_4^2, c_3\xi_7^2 = r_3\xi_3^2, \\ \cos(\theta\ell_3)\sqrt{r_3/c_3}\xi_3 + \cos(\theta\ell_4)\sqrt{r_4/c_4}\xi_4 = \xi_8. \end{cases}$$

Let $\theta_k = \ell^{-1}2kc\pi, k \in \mathbb{Z}$, where c is the least common multiple of c_3 and c_4 , then

$$\begin{cases} \omega = 0, \xi_1 = \xi_2 = \xi_6 = 0, \\ \sin(\theta_k\ell_j) = 0, j \in \{3, 4, 5, 7, 8\}, \\ \xi_5 = \pm\sqrt{r_4/c_4}\xi_4, \xi_7 = \pm\sqrt{r_3/c_3}\xi_3, \\ \xi_8 = \sqrt{r_3/c_3}\xi_3 + \sqrt{r_4/c_4}\xi_4, \end{cases}$$

where ξ_3 and ξ_4 are any constants for each $k \in \mathbb{Z}$ and $\xi_3\xi_4 \neq 0$. So, $\lambda_k = i\ell^{-1}2kc\pi\,(k \in \mathbb{Z})$, is the purely imaginary eigenvalue of the controlled network (96). According to (84), for each $k \in \mathbb{Z}$, the corresponding eigenfunctions is

$$\varphi^{(k)}(x) = \left(0, 0, \sqrt{\frac{r_3\ell}{c_3}}\sin(\theta_k\ell_3 x)\xi_3, \sqrt{\frac{r_4\ell}{c_4}}\sin(\theta_k\ell_4 x)\xi_4,\right.$$
$$\pm\sin(\theta_k\ell x)\sqrt{\frac{r_4\ell}{c_4}}\xi_4, 0, \pm\sin(\theta_k\ell x)\sqrt{\frac{r_3\ell}{c_3}}\xi_3,$$
$$\left.\sin(\theta_k\ell x)\left(\sqrt{\frac{r_3\ell}{c_3}}\xi_3 + \sqrt{\frac{r_4\ell}{c_4}}\xi_4\right)\right)^{\mathrm{T}}.$$

So, the four linear independent eigenfunctions are

$$(0, 0, \sin(\theta_k\ell_3 x), 0, 0, 0, \pm\sin(\theta_k\ell x), \sin(\theta_k\ell x))^{\mathrm{T}}$$

and

$$(0, 0, 0, \sin(\theta_k\ell_4 x), \pm\sin(\theta_k\ell x), 0, 0, \sin(\theta_k\ell x))^{\mathrm{T}}.$$

Therefore the controlled network (96) is not stable when (ℓ_1, ℓ_2), (ℓ_2, ℓ_6) and (ℓ_6, ℓ_1) satisfy the relation "IR$^{(1,1)}$", $\ell_5 = \ell_7 = \ell_8 = \ell$, $\ell_3 = r_3\ell/c_3$, $\ell_4 = r_4\ell/c_4$, where c_3, r_3, c_4 and r_4 are integers.

From the above examples we can see that the four conditions in Theorem 16 have the strict logicality and regularity although they look like very complex. Once we know the connection manner of the network, we can determine sets J_Ψ^+, J_Ψ^-, $\mathcal{I}_E^+(p)$, $\mathcal{I}_E^-(p)$, $\mathcal{I}_\beta$, $\mathcal{D}$, and write down concrete conditions, e.g. (92) and (98). Then, by Lemma 20 we can judge the stability of a concrete network. Therefore the conditions in Theorem 16 are very efficient in analysis of asymptotical stability of networks. These conditions are also the criterions for designing the 1-D wave networks with circuits and parallel edges. In elastic strings network, these conditions reflect whether there exists resonance between any two strings with fixed ends, according to Remark 19. In addition, when an 1-D wave network contain self-loops, we can add a virtual vertex p_j and a virtual controller on every self-loop. The virtual controller means that the corresponding parameter β_j and γ_j in the feedback control law (15) vanish, i.e., $\beta_j = \gamma_j = 0$. So, these self-loops become the parallel edges and this modified network has no self-loop. Thus, these results obtained in this paper can also be applied to the 1-D wave networks with self-loops, which is very interesting problem. Finally we remark that the process of the stability analysis proposed in the present paper can be coded by any computer language, which will be our further work. So it can be applied easily to analysis of the concrete networks. In addition, we remark that the condition (26) is a necessary condition for the controlled network to form a Riesz basis system, but when it dose not hold, the system may still be stable. This also will be content of our discussion further.

6 Appendix

In this appendix, we introduce some notions of graph and basic results used in the paper.

Definition 25. *The number of edges joining p_i and p_j is called the degree of p_i and p_j, denoted by $\deg(p_i, p_j)$, the set defined by*

$$\mathcal{I}_E(p_i, p_j) = \{k \in \mathcal{I}_E \,|\, \epsilon_k \text{ joins } p_i \text{ and } p_j,\ \epsilon_k \in E\},$$

is called the index set of incident between p_i and p_j.

Obviously, $\deg(p_i, p_j) = \#\mathcal{I}_E(p_i, p_j)$ and

$$\mathcal{I}_E(p_i) = \mathcal{I}_E^-(p_i) \cup \mathcal{I}_E^+(p_i) = \bigcup_{j \in \mathcal{I}_V} \mathcal{I}_E(p_i, p_j).$$

Definition 26. *Let $G = (V, E)$ be a connected digraph without loops. The adjacency matrix of G is the $m \times m$ matrix $A_G = (a_{i,j})$, where $a_{i,j} = \deg(p_i, p_j)$ is the number of edges in G joining p_i and p_j.*

From the definition it follows that

$$\Psi\Psi^{\mathrm{T}} = D_G - A_G \tag{102}$$

where $D_G = \operatorname{diag}\{\deg(p_1), \deg(p_2), \ldots, \deg(p_m)\}$.

Proposition 27. *Let G be a connected digraph without loops, Ψ and $A_G = (a_{i,j})$ be the incidence matrix and the adjacency matrix, respectively. Then each column (or row) of A_G has at least a nonzero element, and $\Psi\Psi^{\mathrm{T}}$ is an irreducible and diagonal predominant symmetrical matrix.*

Proof If $a_{k,j} = 0$ for all $j \in \{1, 2, \ldots, m\}$, then p_k is not incident with the other vertexes, which contradicts G is a connected digraph. So, each column of A_G has at least one nonzero element.

Since G is connected graph, for any two vertices, p_i and p_j, there exists a sequence of edges that joins p_i and p_j, that is, there exists a sequence consisting of nonzero numbers:

$$a_{i,i_1}, a_{i_1,i_2}, \ldots, a_{i_k,j}.$$

Thus, according to (102) and [33, Theorem 2.3.5], $\Psi\Psi^{\mathrm{T}}$ is irreducible. From (102) and the definition of A_G, it follows that $\Psi\Psi^{\mathrm{T}}$ is a diagonal predominant symmetrical matrix. □

Lemma 28. *Let G be a connected graph without loops, and Ψ be its incidence matrix. Let $W = \operatorname{diag}\{w_1, \ldots, w_n\}$, $w_k > 0$, $k = 1, 2, \ldots, n$. Thus,*

- **a** *the $m \times m$ matrix $\Psi W \Psi^{\mathrm{T}}$ is irreducible and diagonal predominant;*
- **b** *the $m \times m$ matrix $\gamma + \Psi W \Psi^{\mathrm{T}}$ is symmetric and positive definite matrix, where $\gamma = \operatorname{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \geq 0$ with $\sum_{k=1}^m \gamma_k > 0$;*
- **c** *Let $J = \{j_1, j_2, \ldots, j_r\}$, $r < m$, where $j_1, j_2, \ldots, j_r$ distinguish indices in $\mathcal{I}_V$, and P_J be a linear transformation from $\mathbb{C}^m$ to $\mathbb{C}^r$ defined by $P_J = (e_{j_1}^{\mathrm{T}}, e_{j_2}^{\mathrm{T}}, \ldots, e_{j_r}^{\mathrm{T}})^{\mathrm{T}}$, where e_{j_k} is the j_k-th column of the identity matrix of order m, $k = 1, 2, \ldots, r$. Then $P_J \Psi W \Psi^{\mathrm{T}} P_J^{\mathrm{T}}$ is a $r \times r$ symmetric positive definite matrix.*

Proof Let

$$\Psi = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,n} \\ \psi_{2,1} & \psi_{2,2} & \cdots & \psi_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{m,1} & \psi_{m,2} & \cdots & \psi_{m,n} \end{pmatrix},$$

then the (i, j) entry of the matrix $\Psi W \Psi^{\mathrm{T}}$ is $\sum\limits_{k=1}^{n} \psi_{i,k} w_k \psi_{j,k}$. We note that the sum of entries of each column (or row) of $\Psi W \Psi^{\mathrm{T}}$ is zero, so

$$\sum_{k=1}^{n} \psi_{j,k} w_k \psi_{j,k} = -\sum_{i=1, i\neq j}^{m} \sum_{k=1}^{n} \psi_{i,k} w_k \psi_{j,k}.$$

From the relation

$$\sum_{k=1}^{n} \psi_{i,k} w_k \psi_{j,k} = \begin{cases} 0, & \text{if } p_i \text{ is not incident with } p_j, \\ -\sum\limits_{k \in \mathcal{I}_E(p_i,p_j)} w_k, & \text{if } p_i \text{ is incident with } p_j, \\ \sum\limits_{k \in \mathcal{I}_E(p_i)} w_k, & \text{if } i = j, \end{cases} \tag{103}$$

it follows that $\Psi W \Psi^{\mathrm{T}}$ is a diagonal predominant matrix.

Similar to the proof of Proposition 27, it can be shown that $\Psi W \Psi^{\mathrm{T}}$ is irreducible.

Next, we shall prove (b). Since $\gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \geq 0$ and $\sum_{k=1}^{m} \gamma_k > 0$, $\gamma + \Psi W \Psi^{\mathrm{T}}$ is a weak diagonal predominant matrix. Thus, by [33, Theorem 2.3.10], it follows that $\gamma + \Psi W \Psi^{\mathrm{T}}$ a symmetric and positive definite matrix.

To prove (c), we note at first that $P_J \Psi W \Psi^{\mathrm{T}} P_J^{\mathrm{T}}$ is a $r \times r$ symmetrical matrix obtained by eliminating all rows and columns but j_1-th, j_2-th, $\ldots$ and j_r-th row and column from $\Psi W \Psi^{\mathrm{T}}$, whose i-th diagonal entry is $\sum\limits_{k \in \mathcal{I}_E(p_{j_i})} w_k$, $i = 1, 2, \ldots, r$ and whose (i, s) entry equals to $\sum\limits_{k=1}^{n} \psi_{j_i,k} w_k \psi_{j_s,k}$, $i, s = 1, 2, \ldots, r$. So according to (a) and [33, Theorem 2.3.5], $P_J \Psi W \Psi^{\mathrm{T}} P_J^{\mathrm{T}}$ also is irreducible.

By (103), for $k \in \{1, 2, \ldots, r\}$,

$$\begin{aligned} \sum_{s \in \mathcal{I}_E(p_{j_k})} w_s &= -\sum_{i=1, i \neq j_k}^{m} \sum_{s=1}^{n} \psi_{i,s} w_s \psi_{j_k,s} \\ &\geq -\sum_{i=1, i \neq j_k}^{r} \sum_{s=1}^{n} \psi_{j_i,s} w_s \psi_{j_k,s}, \end{aligned}$$

and there is at least an inequality such that the relation " $>$ " holds. Otherwise, there exists a row (column) whose elements are all zeros and that is eliminated from $\Psi W \Psi^{\mathrm{T}}$. Thus, from (102), it follows that $a_{k,j} = 0, \forall k \in \mathcal{I} = \{1, 2, \ldots, m\}$, which is in contradiction to the fact that G is a connected. Thus, $P_J \Psi W \Psi^{\mathrm{T}} P_J^{\mathrm{T}}$ is a weak diagonal predominant symmetrical matrix. Therefore, by [33, Theorem 2.3.10], $P_J \Psi W \Psi^{\mathrm{T}} P_J^{\mathrm{T}}$ is a symmetric and positive definite matrix. □

Lemma 29. *Assume that G is a digraph without loop, Ψ^+ and Ψ^- are its outgoing incidence matrix and incoming incidence matrix, respectively. Let P_J be defined by (**c**) in Lemma 28 and $W = \mathrm{diag}\{w_1, \ldots, w_n\}$. Then,*

(1) $(\Psi^-) W (\Psi^-)^{\mathrm{T}} =$

$$\mathrm{diag}\left\{ \sum_{s \in \mathcal{I}_E^-(p_1)} w_s, \ldots, \sum_{s \in \mathcal{I}_E^-(p_i)} w_s, \ldots, \sum_{s \in \mathcal{I}_E^-(p_m)} w_s \right\}.$$

(2) $(\Psi^+) W (\Psi^+)^{\mathrm{T}} =$

$$\mathrm{diag}\left\{ \sum_{s \in \mathcal{I}_E^+(p_1)} w_s, \ldots, \sum_{s \in \mathcal{I}_E^+(p_i)} w_s, \ldots, \sum_{s \in \mathcal{I}_E^+(p_m)} w_s \right\}.$$

Proof Let

$$\Psi^- = \begin{pmatrix} \psi_{1,1}^- & \psi_{1,2}^- & \cdots & \psi_{1,n}^- \\ \psi_{2,1}^- & \psi_{2,2}^- & \cdots & \psi_{2,n}^- \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{m,1}^- & \psi_{m,2}^- & \cdots & \psi_{m,n}^- \end{pmatrix},$$

then the (i, j) entry of the matrix $(\Psi^-) W (\Psi^-)^{\mathrm{T}}$ is $\sum\limits_{k=1}^{n} \psi_{i,k}^- w_k \psi_{j,k}^-$, which satisfies

$$\sum_{k=1}^{n} \psi_{i,k}^- w_k \psi_{j,k}^- = \begin{cases} \sum\limits_{k \in \mathcal{I}_E^-(p_i)} w_k, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

since each edge ϵ_k has one head or tail. Hence the equality in (**1**) holds. The equality in (**2**) can be proven similarly. □

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