Approximate Solution of Systems of Singular Integro-Differential Equations by Reduction Method in Generalized Holder spaces

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Abstract: The computation schemes of reduction method for approximate solution of systems of singular integro- differential equations have been elaborated. The equations are defined on an arbitrary smooth closed contour of complex plane. Estimates of the rate of convergence are obtained in generalized Hölder spaces.

Key–Words:Reduction Method, Generalized Holder Spaces, systems of singular integro-differential equations

1 Introduction

Singular integral equations (SIE) and singular integro- differential equations with Cauchy kernels (SIDE) and their systems arise in different problems of elasticity theory, aerodynamics, mechanics, thermo elasticity, queuing analysis, mathematical biology. (see [1]-[5] and the literature cited therein). The general theory of SIE and SIDE has been widely investigated in the last decades [6]-[10].

It is well known that problem (4)-(5) admits a closed-form solution only in rare special cases. Even in these cases, evaluating the solution numerically can be very complicated and laborious. In this connection, it is of interest to elaborate the approximate methods for problem (4)-(5) with the corresponding theoretical background.

In this article we study the reduction method for approximative solution of systems of SIDE. We prove the convergence in Generalized Holder spaces. Note that, for the case of the unit circle, this problem was studied in a number of papers (see [11]-[14] and the bibliography therein), and, in the case arbitrary smooth closed contours, problem (4)-(5) was solved approximately for one-dimensional SIDE and one dimensional SIE by collocation method in Lebesgue, Holder spaces [16]-[20].

Transition to another contour, different from the standard one, implies many difficulties. It should be noted that conformal mapping from the arbitrary smooth closed contour to the unit circle does not solve the problem. Moreover, it makes more difficult.

- The coefficients, kernel and right part of transformed equation lose their smoothness;
- The power of smoothness appears in convergence speed of collocation method. So that the evaluations of convergence speed will depend from particular contour;
- The numerical schemes of researched methods become more difficult. The singularity appears in new kernel.

In Section 2 we introduce the main definitions and notations. We present the numerical scheme of reduction method in Section 3. In Section 4 we formulate auxiliary results. We use these results to prove the convergence theorems in Section 5. We prove the convergence theorem of reduction method in Section 5.

2 Main definitions

Let Γ be a smooth closed contour bounding a simply connected domain D^+ that contains thepoint $z=0$, and let $D^- = C\setminus\{D^+\cup\Gamma\}$. The class of such contours will be denoted by Λ. Let $w=\Phi(z)$ be a conformal function mapping D^- onto the domain $|w|>1$ such that $\Phi(\infty)=\infty$, and $\lim\limits_{z\to\infty} z^{-1}\Phi(z)=\alpha>0$, and let $z=\phi(w)$ be the inverse function of $\Phi(z)$. Further, let a function $w=F(z)$ be a conformal mapping of D^+ onto the domain $|w|>1$ such that $F(0)=\infty$ and $\lim\limits_{z\to 0} zF(z)=\beta>0$, and let $z=\varphi(w)$ be the inverse function. In a neighborhood of the point at infinity, the function $\Phi(z)$ can be expanded in a series $\Phi(z)=\alpha/z+\alpha_0+\alpha_1/z+\alpha_2/z^2+\ldots$, and the inverse function has the form $z=\phi(w)=\gamma w+\gamma_0+\gamma_1/w+\gamma_2/w^2+\ldots$, $|w|>1$, where $\gamma=1/\alpha>0$. In a neighborhood of zero, the function $F(z)$ admits the expansion $F(z)=\beta z^{-1}+\beta_0+\beta_1 z+\beta_2 z^2+\ldots$. Throughout the following, one can assume without loss of generality that $\alpha=1$ and $\beta=1$ [21]. By $\Phi_k(z)$ $(k=0,1,2,\ldots)$ we denote the polynomial comprising the terms with nonnegative powers of z in the Laurent expansion of the function $[\Phi(z)]^k$, and by $F_k(1/z)$ $(k=1,2,\ldots)$ we denote the polynomial comprising the terms with negative powers of z in the expansion of $[F(z)]^k$. Let S_n be the operator that takes each continuous function $g(t)$ on Γ to the nth partial sum of its Faber-Laurent series:

$$(S_n g)(t)=\sum_{k=0}^{n} a_k\Phi_k(t)$$

$$+\sum_{k=1}^{n} b_k F_k\Big(\frac{1}{t}\Big),\quad t\in\Gamma,$$

$$a_k=\frac{1}{2\pi i}\int\limits_{\Gamma_0}\frac{g(\phi(\tau))}{\tau^{k+1}}d\tau,\quad k=0,1,2,\ldots,$$

$$b_k=\frac{1}{2\pi i}\int\limits_{\Gamma_0}\frac{g(\varphi(\tau))}{\tau^{k+1}}d\tau,\quad k=1,2,\ldots$$

By $\omega(\delta)$ $(\delta\in(0,h])$ $h=$ *diam*(Γ) we denote the arbitrary module of continuity. By $[H(\omega)]_m$ we denote Banach space m-dimensional vector functions(v.f) satisfying on Γ the Holder condition[22]. The norm is defined as

$$\forall g(t) = \{g_1(t), \dots, g_m(t)\},$$

$$||g||_{\omega,m} = \sum_{k=1}^{m} (||g_k||_C + H(g_k, \omega)) \quad (1)$$

$$||g||_C = \max_{t\in\Gamma} |g(t)|,$$

$$H(g;\omega) = \sup_{\sigma\in(0;l]} \frac{\omega(g;\sigma)}{\omega(\sigma)},$$

$\omega(g;\sigma)$ is the module of continuity of function $g(t)$ on Γ. We consider only the spaces $[H(\omega)]_m$ with modules of continuity satisfying the Barry-Stechkin conditions [22]:

$$\int_0^h \frac{\omega(\xi)}{\xi} < \infty, \qquad (2)$$

$$\int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + \delta \int_\delta^h \frac{\omega(\xi)}{\xi^2} d\xi = O(\omega(\delta)),\ \delta \to +0. \qquad (3)$$

In this case the singular integral operator with Cauchy kernel is bounded in Generalized Holder spaces[22]. By $[H^{(r)}(\omega)]_m,\ r \geq 0\ ([H^0(\omega)]_m = [H(\omega))]_m$ we denote the spaces of $r-$ times continuous-differentiable m-dimensional vector functions. The $r-$order derivatives of these v.f. are elements of space $[H(\omega)]_m$. Recall that if $\omega(\delta) = \delta^\alpha,\ \alpha \in (0;h]$, then $H(\omega) = H_\alpha$ is a classical Hölder space with exponent α. The space $[H(\omega)]_m$ is a Banach nonseparable space. So the approximation of whole class of functions $[H(\omega)]_m$ by norm (1) with the help of finite-dimensional approximation is impossible. But the problem can be solved in some subset of $[H(\omega)]_m$. Let ω_1 and ω_2 be two modulus of continuity satisfying the conditions (2) and (3). We suppose that the function

$$\Phi(\delta) = \frac{\omega_1(\delta)}{\omega_2(\delta)}$$

$\delta \in (0;h]$ is nondecreasing on $(0;h]$ and $\lim_{\delta\to 0} \Phi(\delta) = 0$.

3 Numerical schemes

We consider the system of the SIDE in $[H(\omega)]_m$

$$(Mx \equiv) \sum_{r=0}^{q} \Big[\tilde{A}_r(t)x^{(r)}(t) + \tilde{B}_r(t)\frac{1}{\pi i}\int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i}\int_\Gamma K_r(t,\tau)\cdot x^{(r)}(\tau)d\tau\Big] = f(t), \quad t \in \Gamma, \qquad (4)$$

where $\tilde{A}_r(t), \tilde{B}_r(t),\ K_r(t,\tau)\ (r = 0,\dots,q)$ are known $m \times m$ matrix functions(m.f.), the elements of the m.f. belong to $[H(\omega)]_m$, $f(t)$ is a known $m-$dimensional v.f. in $[H(\omega)]_m$, $x^{(0)}(t) = x(t)$ is an unknown v.f. in $[H(\omega)]_m$, $x^{(r)}(t) = \frac{d^r x(t)}{dt^r}$ $(r = 1,\dots,q)$, and q is a positive integer. We suppose that the v.f. $x^{(q)}(t)$ belongs to $[H(\omega)]_m$, that is,

$$x^{(k)}(t) \in [H(\omega)]_m, \qquad k = 0,\dots,q-1.$$

We search for a solution of (4) in the class of v.f. satisfying the condition

$$\frac{1}{2\pi i}\int_\Gamma x(\tau)\tau^{-k-1}d\tau = 0,$$

$$k = 0, \ldots, q-1. \tag{5}$$

We note that the solutions of SIDE (4) can differ by a constant [7, 15]. In this case we cannot investigate the solution of SIDE (4) directly. That is why we introduce additional conditions (5) for v.f. $x(t)$.

We denote the system (4) with conditions (5) as problem "(4)-(5)". Using the Riesz operators $P = \frac{1}{2}(I + S)$, where I is the identity operator and $(Sx)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{x(\tau)}{\tau - t} d\tau$ is the singular operator (with Cauchy kernel), we rewrite the system (4) in the following form:

$$(Mx) \equiv \sum_{r=0}^{q} \Big[A_r(t)(Px^{(r)}(t)$$
$$+B_r(t)(Qx^{(r)}(t)$$
$$+\frac{1}{2\pi i}\int_{\Gamma} K_r(t,\tau)\cdot x^{(r)}(\tau)d\tau\Big] \tag{6}$$
$$= f(t),\, t \in \Gamma,$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = 0, \ldots, q$ are $m \times m$ m.f. The elements belong to $[H(\omega)]_m$.
We seek an approximate solution of problem (4)-(5) in the form of a polynomial

$$x_n(t) = t^q \sum_{k=0}^{n} \alpha_k^{(n)} \Phi_k(t) + \sum_{k=1}^{n} \alpha_{-k}^{(n)} F_k\Big(\frac{1}{t}\Big),$$
$$t \in \Gamma, \tag{7}$$

with unknown m-dimensional numerical vectors $\alpha_k = \alpha_k^{(n)}$, $k = -n, \ldots, n$. The m-dimensional numerical vectors α_k, $k = \overline{-n, n}$, are found from the condition:

$$S_n[Mx_n - f] = 0,$$
$$S_n M S_n x_n = S_n f, \tag{8}$$

for the unknown v.f $x_n(t)$ of the form (7). Note that Eq. (8) is a system of $(2n+1) * m$ linear algebraic equations(SLAE) with $(2n+1) * m$ unknowns α_k, $k = -n, \ldots, n$. Note that the matrix of this system is determined by the Faber-Laurent coefficients of the m.f. $A_r(t)$ and $B_r(t)$:

$$\frac{1}{2\pi i}\int\limits_{\Gamma} h_r(t,\tau)\Phi_k(\tau)\, d\tau,\ k = \overline{0, n},$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma} h_r(t,\tau) F_k\Big(\frac{1}{\tau}\Big)\, d\tau,\ k = \overline{1, n},\ r = \overline{0, q}.$$

In what follows, we give a theoretical justification of the reduction method, i.e., derive conditions providing the solvability (starting from some indices n) of Eq. (8) and the convergence of the approximate solutions (7) to the exact solution $x(t)$ of problem (4)-(5).

Let $[\overset{o}{H}{}^{(q)}(\omega_2)]_m$ be a subspace of $[H^{(q)}(\omega_2)]_m$ space. The elements from $[\overset{o}{H}{}^{(q)}(\omega_2)]_m$ satisfy the condition (5) with the norm as in $[H^{(q)}(\omega_2)]_m$.

Theorem 1 *Let the following conditions be satisfied:*

1) M.F. $A_r(t)$, $B_r(t)$ *and* $K_r(t,\tau)$, $r = 0, \ldots, q$, *belong to the space* $[H(\omega_1)]_m$;

2) $\mathrm{Det}(A_q(t)) \neq 0\, \mathrm{Det}(B_q(t))$;

3) the left partial indexes of M.F. $A_q(t)$ are equal to zero and right partial indexes of M.F. B_q are equal to q;

4) the operator $M : [\overset{o}{H}{}^{(q)}(\omega_2)]_m \to [H(\omega_2)]_m$.

5) $\Phi(\delta) = \dfrac{\omega_1(\delta)}{\omega_2(\delta)}$ is nondecreasing on $(0; h]$.

If

$$\lim_{\delta\to+0}\Phi(\delta)\ln^2(\delta) = 0$$

then starting from indices $n \geq n_1$ the SLAE (8) of reduction method is uniquely solvable. The approximate solutions $x_n(t)$ given by formula (7) converge in the norm of space $[H^{(q)}(\omega_2)]_m$ to the exact solution of problem (4)-(5). The following estimation is true:

$$||x - x_n||^m_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2 n\right). \quad (9)$$

4 Auxiliary Results

The vector functions $\dfrac{d^q(Px)(t)}{dt^q}$ and $\dfrac{d^q(Qx)(t)}{dt^q}$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\left.\begin{array}{ll} \dfrac{d^q(Px)(t)}{dt^q} = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{v(\tau)}{\tau - t}d\tau, & t \in F^+, \\ \dfrac{d^q(Qx)(t)}{dt^q} = \dfrac{t^{-q}}{2\pi i}\displaystyle\int_\Gamma \dfrac{v(\tau)}{\tau - t}d\tau, & t \in F^-. \end{array}\right\} \quad (10)$$

Using the integral representation (10) we reduce the problem (4)-(5) to an equivalent system of SIE (in terms of solvability):

$$(\Upsilon v \equiv) C(t)v(t) + \frac{D(t)}{\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_\Gamma h(t,\tau)v(\tau)d\tau = f(t), \quad t \in \Gamma, \quad (11)$$

for unknowns $v(t)$ where

$$C(t) = \frac{1}{2}[A_q(t) + t^{-q}B_q(t)],$$

$$D(t) = \frac{1}{2}[A_q(t) - t^{-q}B_q(t)], \quad (12)$$

$$h(t,\tau) = \frac{1}{2}\left[K_q(t,\tau) + K_q(t,\tau)\tau^{-n}\right] - \frac{1}{2\pi i}\int_\Gamma \left[K_q(t,\bar t) - K_q(t,\bar t)\bar t^{-n}\right]\frac{d\bar t}{\bar t - \tau}$$
$$+ \sum_{j=0}^{q-1}\left[A_j(t)\tilde M_j(t,\tau) + \int_\Gamma K_j(t,\bar t)\tilde M_j(\bar t,\tau)d\bar t\right]$$
$$- \sum_{j=0}^{q-1}\left[B_j(t)\tilde N_j(t,\tau) + \int_\Gamma K_j(t,\bar t)\tilde N_j(\bar t,\tau)d\bar t\right], \quad (13)$$

where $\tilde M_j(t,\tau), \tilde N_j(t,\tau)$ $j = 0,\ldots,q$ are known Hölder continuous M.F. An explicit form for these functions is given in [15]. By virtue of the properties of the m.f. $\tilde M_j(t,\tau)$, $\tilde N_j(t,\tau)$, $K_j(t,\tau), A_j(t), B_j(t)$, $j = 0,\ldots,q$, we obtain that the m.f. $h(t,\tau)$ is a Hölder continuous M.F. Note that the right hand sides in (11) and (4) coincide by conditions (5).

Lemma 2 *The system of SIE (11) and problem (4)-(5) are equivalent in terms of solvability. That is, for each solution*

$v(t)$ of system of SIE (11), there is a solution of problem (4)-(5), determined by the formulae

$$(Px)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)[(\tau-t)^{q-1}$$

$$\log\left(1-\frac{t}{\tau}\right) + \sum_{k=1}^{q-1}\alpha_k\tau^{q-k-1}t^k]d\tau,$$

$$(Qx)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)\tau^{-q}[(\tau-t)^{q-1}$$

$$\log\left(1-\frac{\tau}{t}\right) + \sum_{k=1}^{q-2}\beta_k\tau^{q-k-1}t^k]d\tau, \quad (14)$$

where $\alpha_k, k = 1, \ldots, q-1$, and $\beta_k, k = 1, \ldots, q-2$ are vector numbers. On the other hand, for each solution $x(t)$ of the problem (4)-(5) there is a solution $v(t)$

$$v(t) = \frac{d^q(Px)(t)}{dt^q} + t^q\frac{d^q(Qx)(t)}{dt^q},$$

of system of SIE (11). Furthermore, for linearly independent solutions of (11), there are corresponding linearly independent solutions of problem (4)-(5) from (14) and vice versa.

In formula (14), both $\log(1-t/\tau)$ and $\log(1-\tau/t)$, for given τ, there are branches that vanish at $t = 0$ and $t = \infty$, respectively. We formulate the theorems about the theoretical background of numerical schemes of the reduction for system of SIE

Theorem 3 *Let the following conditions be satisfied:*

a) M.F. $C(t), D(t)$ and $h(t,\tau) \in [H(\omega_1)]_m$;

b) $det(C(t)) \neq 0$, $det(D(t)) \neq 0$, $t \in \Gamma$;

c) the left partial indexes of m.f. $C(t)$ right partial indexes of m.f.$D(t)$ are equal to zero;

d) operator $\Upsilon = aP + bQ + H$ be invertible in $[H(\omega_2)]_m$, H is integral operator with kernel $h(t,\tau)$, P and Q are Riesz projectors $P = \frac{1}{2}(I+S)$, $Q = \frac{1}{2}(I-S)$, S is a singular operator with Cauchy kernel.

e) $\Phi(\delta) = \dfrac{\omega_1(\delta)}{\omega_2(\delta)}$ is nondecreasing on $(0;h]$.

If

$$\lim_{\delta\to+0}\Phi(\delta)\ln^2(\delta) = 0$$

then the operator of the reduction method

$$\Upsilon_n = S_n[aP + bQ + H]S_n,$$

of operator $\Upsilon v = f$ for large enough numbers $(n \geq n_0)$ is invertible in the space $[H(\omega_2)]_m$ and the approximate solutions $v_n(t) = \Upsilon_n^{-1}S_n f$ converges to the function $v = \Upsilon^{-1}f$. The following estimation is true:

$$||v - v_n||^m_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2\right). \quad (15)$$

5 Proof of convergence theorem

In this section we prove the Theorem 1.
Proof We should show that for numbers $n \geq n_0$ large enough the operator is invertible. The operator M acts from the

subspace $[\overset{o}{X}_n]_m = t^q P[X_n]_m + Q[X_n]_m$ (the norm defined as in $[H(\omega_2)^{(q)}]_m$) to the space $[X_n]_m = S_n[H_{\omega_2}]_m$ of m dimensional polynomials of the form $\sum_{k=-n}^{n} r_k t^k$ (the norm as in $[H(\omega_2)]_m$).

In a similar way, by using formulas (10), we represent the v.f.

$$d^q(P(x_n)(t))/dt^q, \quad d^q(Q(x_n)(t))/dt^q$$

by Cauchy type integrals with the same density $v_n(t)$:

$$\frac{d^q(P(x_n)(t))}{dt^q} = \frac{1}{2\pi i}\int_\Gamma \frac{v_n(\tau)}{\tau - t}d\tau,\ t \in F^+,$$

$$\frac{d^q(Q(x_n)(t))}{dt^q} = \frac{t^{-q}}{2\pi i}\int_\Gamma \frac{v_n(\tau)}{\tau - t}d\tau,\ t \in F^-. \quad (16)$$

By taking account of the formulas $(Px)^{(r)}(t) = P(x^{(r)})(t)$ and $(Qx)^{(r)}(t) = Q(x^{(r)})(t)$, $r = 1, 2, \ldots,$ and the relations

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!}t^{k+q-r},\ k = 0, \ldots, n,$$

$$(t^{-k})^{(r)} = (-1)^r \frac{(k+r-1)!}{(k-1)!}t^{-k-r},\ k = 1, \ldots, n,$$

from (16), we obtain

$$v_n(t) = \sum_{k=0}^{n} \frac{(k+q)!}{k!}t^k\xi_k + (-1)^q \sum_{k=1}^{n} \frac{(k+q-1)!}{(k-1)!}t^{-k}\xi_{-k}$$

Consequently $v_n(t) \in [X_n]_m$; here we have used the fact that the polynomials $x_n(t)$, given by (7) can be represented uniquely in the form

$$t^q \sum_{k=0}^{n} \xi_k t^k + \sum_{k=-n}^{-1} \xi_k t^k.$$

Using of the representations (16), Eq. (8), as well as the problem (4)-(5) can be reduced to an equivalent equation (in the sense solvability)

$$S_n R S_n v_n = S_n f, \quad (17)$$

Treated as an equation in the subspace $[X_n]_m$. Obviously, Eq. (17) is the equation of the method of reduction over Faber-Laurent polynomials for the singular integral equation (11), and for singular integral equations, the method of reduction over Faber-Laurent polynomials was considered in [17], where sufficient conditions for the solvability and convergence of this method were obtained. Assumptions in Theorem 3 provide the validity of all assumptions in Theorem 1. We have that the Eq. (17) with $n \geq n_1$ is uniquely solvable; moreover, the approximate solutions $v_n(t)$ of this equation converge to the exact solution $v(t)$ of the system of singular integral equation (11) in the norm of the space $[H(\omega_2)]_m$ as $n \to \infty$:

$$\|v_n - v\|^m_{\omega_2,q} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2(\delta)\right). \quad (18)$$

v.f. $x_n(t)$ can be expressed via the v.f. $v_n(t)$ by formulas (14). From definition of the norm in the space $[H^{(q)}_{\omega_2}]_m$ together with (18) implies estimate (9).

The proof of Theorem 1 is complete.

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