Application Of A Generalized Bernoulli Sub-ODE Method For Finding Traveling Solutions Of Some Nonlinear Equations

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Abstract: In this paper, a generalized Bernoulli sub-ODE method is proposed to construct exact traveling solutions of nonlinear evolution equations. We apply the method to establish traveling solutions of the variant Boussinseq equations, (2+1)-dimensional NNV equations and (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations. As a result, some new exact traveling wave solutions are found.

Key–Words: Bernoulli sub-ODE method, traveling wave solutions, variant Boussinseq equations, NNV equations, Boussinesq and Kadomtsev-Petviashvili equations.

1 Introduction

It is well known that nonlinear evolution equations (NLEEs) are widely used to describe many complex physical phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. So, the powerful and efficient methods to find analytic solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the homogeneous balance method [1, 2], the hyperbolic tangent expansion method [3, 4], the trial function method [5], the tanh-method [6-8], the non-linear transform method [9], the inverse scattering transform [10], the Backlund transform [11, 12], the Hirota's bilinear method [13, 14], the generalized Riccati equation method [15, 16], the Weierstrass elliptic function method [17], the theta function method [18-20], the sine-cosine method [21], the Jacobi elliptic function expansion [22, 23], the complex hyperbolic function method [24-26], the truncated Painleve expansion [27], the F-expansion method [28], the rank analysis method [29], the exp-function expansion method [30], the (G'/G)-expansion method [31-40] and so on.

In [41], we proposed a new Bernoulli sub-ODE method to construct exact traveling wave solutions for NLEEs. In this paper, we will apply the Bernoulli sub-ODE method to construct exact traveling wave solutions for some special nonlinear equations. First, we reduce the nonlinear equations to ODEs by a traveling wave variable transformation. Second, we suppose the solution can be expressed as an polynomial of single variable G, where $G = G(\xi)$ satisfied the Bernoulli equation. Then the degree of the polynomial can be determined by the homogeneous balance method, and the coefficients can be obtained by solving a set of algebraic equations.

The rest of the paper is organized as follows. In Section 2, we describe the Bernoulli sub-ODE method for finding traveling wave solutions of nonlinear evolution equations, and give the main steps of the method. In the subsequent sections, we will apply the method to find exact traveling wave solutions of the variant Boussinseq equation, (2+1)-dimensional NNV equations and (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations. In the last Section, some conclusions are presented.

2 Description Of The Bernoulli Sub-ODE Method

In this section we describe the Bernoulli Sub-ODE Method.

First we present the solutions of the following

ODE:

$$G' + \lambda G = \mu G^2, \quad (1)$$

where $\lambda \neq 0$, $G = G(\xi)$.

When $\mu \neq 0$, Eq. (1) is the type of Bernoulli equation, and we can obtain the solution as

$$G = \frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}, \quad (2)$$

where d is an arbitrary constant.

When $\mu = 0$, the solution of Eq. (1) is given by

$$G = de^{-\lambda\xi}, \quad (3)$$

where d is an arbitrary constant.

Suppose that a nonlinear equation, say in two or three independent variables $x,\ y,\ t$, is given by

$$P(u, u_t, u_x, u_y, , u_{tt}, u_{xt}, u_{xx}, u_{xy} ...) = 0, \quad (4)$$

where $u = u(x, y, t)$ is an unknown function, P is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By using the solutions of Eq. (1), we can construct a serials of exact solutions of nonlinear equations:

Step 1. We suppose that

$$u(x,\ y,\ t) = u(\xi), \quad \xi = \xi(x, y, t). \quad (5)$$

The traveling wave variable (5) permits us reducing (4) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \quad (6)$$

Step 2. Suppose that the solution of (6) can be expressed by a polynomial in G as follows:

$$u(\xi) = a_m G^m + a_{m-1} G^{m-1} +\ ... + a_0, \quad (7)$$

where $G = G(\xi)$ satisfies Eq. (1), and $a_m,\ a_{m-1}$ ..., $a_0,\ \mu$ are constants to be determined later with $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (6).

Step 3. Substituting (7) into (6) and using (1), collecting all terms with the same order of G together, the left-hand side of (6) is converted to another polynomial in G. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_m,\ a_{m-1} ...,\ k,\ c,\ \lambda$ and μ.

Step 4. Solving the algebraic equations system in Step 3, and by using the solutions of Eq. (1), we can construct the traveling wave solutions of the nonlinear evolution equation (6).

In the following sections, we will apply the method described above to some examples.

3 Application Of The Bernoulli Sub-ODE Method For The Variant Boussinseq Equations

In this section, we will consider the variant Boussinseq equations [42, 43]:

$$u_t + uu_x + v_x + \alpha u_{xxt} = 0, \quad (8)$$

$$v_t + (uv)_x + \beta u_{xxx} = 0, \quad (9)$$

where α and β are arbitrary constants, $\beta > 0$.

Supposing that

$$\xi = k(x - ct), \quad (10)$$

by (10), (8) and (9) are converted into ODEs

$$-cu' + uu' + v' - \alpha k^2 cu''' = 0 \quad (11)$$

$$-cv' + (uv)' + \beta k^2 u''' = 0. \quad (12)$$

Integrating (11) and (12) once, we have

$$-cu + \frac{1}{2}u^2 + v - \alpha k^2 cu'' = g_1, \quad (13)$$

$$-cv + uv + \beta k^2 u'' = g_2, \quad (14)$$

where g_1 and g_2 are the integration constants.

Suppose that the solution of (13) and (14) can be expressed by a polynomial in G as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i G^i, \quad (15)$$

$$v(\xi) = \sum_{i=0}^{n} b_i G^i, \quad (16)$$

where a_i, b_i are constants, $G = G(\xi)$ satisfies Eq. (1).

Balancing the order of u^2 and v in Eq. (13), the order of u'' and uv in Eq. (14), then we can obtain $2m = n,\ n + 2 = m + n \Rightarrow\ m = 1,\ n = 2$, so Eqs. (15) and (16) can be rewritten as

$$u(\xi) = a_2 G^2 + a_1 G + a_0,\ a_2 \neq 0, \quad (17)$$

$$v(\xi) = b_2 G^2 + b_1 G + b_0,\ b_2 \neq 0, \quad (18)$$

where $a_1,\ a_0,\ b_2,\ b_1,\ b_0$ are constants to be determined later.

Substituting (17) and (18) into (13) and (14) and collecting all the terms with the same power of G together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq. (13):

$G^0: \ -ca_0 - g_1 + \frac{1}{2}a_0^2 + b_0 = 0.$

$G^1: \ b_1 + a_0a_1 - ca_1 - \alpha k^2 ca_1\lambda^2 = 0.$

$G^2: \ -ca_2 + b_2 + 3\alpha k^2 ca_1\mu\lambda + \frac{1}{2}a_1^2 - 4\alpha k^2 ca_2\lambda^2 + a_0a_2 = 0.$

$G^3: \ a_1a_2 - 2\alpha k^2 ca_1\mu^2 + 10\alpha k^2 ca_2\mu\lambda = 0.$

$G^4: \ \frac{1}{2}a_2^2 - 6\alpha k^2 ca_2\mu^2 = 0.$

For Eq. (14):
$G^0: \ -cb_0 - g_2 + a_0b_0 = 0.$

$G^1: \ b_1a_0 + a_1b_0 - cb_1 + \beta k^2 a_1\lambda^2 = 0.$

$G^2: \ -cb_2 + a_1b_1 - 3betak^2 a_1\mu\lambda + 4\beta k^2 a_2\lambda^2 + a_2b_0 + a_0b_2 = 0.$

$G^3: \ 2\beta k^2 a_1\mu^2 + a_1b_2 - 10\beta k^2 a_2\mu\lambda + a_2b_1 = 0.$

$G^4: \ a_2b_2 + 6\beta k^2 a_2\mu^2 = 0.$

Solving the algebraic equations above yields:

$$\begin{aligned} &a_2 = 12\alpha k^2 c\mu^2, \\ &a_1 = -12\alpha k^2 c\mu\lambda, \\ &a_0 = \tfrac{1}{2}(\tfrac{\beta + 2\alpha c^2 + 2\lambda^2\alpha^2k^2c^2}{\alpha c}), \\ &b_2 = -6\beta k^2\mu^2, \ b_1 = 6\beta k^2\mu\lambda, \\ &b_0 = -\tfrac{\beta}{4}(\tfrac{-\beta + 2\lambda^2\alpha^2k^2c^2}{\alpha^2c^2}), \\ &g_2 = -\tfrac{\beta}{8}(\tfrac{-\beta^2 + 4\lambda^4\alpha^4k^4c^4}{\alpha^3c^3}), \\ &g_1 = \tfrac{1}{8}(\tfrac{-4c^4\alpha^2 + 3\beta^2 + 4\lambda^4\alpha^4k^4c^4}{\alpha^2c^2}). \end{aligned} \tag{19}$$

Combining with (2) and (3), under the conditions $\mu \neq 0$, we can obtain the traveling wave solutions of the variant Boussinseq equations (8) and (9) as follows:

$$\begin{aligned} u_1(\xi) = &\frac{1}{2}(\frac{\beta + 2\alpha c^2 + 2\lambda^2\alpha^2k^2c^2}{\alpha c}) \\ &-12\alpha k^2 c\mu\lambda(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) \\ &+12\alpha k^2 c\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 \end{aligned} \tag{20}$$

$$\begin{aligned} v_1(\xi) = &-\frac{\beta}{4}(\frac{-\beta + 2\lambda^2\alpha^2k^2c^2}{\alpha^2c^2}) \\ &+6\beta k^2\mu\lambda(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) \\ &-6\beta k^2\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 \end{aligned} \tag{21}$$

Remark 1 *When $\mu = 0$, we obtain the trivial solutions.*

Remark 2 *The exact traveling wave solutions (20)-(21) of the variant Boussinseq equations are different from the results in [42, 43], and have not been reported by other authors to our best knowledge.*

4 Application Of The Bernoulli Sub-ODE Method For (2+1)-dimensional NNV Equations

In this section, we consider the (2+1)-dimensional NNV equations [44–46]:

$$\begin{aligned} &u_t + au_{xxx} + bu_{yyy} + cu_x + du_y \\ &= 3a(uv)_x + 3b(uw)_y, \end{aligned} \tag{22}$$

$$u_x = v_y, \tag{23}$$

$$u_y = w_x. \tag{24}$$

Suppose that

$$\xi = kx + ly + \omega t. \tag{25}$$

By (25), (22), (23) and (24) are converted into ODEs

$$\begin{aligned} &\omega u' + ak^3u''' + bl^3u''' + cku' + dlu' \\ &= 3ak(uv)' + 3bl(uw)', \end{aligned} \tag{26}$$

$$ku' = lv', \tag{27}$$

$$lu' = kw'. \tag{28}$$

Integrating (26), (27) and (28) once times, we have

$$\begin{aligned} &\omega u + ak^3u'' + bl^3u'' + cku + dlu \\ &= 3akuv + 3bluw + g_1, \end{aligned} \tag{29}$$

$$ku = lv + g_2, \tag{30}$$

$$lu = kw + g_3, \tag{31}$$

where $g_1, \ g_2, \ g_3$ are the integration constants.

Suppose that the solutions of (29), (30) and (31) can be expressed by polynomials in G as follows:

$$u(\xi) = \sum_{i=0}^{m} a_iG^i, \tag{32}$$

$$v(\xi) = \sum_{i=0}^{n} b_iG^i, \tag{33}$$

$$w(\xi)=\sum_{i=0}^{s}c_iG^i, \qquad (34)$$

where $a_i,\ b_i,\ c_i$ are constants, $G=G(\xi)$ satisfies Eq. (1).

Balancing the order of u'' and uv in Eq. (29), the order of u and v in Eq. (30), the order of u and w in Eq. (31), then we can obtain $m+2=m+n,\ m=n,\ m=s\Rightarrow\ m=n=s=2$, so Eq.(32), (33) and (34) can be rewritten as

$$u(\xi)=a_2G^2+a_1G+a_0,\ a_2\neq 0, \qquad (35)$$

$$v(\xi)=b_2G^2+b_1G+b_0,\ b_2\neq 0, \qquad (36)$$

$$w(\xi)=c_2G^2+c_1G+c_0,\ c_2\neq 0, \qquad (37)$$

where $a_2,\ a_1,\ a_0,\ b_2,\ b_1,\ b_0,\ c_2,\ c_1,\ c_0$ are constants to be determined later.

Substituting (35), (36) and (37) into (29), (30) and (31) and collecting all the terms with the same power of G together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq. (29):

$$\begin{aligned}
G^0:\quad & cka_0+dla_0-3aka_0b_0-3bla_0c_0\\
& -g_0+\omega a_0=0.\\
G^1:\quad & -3aka_0b_1-3bla_0c_1+dla_1+ak^3a_1\lambda^2\\
& +cka_1-3aka_1b_0+bl^3a_1\lambda^2-3bla_1c_0\\
& +\omega a_1=0.\\
G^2:\quad & -3ak^3a_1\mu\lambda+cka_2-3aka_0b_2-3bla_0c_2\\
& +4ak^3a_2\lambda^2-3aka_1b_1-3bl^3a_1\mu\lambda+\omega a_2\\
& +4bl^3a_2\lambda^2-3bla_1c_1+dla_2-3aka_2b_0\\
& -3bla_2c_0=0.\\
G^3:\quad & -3aka_1b_2+2bl^3a_1\mu^2+2ak^3a_1\mu^2\\
& -3aka_2b_1-3bla_1c_2-10ak^3a_2\mu\lambda\\
& -10bl^3a_2\mu\lambda-3bla_2c_1=0.\\
G^4:\quad & -3aka_2b_2-3bla_2c_2+6ak^3a_2\mu^2\\
& +6bl^3a_2\mu^2=0.
\end{aligned}$$

For Eq.(30):

$$\begin{aligned}
G^0:\quad & ka_0-lb_0-g_2=0.\\
G^1:\quad & ka_1-lb_1=0.\\
G^2:\quad & ka_2-lb_2=0.
\end{aligned}$$

For Eq. (31):

$$\begin{aligned}
G^0:\quad & la_0-kc_0-g_3=0.\\
G^1:\quad & la_1-kc_1=0.\\
G^2:\quad & la_2-kc_2=0.
\end{aligned}$$

Solving the algebraic equations above yields:

$$\begin{aligned}
& a_2=2lk\mu^2,\ a_1=-2l\mu\lambda k,\ a_0=a_0,\\
& b_2=2k^2\mu^2,\ \ b_1=-2\mu k^2\lambda,\ b_0=b_0,\\
& c_2=2\mu^2l^2,\ c_1=-2\mu l^2\lambda,\ k=k,\ \ l=l,\ \omega=\omega,\\
& c_0=\frac{1}{3}\frac{-3ak^3a_0-3bl^3a_0+dl^2k+ak^4l\lambda^2}{bl^2k}\\
& \quad+\frac{1}{3}\frac{ck^2l-3ak^2lb_0+bl^4\lambda^2k+\omega lk}{bl^2k},\\
& g_1=-a_0\frac{-3ak^3a_0-3bl^3a_0+ak^4l\lambda^2+bl^4\lambda^2k}{lk},\\
& g_2=ka_0-lb_0,\\
& g_3=-\frac{1}{3}\frac{-6bl^3a_0-3ak^3a_0+dl^2k+ak^4l\lambda^2}{l^2b}\\
& \quad-\frac{1}{3}\frac{ck^2l-3ak^2lb_0+bl^4\lambda^2k+\omega lk}{l^2b},
\end{aligned} \qquad (38)$$

where $k,\ l,\ \omega,\ a_0,\ b_0$ are arbitrary constants.

Under the condition $\mu\neq 0$, combining with (2) and (3), we can obtain the traveling wave solutions of the (2+1)-dimensional NNV equations (22)-(24) as follows:

$$u(\xi)=2lk\mu^2\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big)^2-2l\mu\lambda k\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big)+a_0 \qquad (39)$$

$$v(\xi)=2k^2\mu^2\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big)^2-2\mu k^2\lambda\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big)+b_0 \qquad (40)$$

$$\begin{aligned}
w(\xi) &= 2\mu^2l^2\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big)^2-2\mu l^2\lambda\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big)\\
&+ \frac{1}{3}\frac{-3ak^3a_0-3bl^3a_0+dl^2k+ak^4l\lambda^2}{bl^2k}\\
&+ \frac{1}{3}\frac{ck^2l-3ak^2lb_0+bl^4\lambda^2k+\omega lk}{bl^2k}
\end{aligned} \qquad (41)$$

where $\xi=kx+ly+\omega t$.

Remark 3 *Some authors have reported some exact solutions for the (2+1)-dimensional NNV equations in [44-46]. To our best knowledge, our results (39)-(41) have not been reported so far in the literature.*

Remark 4 *When $\mu=0$, we obtain the trivial solutions.*

5 Application Of The Bernoulli Sub-ODE Method For (2+1) dimensional Boussinesq and Kadomtsev-Petviashvili equations

In this section we will consider the following (2+1) dimensional Boussinesq and Kadomtsev-Petviashvili equations [47] :

$$u_y = q_x, \tag{42}$$
$$v_x = q_y, \tag{43}$$
$$q_t = q_{xxx} + q_{yyy} + 6(qu)_x + 6(qv)_y. \tag{44}$$

In order to obtain the traveling wave solutions of (42)-(44), we suppose that

$$\begin{array}{l} u(x,\ y,\ t) = u(\xi), \\ v(x,\ y,\ t) = v(\xi), \\ q(x,\ y,\ t) = q(\xi), \\ \xi = ax + dy - ct, \end{array} \tag{45}$$

where a, d, c are constants that to be determined later.

Using the wave variable (45), (42)-(44) can be converted into ODEs

$$du' - aq' = 0, \tag{46}$$
$$av' - dq' = 0, \tag{47}$$
$$\begin{array}{l}(a^3 + d^3)q''' - cq' - 6auq' \\ \quad -6aqu' - 6dvq' - 6dqv' = 0.\end{array} \tag{48}$$

Integrating the ODEs above, we obtain

$$du - aq = g_1, \tag{49}$$
$$av - dq = g_2, \tag{50}$$
$$(a^3 + d^3)q'' - cq - 6auq - 6dvq = g_3. \tag{51}$$

Supposing that the solutions of the ODEs above can be expressed by a polynomial in G as follows:

$$u(\xi) = \sum_{i=0}^{l} a_i G^i, \tag{52}$$

$$v(\xi) = \sum_{i=0}^{m} b_i G^i, \tag{53}$$

$$q(\xi) = \sum_{i=0}^{n} c_i G^i, \tag{54}$$

where a_i, b_i, c_i are constants, and $G = G(\xi)$ satisfies Eq. (1).

Balancing the order of u' and q' in Eq. (52), the order of v' and q' in Eq. (53) and the order of q''' and vq' in Eq. (54), we have $l+1 = n+1,\ m+1 = n+1,\ n+3 = m+n+1 \ \Rightarrow\ l = m = n = 2$. So Eq.(52)-(54) can be rewritten as

$$u(\xi) = a_2G^2 + a_1G + a_0,\ a_2 \neq 0, \tag{55}$$
$$v(\xi) = b_2G^2 + b_1G + b_0,\ b_2 \neq 0, \tag{56}$$
$$u(\xi) = c_2G^2 + c_1G + c_0,\ c_2 \neq 0, \tag{57}$$

where $a_i,\ b_i,\ c_i$ are constants to be determined later.

Substituting (55)-(57) into the ODEs (49)-(51), collecting all terms with the same power of G together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq. (49):

$G^0:\ a_0d - ac_0 - g_1 = 0.$

$G^1:\ a_1d - ac_1 = 0.$

$G^2:\ a_2d - ac_2 = 0.$

For Eq. (50):

$G^0:\ ab_0 - g_2 - dc_0 = 0.$

$G^1:\ ab_1 - dc_1 = 0.$

$G^2:\ -dc_2 + ab_2 = 0.$

For Eq. (51):

$G^0:\ -g_3 - cc_0 - 6db_0c_0 - 6aa_0c_0 = 0.$

$$G^1:\ -6aa_1c_0 - 6db_1c_0 + a^3c_1\lambda^2 - 6db_0c_1 \\ -6aa_0c_1 + d^3c_1\lambda^2 - cc_1 = 0.$$

$$G^2:\ 4a^3c_2\lambda^2 - 6aa_0c_2 - 6db_1c_1 - 6aa_1c_1 \\ -6aa_2c_0 + 4d^3c_2\lambda^2 - cc_2 - 3d^3c_1\mu\lambda \\ -6db_0c_2 - 3a^3c_1\mu\lambda - 6db_2c_0 = 0.$$

$$G^3:\ -6aa_2c_1 - 6db_2c_1 + 2a^3c_1\mu^2 - 10d^3c_2\mu\lambda \\ -6aa_1c_2 - 10a^3c_2\mu\lambda - 6db_1c_2 \\ +2d^3c_1\mu^2 = 0.$$

$G^4:\ 6d^3c_2\mu^2 - 6aa_2c_2 + 6a^3c_2\mu^2 - 6db_2c_2 = 0.$

Solving the algebraic equations above yields:

Case 1:

$$a_0 = a_0,\ a_1 = -\mu\lambda a^2,\ a_2 = a^2\mu^2,$$

$$b_0 = b_0,\ b_1 = -d^2\mu\lambda,\ b_2 = d^2\mu^2,\ a = a,$$

$$c_0 = c_0,\ c_1 = -d\mu\lambda a,\ c_2 = d\mu^2 a,$$

$$g_2 = ab_0 - dc_0,\ g_1 = da_0 - ac_0,\ d = d,$$

$$c = \frac{-6a^3c_0 - 6d^3c_0 + a^4d\lambda^2 - 6d^2b_0a - 6a^2a_0d + d^4\lambda^2 a}{ad},$$

$$g_3 = -c_0\frac{-6a^3c_0 - 6d^3c_0 + a^4d\lambda^2 + d^4\lambda^2 a}{ad}, \tag{58}$$

where $a_0,\ b_0,\ c_0,\ a,\ d$ are arbitrary constants.

Assume $\mu \neq 0$, then substituting the results above into (55)-(57), combining with (2) we can obtain the traveling wave solution of (2+1) dimensional BKP equation as follows:

$$u_1(\xi) = a^2\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 - \mu\lambda a^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + a_0, \tag{59}$$

$$v_1(\xi) = d^2\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 - d^2\mu\lambda(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + b_0, \tag{60}$$

$$q_1(\xi) = d\mu^2 a(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 - d\mu\lambda a(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + c_0, \tag{61}$$

where

$$\xi = ax + dy - \frac{-6a^3c_0 - 6d^3c_0 + a^4d\lambda^2}{ad}t - \frac{-6d^2b_0a - 6a^2a_0d + d^4\lambda^2 a}{ad}t. \tag{62}$$

Case 2:

$$\begin{aligned} &a_0 = a_0,\ a_1 = a_1,\ a_2 = d^2\mu^2,\\ &b_0 = b_0,\ b_1 = a_1,\ b_2 = d^2\mu^2,\\ &d = d,\ c = 6d(-b_0 + a_0),\\ &c_0 = c_0,\ c_1 = -a_1,\ c_2 = -d^2\mu^2,\\ &g_2 = -db_0 - dc_0,\ g_1 = da_0 + dc_0,\\ &a = -d,\ g_3 = 0, \end{aligned} \tag{63}$$

where $a_0,\ b_0,\ c_0,\ a_1,\ d$ are arbitrary constants.

Similarly, under the condition $\mu \neq 0$, we can obtain traveling wave solutions of (2+1) dimensional Boussinesq and Kadomtsev-Petviashvili equations as follows:

$$u_2(\xi) = d^2\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 + a_1(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + a_0, \tag{64}$$

$$v_2(\xi) = d^2\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 + a_1(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + b_0, \tag{65}$$

$$q_2(\xi) = -d^2\mu^2(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}})^2 - a_1(\frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}) + c_0, \tag{66}$$

where

$$\xi = -dx + dy - 6d(-b_0 + a_0)t. \tag{67}$$

Case 3:

$$\begin{aligned} &a_0 = a_0,\ a_1 = \tfrac{1}{2}d^2\mu\lambda(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i),\\ &a_2 = d^2\mu^2(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i),\\ &b_0 = b_0,\ b_1 = \tfrac{1}{2}\mu\lambda d^2,\ b_2 = d^2\mu^2,\\ &c_0 = c_0,\ c_1 = \tfrac{1}{2}d^2\mu\lambda(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i),\\ &c_2 = d^2\mu^2(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i),\\ &d = d,\ a = (\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)d,\\ &c = -6da_0(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i) - 6db_0,\\ &g_1 = da_0 - dc_0(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i),\\ &g_2 = db_0(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i) - dc_0,\\ &g_3 = 0, \end{aligned} \tag{68}$$

where $a_0,\ b_0,\ c_0,\ d$ are arbitrary constants.

Thus

$$\begin{aligned} u_3(\xi) = {} & d^2\mu^2(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})^2\\ & +\tfrac{1}{2}d^2\mu\lambda(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}) + a_0, \end{aligned} \tag{69}$$

$$v_3(\xi) = d^2\mu^2(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})^2 + \tfrac{1}{2}\mu\lambda d^2(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}) + b_0, \tag{70}$$

$$\begin{aligned} q_3(\xi) = {} & d^2\mu^2(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})^2\\ & +\tfrac{1}{2}d^2\mu\lambda(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}) + c_0, \end{aligned} \tag{71}$$

$$\xi = (\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i)dx + dy + [6da_0(\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i) + 6db_0]t, \tag{72}$$

where $\mu \neq 0$.

Case 4:

$$\begin{aligned} & a_0 = a_0, a_1 = \tfrac{1}{2}d^2\mu\lambda(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i), \\ & a_2 = d^2\mu^2(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i) \\ & b_0 = b_0, b_1 = -a_1(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i), \\ & b_2 = d^2\mu^2 \\ & c_0 = c_0, c_1 = -a_1(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i), \\ & c_2 = d^2\mu^2(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i) \\ & d = d, a = (\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)d, \\ & c = -6da_0(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i) - 6db_0 \\ & g_1 = da_0 - dc_0(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i), \\ & g_2 = db_0(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i) - dc_0, \\ & g_3 = 0, \end{aligned} \tag{73}$$

where a_0, b_0, c_0, d are arbitrary constants.

Then

$$\begin{aligned} u_4(\xi) &= d^2\mu^2(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})^2 \\ &+\tfrac{1}{2}d^2\mu\lambda(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}) + a_0, \end{aligned} \tag{74}$$

$$\begin{aligned} v_4(\xi) &= d^2\mu^2(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})^2 \\ &-a_1(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}) + b_0, \end{aligned} \tag{75}$$

$$\begin{aligned} & q_4(\xi) = d^2\mu^2(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}})^2 \\ & -a_1(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3}i)(\tfrac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}) + c_0, \end{aligned} \tag{76}$$

$$\xi = (\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i)dx+dy+[6da_0(\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i)+6db_0]t, \tag{77}$$

where $\mu \neq 0$.

Remark 5 *When $\mu = 0$, we obtain the trivial solutions. The traveling wave solutions established for (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations (59)-(61), (64)-(66), (69)-(73), (74)-(76) have not been reported by other authors to our best knowledge.*

6 Comparison with Zayed' results

In [46,43], Zayed solved the (2+1)-dimensional NNV equations and the variant Boussinseq equations by using the (G'/G) expansion method respectively. In this section, we will present some comparisons between the established results in Section 3-4 and Zayed' results.

Let μ_1, λ_1 represent μ, λ in [46] respectively. Then in (39)-(41), considering d, μ, λ, a_0, b_0, l, k, ω are arbitrary constants, if we take

$$d = \frac{A+B}{\sqrt{\lambda_1^2 - 4\mu_1}},\ \mu = B - A,\ \lambda = \sqrt{\lambda_1^2 - 4\mu_1},$$

$$a_0 = \alpha_0 - 2\mu_1,\ l = 1,\ k = 1,\ b_0 = -2\mu_1,$$

$$\omega = 3(a+b)(\alpha_0-2\mu_1) - \frac{A+B}{\sqrt{\lambda_1^2 - 4\mu_1}} - (a+b)(\lambda_1^2-4\mu_1)$$

$$-c - 6a\mu_1 - 3b(\gamma_0 - \frac{1}{2}\lambda_1^2),$$

where A, B, α_0, γ_0 are defined in [46], our solutions (39)-(41) reduce to the solutions derived in [46, (3.32)-(3.34)]. Furthermore, under the condition $A = 0$, $B \neq 0$, $\lambda_1 > 0$, $\mu_1 = 0$, (39)-(41) reduce the solitary solutions in [46, (3.41)-(3.43)]. If we take

$$d = \frac{iA+B}{i\sqrt{4\mu_1 - \lambda_1^2}},\ \mu = iA - B,\ \lambda = i\sqrt{4\mu_1 - \lambda_1^2},$$

$$a_0 = \alpha_0 - 2\mu_1,\ l = 1,\ k = 1,\ b_0 = -2\mu_1,$$

$$\omega = 3(a+b)(\alpha_0-2\mu_1) - \frac{iA+B}{i\sqrt{\lambda_1^2 - 4\mu_1}} - (a+b)(\lambda_1^2-4\mu_1)$$

$$-c - 6a\mu_1 - 3b(\gamma_0 - \frac{1}{2}\lambda_1^2),$$

then our solutions (39)-(41) reduce to the solutions derived in [46, (3.35)-(3.37)]. So in this way, our results (39)-(41) extend Zayed' results for the (2+1)-dimensional NNV equations in [46] .

For the variant Boussinseq equations, we note that our solutions (20)-(21) are different solutions from the results in [43, (33)-(38)].

7 Conclusions

In this paper we have seen that some new traveling wave solutions of the variant Boussinseq equations, (2+1)-dimensional NNV equations and (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations are successfully found by using the Bernoulli sub-ODE method. The main points of the method are that assuming the solution of the ODE reduced by using the traveling wave variable as well as integrating can be expressed by an m-th degree polynomial in G, where $G = G(\xi)$ is the general solutions of a Bernoulli sub-ODE equation. The positive integer m can be determined by the general homogeneous balance method, and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations.

Compared to the methods used before, one can see that this method is concise and effective. Also this method can be applied to other nonlinear problems.

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