Shape-Preserving Rational Bi-Cubic Spline for Monotone Surface Data

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Abstract: - In this paper, we extended the rational cubic function to rational bi-cubic function that presents a smooth, visually pleasant and interactive view of monotonicity preserving surface. Moreover, it involves six free parameters in its description. These free parameters are arranged in such a way where two of these are constrained to preserve the monotonicity, while the remaining other four free parameters are left free to designer for the refinement of monotone surface as desired. The scheme under discussion is C^1, flexible, simple, local and economical as compared to existing schemes. Numerical examples are provided to demonstrate that the anticipated scheme is interactive and smooth.

Key-Words: - Shape preserving interpolation, Rational cubic function, Rational Bi-cubic function, Monotone surface, Monotone surface data, Free parameters.

1 Introduction

Spline interpolation plays a significant role in Computer Graphics, Computer Aided Geometric Design, Engineering, Font Designing, Data Visualization, Shape preservation, Shape control and so many others. The data have some special shape characteristics e.g. monotonicity, positivity, convexity and of course it is often needed to generate a monotonicity preserving interpolating curve and surface according to the given monotone data. The aspiration of this paper is to preserve the hereditary attribute of data that is the monotonicity.

Monotonicity is a substantial shape characteristic of data. Many physical situations exist where entities are taken only monotone, for instance, monotonicity is applied in the specification of Digital to Analog Converters (DACs), Analog to Digital Converters (ADCs) and sensors [17]. These devices are used to control system applications where only monotonicity is acceptable. Erythrocyte sedimentation rates (ESR) in cancer patients, uric acid level in gout patients, approximation of couples and quasi couples in statistics, rate of dissemination of drug in blood , empirical option of pricing models in finance and the curves and surfaces of Dose-response in biochemistry and pharmacology [4] are good examples of monotone data. As well as in Engineering, the study of tensile strength of the material which give the monotone data because the tensile strength of a material can be defined as the maximum force that a material can withstand before breaking , The forced applied usually is called stress and is studied alongside the stretch of the material referred as strain [3].

The problem of monotonicity preserving interpolation has been considered by a number of authors [1-17] and references therein. Beatson and Ziegler [2] presented a visualization of monotone data arranged over a rectangular grid by C^1 monotone quadratic spline. Necessary and sufficient constraints on functional and derivative values were derived. Carlson and Fritsch [5] upgraded their results of univariate monotone interpolation to bivariate monotone interpolation for regular data. The interpolating function was determined by first partial derivatives and first mixed partial derivatives (twist) at mesh points. They derived necessary and sufficient conditions on these derivatives. As a result on a single rectangular element bi-cubic polynomial is monotone.

Casciola and Romani [6] extended NURBS (Non-Uniform Rational B-Spline) version of the rational interpolating spline with tension parameters for rectangular topology case. The scheme allowed the user to interactively modify the resulting surface by a set of tension parameters. Costantini and Fontanella [7] developed a method for constructing shape preserving surfaces interpolating arbitrary sets of data on rectangular grid by using monotone and or convex splines, having degree n and order of continuity k .The interpolating splines were obtained by using Bernstein polynomials of suitable continuous piecewise linear functions. The presented work is useful in developing algorithms for the construction of shape preserving splines interpolation for arbitrary set of data points.

Floater and Peña [8] defined and study the three kinds of monotonicity preservation of systems of bivariate functions on a triangle by using Bernstein polynomials with some geometric applications. Hussain and Maria [10] developed the schemes with two shape parameters to visualize the monotone data in the view of monotone curves and surfaces. They extended to partially blended rational bi-cubic function (Coons patches). The schemes are local but unfortunately did not give the flexibility to the user for the refinement of curves and surfaces as desired. Hussain, et al [11] extended the GPRC (General Piecewise Rational Cubic) function to partially blended rational bi-cubic function (Coon patches) with sixteen shape parameters (in each patch). The authors derived the constraints for the eight shape parameters to preserve the shape of monotone data, while the remaining eight free parameters were left to user for the refinement of surfaces. The constrained parameters in the scheme depend on each other.

Hussain, et al [15] extended the rational cubic function to partially blended rational bi-cubic function with twelve free parameters. Data dependent sufficient constraints were derived for the four of these parameters to preserve the monotonicity, while the remaining of these free parameters were left free for user's choice. The scheme [11, 15] are expensive and time consuming due to bunch of free parameters. Sarfraz, et al [16] presented a scheme to visualize the shape of monotone data by bi-cubic function. The authors derived simple constraints on the free parameters to preserve the shape of the data. The monotonicity scheme does enable derivative specification, but fails to maintain smoothness of surface.

In this paper, we have presented an efficient monotone data visualization scheme. The C^1 rational cubic function with three free parameters is extended into rational bi-cubic function. The rational bi-cubic function involves six free parameters in each patch for its representation. These free parameters are arranged in such a manner where two of these are constrained to preserve the shape of monotone data, while the remaining free parameters are left free to designer's choice for refine of surfaces as desired. Our scheme has a number of advantages over the existing schemes.

- The developed scheme has been demonstrated through different numerical examples and observed that the scheme is not only C^1, local, computationally economical, easy to compute, time saving but also visually pleasant as compared to existing schemes [10-15].
- In [10,12-14], the schemes do not allow the user to refine the monotone surface as desired while for more pleasing surface (and still having the monotone shape preserved) an additional modifications is required, this task is more easily done in this paper by a simple adjustment of free parameters in the rational bi-cubic interpolation on designer's choice.
- Data dependent sufficient constraints for free parameters are derived to preserve the monotonicity of monotone data. This scheme works well for both equally and unequally space data.
- In [11], the author were derived the simple data dependent constraints for free parameters but these constraints parameters are depend on each other, while in this paper the scheme is computationally time saving due to independence of the constraints parameters.
- In [3,6,7,9], the authors developed the schemes to achieve the desired shape of data by inserting extra knots between any two knots in the interval while we preserve the shape of monotone data by only imposing constraints on free parameters without any extra knots.

- In this paper, there is no need of necessary and sufficient conditions on functional and derivative values like [2] and [5].

The remaining part of the paper is arranged as follows: Review of rational cubic function with three shape parameters is discussed in section 2. The rational bi-cubic function is given in section 3. The choice of derivative for the construction of smooth monotone surface is discussed in section 4. Monotonicity preserving rational bi cubic surface interpolation is discussed in section 5. Finally, numerical examples and conclusion are given in sections 6 and 7, respectively.

2 Review of Rational Cubic Spline Function

Let $\{(x_i, f_i): i = 0,1,2,...,n\}$ be the given set of data points such as $x_0 < x_1 < x_2 < ... < x_n$. The rational cubic function with three free parameters [1], in each subinterval $I_i = [x_i, x_{i+1}]$, $i = 0,1,2,...,n-1$ is defined as:

$$S_i(x) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i\beta_i}{q_i(\theta)} \tag{1}$$

where $\theta = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$ and u_i, v_i, w_i are the positive free parameters. The following interpolatory conditions are imposed for the C^1 continuity of the piecewise rational cubic function (1),

$$\begin{cases} S_i(x_i) = f_i, & S_i(x_{i+1}) = f_{i+1} \\ S_i'(x_i) = d_i, & S_i'(x_{i+1}) = d_{i+1} \end{cases} \tag{2}$$

where $S_i'(x)$ denotes the derivative with respect to x and d_i denotes the derivatives estimated at knots. The C^1 continuity conditions defined in equation (2) claim the following values of unknown β_i, $i = 0,1,2,3$

$$\begin{aligned} \beta_0 &= u_i f_i \\ \beta_1 &= w_i f_i + u_i h_i d_i \\ \beta_2 &= w_i f_{i+1} - v_i h_i d_{i+1} \\ \beta_3 &= v_i f_{i+1} \end{aligned} \tag{3}$$

Using equation (3), the piecewise rational cubic function (1) reformulated as:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} \tag{4}$$

$$p_i(\theta) = \begin{cases} u_i f_i(1-\theta)^3 + (w_i f_i + u_i h_i d_i)\theta(1-\theta)^2 \\ +(w_i f_{i+1} - v_i h_i d_{i+1})\theta^2(1-\theta) + v_i f_{i+1}\theta^3 \end{cases}$$

$$q_i(\theta) = u_i(1-\theta)^3 + w_i\theta(1-\theta) + v_i\theta^3$$

Remark 1.1: For the values of free parameters set as: $u_i = 1$, $v_i = 1$ and $w_i = 3$, then the C^1 piecewise rational cubic function (4) reduces to standard cubic Hermite spline.

M. Abbas, et al [1] developed the following result for the monotonicity-preserving of 2D monotone data.

Theorem 2.1 [M. Abbas, et al [1]]
The C^1 piecewise rational cubic function (4) preserves the monotonicity of monotone data, if in each subinterval $I_i = [x_i, x_{i+1}]$, $i = 0,1,2,....,n$, the free parameters satisfy the following sufficient conditions

$$w_i > max\left\{0, \frac{u_i d_i}{\Delta_i}, \frac{v_i d_{i+1}}{\Delta_i}, \frac{u_i d_i + v_i d_{i+1}}{\Delta_i}\right\}, u_i > 0, v_i > 0.$$

The above result can be rewritten as:

$$w_i = l_i + max\left\{0, \frac{u_i d_i}{\Delta_i}, \frac{v_i d_{i+1}}{\Delta_i}, \frac{u_i d_i + v_i d_{i+1}}{\Delta_i}\right\},$$

$$l_i > 0, u_i > 0 \text{ and } v_i > 0.$$

where $\Delta_i = (f_{i+1} - f_i)/h_i$

3 Rational Bi-Cubic Spline Function

In this section, we extended a C^1 piecewise rational cubic function (4) to rational bi-cubic function $S(x, y)$ over the rectangular domain $\Omega = [a,b] \times [c,d]$. The partition of arbitrary intervals $[a,b]$ and $[c,d]$ can be defined as $\pi: a = x_0 < x_1 < x_2 < ... < x_n = b$, $\hat{\pi}: c = y_0 < y_1 < y_2 < ... < y_m = d$ respectively. The rational bi-cubic function over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0,1,2,...,n-1$, $j = 0,1,2,...,m-1$ defined as:

$$S(x, y) = S_{i,j}(x, y) = B_i(\theta)F(i, j)\hat{B}_j^t(\phi) \tag{5}$$

where,

$$F(i,j)=\begin{pmatrix} F_{i,j} & F_{i,j+1} & F^{y}_{i,j} & F^{y}_{i,j+1} \\ F_{i+1,j} & F_{i+1,j+1} & F^{y}_{i+1,j} & F^{y}_{i+1,j+1} \\ F^{x}_{i,j} & F^{x}_{i,j+1} & F^{xy}_{i,j} & F^{xy}_{i,j+1} \\ F^{x}_{i+1,j} & F^{x}_{i+1,j+1} & F^{xy}_{i+1,j} & F^{xy}_{i+1,j+1} \end{pmatrix} \quad (6)$$

$$B_i(\theta)=\left[b_0(\theta)\, b_1(\theta)\, b_2(\theta)\, b_3(\theta)\right] \quad (7)$$

$$\hat{B}_j(\phi)=\left[\hat{b}_0(\phi)\, b_1(\phi)\, \hat{b}_2(\phi)\, b_3(\phi)\right] \quad (8)$$

with,

$$b_0(\theta)=\frac{(1-\theta)^3 u_{i,j}+\theta(1-\theta)^2 w_{i,j}}{q_i(\theta)}$$

$$b_1(\theta)=\frac{\theta^3 v_{i,j}+\theta^2(1-\theta)w_{i,j}}{q_i(\theta)}$$

$$b_2(\theta)=\frac{\theta(1-\theta)^2 h_i u_{i,j}}{q_i(\theta)}$$

$$b_3(\theta)=-\frac{\theta^2(1-\theta)h_i v_{i,j}}{q_i(\theta)}$$

$$\hat{b}_0(\phi)=\frac{(1-\phi)^3 \hat{u}_{i,j}+\phi(1-\phi)^2 w_{i,j}}{q_j(\phi)}$$

$$\hat{b}_1(\phi)=\frac{\phi^3 \hat{v}_{i,j}+\phi^2(1-\phi)w_{i,j}}{q_j(\phi)}$$

$$\hat{b}_2(\phi)=\frac{\phi(1-\phi)^2 \hat{h}_j \hat{u}_{i,j}}{q_j(\phi)}$$

$$\hat{b}_3(\phi)=-\frac{\phi^2(1-\phi)\hat{h}_j \hat{v}_{i,j}}{q_j(\phi)}$$

$$\begin{cases} q_i(\theta)=u_{i,j}(1-\theta)^3+w_{i,j}\theta(1-\theta)+v_{i,j}\theta^3 \\ q_j(\phi)=\hat{u}_{i,j}(1-\phi)^3+w_{i,j}\phi(1-\phi)+v_{i,j}\phi^3 \end{cases}$$

where $\theta=(x-x_i)/h_i$, $\varphi=(y-y_j)/\hat{h}_j$, $h_i=x_{i+1}-x_i$ and $\hat{h}_j=y_{j+1}-y_j$.

Substituting the equations (6)-(8) in (5), then the rational bi-cubic function (5) defined as:

$$S(x,y)=\frac{(1-\theta)^3 P_{i,j}+\theta(1-\theta)^2 Q_{i,j}+\theta^2(1-\theta)R_{i,j}+\theta^3 T_{i,j}}{(1-\theta)^3 u_{i,j}+\theta(1-\theta)w_{i,j}+\theta^3 v_{i,j}} \quad (9)$$

where,

$$P_{i,j}=\frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k A_k}{q_j(\phi)} \quad (10)$$

with,

$$A_0=\hat{u}_{i,j}u_{i,j}F_{i,j},$$

$$A_1=u_{i,j}(F_{i,j}\hat{w}_{i,j}+F^{y}_{i,j}\hat{h}_j u_{i,j})$$

$$A_2=u_{i,j}(F_{i,j+1}\hat{w}_{i,j}-F^{y}_{i,j+1}\hat{h}_j v_{i,j})$$

$$A_3=u_{i,j}\hat{v}_{i,j}F_{i,j+1}$$

$$Q_{i,j}=\frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k B_k}{q_j(\phi)} \quad (11)$$

such that,

$$B_0=\hat{u}_{i,j}(w_{i,j}F_{i,j}+h_i u_{i,j}F^{x}_{i,j})$$

$$B_1=\hat{w}_{i,j}(w_{i,j}F_{i,j}+h_i u_{i,j}F^{x}_{i,j})+\hat{h}_j u_{i,j}(w_{i,j}F^{y}_{i,j}+h_i u_{i,j}F^{xy}_{i,j})$$

$$B_2=\hat{w}_{i,j}(w_{i,j}F_{i,j+1}+h_i u_{i,j}F^{x}_{i,j+1})-\hat{h}_j v_{i,j}(w_{i,j}F^{y}_{i,j+1}+h_i u_{i,j}F^{xy}_{i,j+1})$$

$$B_3=\hat{v}_{i,j}(w_{i,j}F_{i,j+1}+h_i u_{i,j}F^{x}_{i,j+1})$$

$$R_{i,j}=\frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k C_k}{q_j(\phi)} \quad (12)$$

with,

$$C_0=\hat{u}_{i,j}(F_{i+1,j}w_{i,j}-h_i v_i F^{x}_{i+1,j})$$

$$C_1=\hat{w}_{i,j}(F_{i+1,j}w_{i,j}-h_i v_i F^{x}_{i+1,j})+\hat{h}_j u_{i,j}(F^{y}_{i+1,j}w_{i,j}-h_i v_i F^{xy}_{i+1,j})$$

$$C_2=\begin{cases} \hat{w}_{i,j}(F_{i+1,j+1}w_{i,j}-h_i v_i F^{x}_{i+1,j+1})- \\ \hat{h}_j \hat{v}_{i,j}(F^{y}_{i+1,j+1}w_{i,j}-h_i v_i F^{xy}_{i+1,j+1}) \end{cases}$$

$$C_3=\hat{v}_{i,j}(F_{i+1,j+1}w_{i,j}-h_i v_i F^{x}_{i+1,j+1})$$

$$T_{i,j}=\frac{\sum_{k=0}^{3}(1-\phi)^{3-k}\phi^k D_k}{q_j(\phi)} \quad (13)$$

where,

$$D_0=\hat{u}_{i,j}v_{i,j}F_{i+1,j}$$

$$D_1=v_{i,j}(F_{i+1,j}\hat{w}_{i,j}+F^{y}_{i+1,j}\hat{h}_j u_{i,j})$$

$$D_2=v_{i,j}(F_{i+1,j+1}\hat{w}_{i,j}-F^{y}_{i+1,j+1}\hat{h}_j v_{i,j})$$

$$D_3=v_{i,j}\hat{v}_{i,j}F_{i+1,j+1}$$

4 Determination of derivatives

Usually, the derivatives $F^{x}_{i,j}$ and $F^{y}_{i,j}$ and $F^{xy}_{i,j}$ are not known, so must be calculated either from the given data or by some other sources. Let us denote $F^{x}_{i,j}$ and $F^{y}_{i,j}$ as the first order derivatives with respect to x and y, respectively, at the data point $F_{i,j}$. Similarly, let the mixed derivatives be denoted by $F_{i,j}^{xy}$. Arithmetic mean method, as proposed in [13] is the three-point difference approximation based on arithmetic calculation for the monotone curve manipulation. This method can be oriented and extended for the 3D data visualization as follows:

4.1 Arithmetic Mean Method for 3D data

$$F_{0,j}^{x} = \Delta_{0,j} + \frac{(\Delta_{0,j} - \Delta_{1,j})h_0}{(h_0 + h_1)},$$

$$F_{n,j}^{x} = \Delta_{n-1,j} + \frac{(\Delta_{n-1,j} - \Delta_{n-2,j})h_{n-1}}{(h_{n-1} + h_{n-2})}$$

$$F_{i,j}^{x} = \frac{(\Delta_{i,j} + \Delta_{i-1,j})}{2}, \; i = 1,2,3,...,n-1, \; j = 0,1,2,...,m$$

$$F_{i,0}^{y} = \hat{\Delta}_{i,0} + \frac{(\hat{\Delta}_{i,0} - \Delta_{i,1})\hat{h}_0}{(\hat{h}_0 + h_1)}$$

$$F_{i,m}^{y} = \hat{\Delta}_{i,m-1} + \frac{(\hat{\Delta}_{i,m-1} - \Delta_{i,m-2})\hat{h}_{m-1}}{(\hat{h}_{m-1} + h_{m-2})}$$

$$F_{i,j}^{y} = \frac{(\hat{\Delta}_{i,j} + \Delta_{i,j-1})}{2}, \; i = 0,1,2,...,n, \; j = 1,2,3,...,m-1$$

$$F_{i,j}^{xy} = \frac{1}{2}\left\{ \frac{F_{i+1,j}^{y} - F_{i-1,j}^{y}}{h_{i-1} + h_i} + \frac{F_{i,j+1}^{x} - F_{i,j-1}^{x}}{\hat{h}_{j-1} + h_j} \right\}$$

$$i = 1,2,...,n-1; \quad j = 1,2,...,m-1$$

where $\Delta_{i,j} = \frac{F_{i+1,j} - F_{i,j}}{h_i}$, $\hat{\Delta}_{i,j} = \frac{F_{i,j+1} - F_{i,j}}{\hat{h}_j}$. These arithmetic mean methods are computationally economical and suitable for visualization of data.

5 Monotonicity-Preserving Rational Bi-Cubic Spline Interpolation.

The rational bi-cubic function (5) does not preserve the shape of monotone surface data. So, it is required to assign suitable constraints for the free parameters by some mathematical treatment to preserve the monotonicity of monotone surface data.

Theorem 5.1: The rational bi-cubic function (5) preserves the monotonicity of monotone surface data, if in each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ the free parameters satisfy the following sufficient conditions as: $u_{i,j}, \hat{u}_{i,j} > 0$ and $v_{i,j}, \hat{v}_{i,j} > 0$

$$w_{i,j} > max \left\{ \begin{array}{l} \frac{F_{i,j}^{x}}{\Delta_{i,j}}, -\frac{F_{i,j}^{x}}{2\Delta_{i,j}}, \frac{F_{i+1,j}^{x}}{\Delta_{i,j}}, -\frac{F_{i+1,j}^{x}}{2\Delta_{i,j}}, \\ \frac{F_{i,j+1}^{x}}{\Delta_{i,j+1}}, -\frac{F_{i,j+1}^{x}}{2\Delta_{i,j+1}}, \frac{F_{i+1,j+1}^{x}}{\Delta_{i,j+1}}, -\frac{F_{i+1,j+1}^{x}}{2\Delta_{i,j+1}}, \\ \frac{h_i F_{i+1,j}^{xy}}{F_{i+1,j}^{y}}, \frac{h_i F_{i+1,j+1}^{xy}}{F_{i+1,j+1}^{y}}, -\frac{h_i F_{i,j}^{xy}}{F_{i,j}^{y}}, -\frac{h_i F_{i,j+1}^{xy}}{F_{i,j+1}^{y}} \end{array} \right\}$$

$$\hat{w}_{i,j} > max \left\{ \begin{array}{l} \frac{F_{i,j}^{y}}{\hat{\Delta}_{i,j}}, -\frac{F_{i,j}^{y}}{2\Delta_{i,j}}, \frac{F_{i,j+1}^{y}}{\hat{\Delta}_{i,j}}, -\frac{F_{i,j+1}^{y}}{2\Delta_{i,j}}, \\ \frac{F_{i+1,j}^{y}}{\hat{\Delta}_{i+1,j}}, -\frac{F_{i+1,j}^{y}}{2\Delta_{i+1,j}}, \frac{F_{i+1,j+1}^{y}}{\hat{\Delta}_{i+1,j}}, -\frac{F_{i+1,j+1}^{y}}{2\Delta_{i+1,j}}, \\ -\frac{h_j F_{i,j}^{xy}}{F_{i,j}^{x}}, \frac{h_j F_{i,j+1}^{xy}}{F_{i,j+1}^{x}}, -\frac{h_j F_{i+1,j}^{xy}}{F_{i+1,j}^{x}}, \frac{h_j F_{i+1,j+1}^{xy}}{F_{i+1,j+1}^{x}} \end{array} \right\}$$

The above results are rewritten as:

$$w_{i,j} = l_{i,j} + max \left\{ \begin{array}{l} \frac{F_{i,j}^{x}}{\Delta_{i,j}}, -\frac{F_{i,j}^{x}}{2\Delta_{i,j}}, \frac{F_{i+1,j}^{x}}{\Delta_{i,j}}, -\frac{F_{i+1,j}^{x}}{2\Delta_{i,j}}, \\ \frac{F_{i,j+1}^{x}}{\Delta_{i,j+1}}, -\frac{F_{i,j+1}^{x}}{2\Delta_{i,j+1}}, \frac{F_{i+1,j+1}^{x}}{\Delta_{i,j+1}}, -\frac{F_{i+1,j+1}^{x}}{2\Delta_{i,j+1}}, \\ \frac{h_i F_{i+1,j}^{xy}}{F_{i+1,j}^{y}}, \frac{h_i F_{i+1,j+1}^{xy}}{F_{i+1,j+1}^{y}}, -\frac{h_i F_{i,j}^{xy}}{F_{i,j}^{y}}, -\frac{h_i F_{i,j+1}^{xy}}{F_{i,j+1}^{y}} \end{array} \right\}, \; l_{i,j} > 0$$

$$\hat{w}_{i,j} = m_{i,j} + max \left\{ \begin{array}{l} \frac{F_{i,j}^{y}}{\widehat{\Delta}_{i,j}}, -\frac{F_{i,j}^{y}}{2\Delta_{i,j}}, \frac{F_{i,j+1}^{y}}{\Delta_{i,j}}, -\frac{F_{i,j+1}^{y}}{2\Delta_{i,j}}, \\ \frac{F_{i+1,j}^{y}}{\widehat{\Delta}_{i+1,j}}, -\frac{F_{i+1,j}^{y}}{2\Delta_{i+1,j}}, \frac{F_{i+1,j+1}^{y}}{\Delta_{i+1,j}}, -\frac{F_{i+1,j+1}^{y}}{2\Delta_{i+1,j}}, \\ -\frac{h_j F_{i,j}^{xy}}{F_{i,j}^{x}}, \frac{h_j F_{i,j+1}^{xy}}{F_{i,j+1}^{x}}, -\frac{h_j F_{i+1,j}^{xy}}{F_{i+1,j}^{x}}, \frac{h_j F_{i+1,j+1}^{xy}}{F_{i+1,j+1}^{x}} \end{array} \right\}, \; m_{i,j} > 0$$

Proof:

Let $\{(x_i, y_j, F_{i,j}) : i = 0,1,2,...,n-1, \; j = 0,1,2,...,m-1\}$ be a monotone surface data arranged over rectangular domain $\Omega = [x_0, x_n] \times [y_0, y_m]$. The necessary conditions for monotonicity of monotone surface data are:

$$\begin{cases} F_{i,j} < F_{i+1,j}, \; F_{i,j}^{x} > 0, \; \Delta_{i,j} > 0, \\ \text{whenever } (x_i, y_j) < (x_{i+1}, y_j) \quad \forall i, j \end{cases} \qquad (14)$$

$$\begin{cases} F_{i,j} < F_{i,j+1},\ F_{i,j}^{y} > 0,\ \hat{\Delta}_{i,j} > 0, \\ \text{whenever } (x_i, y_j) < (x_i, y_{j+1}) \qquad \forall i, j \end{cases} \tag{15}$$

and the free parameters are:

$$\begin{cases} u_{i,j} > 0, v_{i,j} > 0, w_{i,j} > 0, \\ \hat{u}_{i,j} > 0,\ \hat{v}_{i,j} > 0, \hat{w}_{i,j} > 0 \end{cases} \tag{16}$$

The rational bi-cubic function $S(x, y)$ defined in (5) preserves the monotonicity of monotone surface data if the following conditions are satisfied as

$$\begin{cases} S_x(x, y) > 0 \\ S_y(x, y) > 0 \end{cases} \quad \forall (x, y) \in \Omega \tag{17}$$

The functions $S_x(x, y)$ and $S_y(x, y)$ represent the first order partial derivations of rational bi-cubic function (5) w. r. t. x and y respectively

$$S_x(x, y) = \frac{\sum_{i=0}^{5} (1-\theta)^{5-i} \theta^i M_i}{(q_{i,j}(\theta))^2 \hat{q}_{i,j}(\varphi)} \tag{18}$$

where,

$$M_0 = \sum_{j=0}^{3} (1-\varphi)^{3-j} \varphi^j M_{0,j}$$

with,

$$M_{0,0} = u_{i,j} \hat{u}_{i,j} F_{i,j}^{x}$$

$$M_{0,1} = u_{i,j} (\hat{w}_{i,j} F_{i,j}^{x} + \hat{h}_j F_{i,j}^{xy})$$

$$M_{0,2} = u_{i,j} (\hat{w}_{i,j} F_{i,j+1}^{x} - \hat{h}_j F_{i,j+1}^{xy}),$$

$$M_{0,3} = u_{i,j} \hat{v}_{i,j} F_{i,j+1}^{x},$$

$$M_1 = \sum_{j=0}^{3} (1-\varphi)^{3-j} \varphi^j M_{1,j}$$

with,

$$M_{1,0} = 2u_{i,j} \hat{u}_{i,j} \left\{ \left(w_{i,j} \Delta_{i,j} - F_{i+1,j}^{x} \right) + 0.5 F_{i,j}^{x} \right\}$$

$$M_{1,1} = u_{i,j} \begin{Bmatrix} \hat{w}_{i,j} (2w_{i,j} \Delta_{i,j} + F_{i,j}^{x}) - 2(w_{i,j} F_{i+1,j}^{x} + \hat{h}_j F_{i+1,j}^{xy}) \\ + \hat{h}_j F_{i,j}^{xy} + \frac{2\hat{h}_j w_{i,j}}{h_i} \left(F_{i+1,j}^{y} - F_{i,j}^{y} \right) \end{Bmatrix}$$

$$M_{1,2} = u_{i,j} \begin{Bmatrix} \hat{w}_{i,j} (2w_{i,j} \Delta_{i,j+1} + F_{i,j+1}^{x}) - 2(w_{i,j} F_{i+1,j+1}^{x} - \hat{h}_j F_{i+1,j+1}^{xy}) \\ - \hat{h}_j F_{i,j+1}^{xy} - \frac{2\hat{h}_j w_{i,j}}{h_i} \left(F_{i+1,j+1}^{y} - F_{i,j+1}^{y} \right) \end{Bmatrix}$$

$$M_{1,3} = 2u_{i,j} \hat{v}_{i,j} \left\{ \left(w_{i,j} \Delta_{i,j+1} - F_{i+1,j+1}^{x} \right) + 0.5 F_{i,j+1}^{x} \right\}$$

$$M_2 = \sum_{j=0}^{3} (1-\varphi)^{3-j} \varphi^j M_{2,j}$$

with,

$$M_{2,0} = \hat{u}_{i,j} \begin{Bmatrix} w_{i,j} \left(w_{i,j} \Delta_{i,j} - F_{i,j}^{x} \right) + 2u_{i,j} \left(w_{i,j} \Delta_{i,j} - F_{i+1,j}^{x} \right) \\ + 3u_{i,j} v_{i,j} \Delta_{i,j} - w_{i,j} F_{i+1,j}^{x} \end{Bmatrix}$$

$$M_{2,1} = \begin{cases} \left(0.5 w_{i,j} + u_{i,j} \right) \begin{Bmatrix} 2\hat{w}_{i,j} w_{i,j} \Delta_{i,j} - 2(w_{i,j} F_{i+1,j}^{x} + \\ \hat{h}_j F_{i+1,j}^{xy}) + \frac{2\hat{h}_j w_{i,j}}{h_i} \left(F_{i+1,j}^{y} - F_{i,j}^{y} \right) \end{Bmatrix} \\ + 3u_{i,j} v_{i,j} w_{i,j} \Delta_{i,j} + \frac{3\hat{h}_j u_{i,j} v_{i,j}}{h_i} \left(F_{i+1,j}^{y} - F_{i,j}^{y} \right) \\ - w_{i,j} (\hat{w}_{i,j} F_{i,j}^{x} + \hat{h}_j F_{i,j}^{xy}) \end{cases}$$

$$M_{2,2} = \begin{cases} \left(0.5 w_{i,j} + v_{i,j} \right) \begin{Bmatrix} 2\hat{w}_{i,j} w_{i,j} \Delta_{i,j+1} - 2(w_{i,j} F_{i+1,j+1}^{x} - \hat{h}_j F_{i+1,j+1}^{xy}) \\ - \frac{2\hat{h}_j w_{i,j}}{h_i} \left(F_{i+1,j+1}^{y} - F_{i,j+1}^{y} \right) \end{Bmatrix} \\ + 3u_{i,j} v_{i,j} w_{i,j} \Delta_{i,j+1} - \frac{3\hat{h}_j u_{i,j} v_{i,j}}{h_i} \left(F_{i+1,j+1}^{y} - F_{i,j+1}^{y} \right) \\ - w_{i,j} (\hat{w}_{i,j} F_{i,j+1}^{x} - \hat{h}_j F_{i,j+1}^{xy}) \end{cases}$$

$$M_{2,3} = \hat{v}_{i,j} \begin{Bmatrix} w_{i,j} \left(w_{i,j} \Delta_{i,j+1} - F_{i,j+1}^{x} \right) + 2u_{i,j} \left(w_{i,j} \Delta_{i,j+1} - F_{i+1,j+1}^{x} \right) \\ + 3u_{i,j} v_{i,j} \Delta_{i,j+1} - w_{i,j} F_{i+1,j+1}^{x} \end{Bmatrix}$$

$$M_3 = \sum_{j=0}^{3} (1-\varphi)^{3-j} \varphi^j M_{3,j}$$

with,

$$M_{3,0} = \hat{u}_{i,j} \begin{Bmatrix} w_{i,j} \left(w_{i,j} \Delta_{i,j} - F_{i+1,j}^{x} \right) + 2v_{i,j} \left(w_{i,j} \Delta_{i,j} - F_{i,j}^{x} \right) \\ + 3u_{i,j} v_{i,j} \Delta_{i,j} - w_{i,j} F_{i,j}^{x} \end{Bmatrix}$$

$$M_{3,1} = \begin{cases} \left(0.5 w_{i,j} + u_{i,j} \right) \begin{Bmatrix} 2\hat{w}_{i,j} w_{i,j} \Delta_{i,j} - 2(w_{i,j} F_{i,j}^{x} + \hat{h}_j F_{i,j}^{xy}) \\ + \frac{2\hat{h}_j w_{i,j}}{h_i} \left(F_{i+1,j}^{y} - F_{i,j}^{y} \right) \end{Bmatrix} \\ + 3u_{i,j} v_{i,j} \hat{w}_{i,j} \Delta_{i,j} + \frac{3\hat{h}_j u_{i,j} v_{i,j}}{h_i} \left(F_{i+1,j}^{y} - F_{i,j}^{y} \right) \\ - w_{i,j} (\hat{w}_{i,j} F_{i+1,j}^{x} + \hat{h}_j F_{i+1,j}^{xy}) \end{cases}$$

$$M_{3,2} = \begin{cases} \left(0.5w_{i,j} + v_{i,j}\right) \begin{Bmatrix} 2\hat{v}_{i,j} w_{i,j} \Delta_{i,j+1} - 2(w_{i,j} F^{x}_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1}) \\ -\dfrac{2\hat{h}_j w_{i,j}}{h_i}\left(F^{y}_{i+1,j+1} - F^{y}_{i,j+1}\right) \end{Bmatrix} \\ +3u_{i,j} v_{i,j} \hat{w}_{i,j} \Delta_{i,j+1} - \dfrac{3\hat{h}_j u_{i,j} v_{i,j}}{h_i}\left(F^{y}_{i+1,j+1} - F^{y}_{i,j+1}\right) \\ -w_{i,j}(\hat{w}_{i,j} F^{x}_{i+1,j+1} - \hat{h}_j F^{xy}_{i+1,j+1}) \end{cases}$$

$$M_{3,3} = \hat{v}_{i,j} \begin{Bmatrix} w_{i,j}\left(w_{i,j}\Delta_{i,j+1} - F^{x}_{i+1,j+1}\right) + 2v_{i,j}\left(w_{i,j}\Delta_{i,j+1} - F^{x}_{i,j+1}\right) \\ +3u_{i,j} v_{i,j} \Delta_{i,j+1} - w_{i,j} F^{x}_{i,j+1} \end{Bmatrix}$$

$$M_4 = \sum_{j=0}^{3} (1-\varphi)^{3-j} \varphi^j M_{4,j}$$

with,

$$M_{4,0} = 2v_{i,j}\hat{u}_{i,j}\left\{\left(w_{i,j}\Delta_{i,j} - F^{x}_{i,j}\right) + 0.5F^{x}_{i+1,j}\right\}$$

$$M_{4,1} = v_{i,j} \begin{Bmatrix} \hat{w}_{i,j}(2w_{i,j}\Delta_{i,j} + F^{x}_{i+1,j}) - 2(w_{i,j}F^{x}_{i,j} + \hat{h}_j F^{xy}_{i,j}) + h_j F^{xy}_{i+1,j} \\ +\dfrac{2\hat{h}_j w_{i,j}}{h_i}\left(F^{y}_{i+1,j} - F^{y}_{i,j}\right) \end{Bmatrix}$$

$$M_{4,2} = v_{i,j} \begin{Bmatrix} \hat{w}_{i,j}(2w_{i,j}\Delta_{i,j+1} + F^{x}_{i+1,j+1}) - 2(w_{i,j}F^{x}_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1}) \\ -\hat{h}_j F^{xy}_{i+1,j+1} - \dfrac{2\hat{h}_j w_{i,j}}{h_i}\left(F^{y}_{i+1,j+1} - F^{y}_{i,j+1}\right) \end{Bmatrix}$$

$$M_{4,3} = 2v_{i,j}\hat{v}_{i,j}\left\{\left(w_{i,j}\Delta_{i,j+1} - F^{x}_{i,j+1}\right) + 0.5F^{x}_{i+1,j+1}\right\}$$

$$M_5 = \sum_{j=0}^{3} (1-\varphi)^{3-j} \varphi^j M_{5,j}$$

where,

$$M_{5,0} = v_{i,j}\hat{u}_{i,j} F^{x}_{i+1,j}$$

$$M_{5,1} = v_{i,j}(\hat{w}_{i,j} F^{x}_{i+1,j} + \hat{h}_j F^{xy}_{i+1,j})$$

$$M_{5,2} = v_{i,j}(\hat{w}_{i,j} F^{x}_{i+1,j+1} - \hat{h}_j F^{xy}_{i+1,j+1})$$

$$M_{5,3} = \hat{v}_{i,j} u_{i,j} F^{x}_{i+1,j+1}$$

$$S_y(x,y) = \frac{\sum_{i=0}^{3} (1-\theta)^{3-i} \theta^i N_i}{q_{i,j}(\theta)\left(\hat{q}_{i,j}(\varphi)\right)^2} \tag{19}$$

where,

$$N_0 = \sum_{j=0}^{5} (1-\varphi)^{5-j} \varphi^j N_{0,j}$$

with,

$$N_{0,0} = u_{i,j}\hat{u}_{i,j} F^{y}_{i,j}$$

$$N_{0,1} = u_{i,j}\hat{u}_{i,j}\left\{\left(2w_{i,j}\hat{\Delta}_{i,j} + F^{y}_{i,j}\right) - 2F^{y}_{i,j+1}\right\}$$

$$N_{0,2} = u_{i,j} \begin{Bmatrix} \hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} - F^{y}_{i,j+1}\right) + 2u_{i,j}\left(w_{i,j}\Delta_{i,j} - F^{y}_{i,j+1}\right) \\ +3\hat{u}_{i,j} v_{i,j}\hat{\Delta}_{i,j} - w_{i,j}F^{y}_{i,j} \end{Bmatrix}$$

$$N_{0,3} = u_{i,j} \begin{Bmatrix} \hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} - F^{y}_{i,j}\right) + 2v_{i,j}\left(w_{i,j}\Delta_{i,j} - F^{y}_{i,j}\right) \\ +3\hat{u}_{i,j} v_{i,j}\hat{\Delta}_{i,j} - w_{i,j}F^{y}_{i,j+1} \end{Bmatrix}$$

$$N_{0,4} = 2u_{i,j}\hat{v}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + F^{y}_{i,j}\right)$$

$$N_{0,5} = u_{i,j}\hat{v}_{i,j}F^{y}_{i,j+1}$$

$$N_1 = \sum_{j=0}^{5} (1-\varphi)^{5-j} \varphi^j N_{1,j}$$

with,

$$N_{1,0} = \hat{u}_{i,j}\left(w_{i,j}F^{y}_{i,j} + h_i F^{xy}_{i,j}\right)$$

$$N_{1,1} = \hat{u}_{i,j} \begin{Bmatrix} \left(w_{i,j}F^{y}_{i,j} + h_i F^{xy}_{i,j}\right) - 2\left(w_{i,j}F^{y}_{i,j+1} + h_i F^{xy}_{i,j+1}\right) \\ +2\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + \dfrac{h_i}{\hat{h}_j}\left(F^{x}_{i,j+1} - F^{x}_{i,j}\right)\right) \end{Bmatrix}$$

$$N_{1,2} = \begin{cases} \left(0.5\hat{w}_{i,j} + u_{i,j}\right) \begin{Bmatrix} 2\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + \dfrac{h_i}{\hat{h}_j}\left(F^{x}_{i,j+1} - F^{x}_{i,j}\right)\right) \\ -2\left(w_{i,j}F^{y}_{i,j+1} + h_i F^{xy}_{i,j+1}\right) \end{Bmatrix} \\ +3\hat{u}_{i,j} v_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + \dfrac{h_i}{\hat{h}_j}\left(F^{x}_{i,j+1} - F^{x}_{i,j}\right)\right) \\ -\hat{w}_{i,j}\left(w_{i,j}F^{y}_{i,j} + h_i F^{xy}_{i,j}\right) \end{cases}$$

$$N_{1,3} = \begin{cases} \left(0.5\hat{w}_{i,j} + v_{i,j}\right) \begin{Bmatrix} 2\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + \dfrac{h_i}{\hat{h}_j}\left(F^{x}_{i,j+1} - F^{x}_{i,j}\right)\right) \\ -2\left(w_{i,j}F^{y}_{i,j} + h_i F^{xy}_{i,j}\right) \end{Bmatrix} \\ +3\hat{u}_{i,j} v_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + \dfrac{h_i}{\hat{h}_j}\left(F^{x}_{i,j+1} - F^{x}_{i,j}\right)\right) \\ -\hat{w}_{i,j}\left(w_{i,j}F^{y}_{i,j+1} + h_i F^{xy}_{i,j+1}\right) \end{cases}$$

$$N_{1,4} = 2\hat{v}_{i,j} \begin{Bmatrix} \left(w_{i,j}F^{y}_{i,j} + h_i F^{xy}_{i,j}\right) + \\ \hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i,j} + \dfrac{h_i}{\hat{h}_j}\left(F^{x}_{i,j+1} - F^{x}_{i,j}\right)\right) \end{Bmatrix}$$

$$N_{1,5} = \hat{v}_{i,j}\left(w_{i,j}F^{y}_{i,j+1} + h_i F^{xy}_{i,j+1}\right)$$

$$N_2 = \sum_{j=0}^{5} (1-\varphi)^{5-j} \varphi^j N_{2,j}$$

with,

$$N_{2,0} = \hat{u}_{i,j}\left(w_{i,j}F^{y}_{i+1,j} - h_i F^{xy}_{i+1,j}\right)$$

$$N_{2,1}=\hat{u}_{i,j}\left\{\begin{array}{l}\left(w_{i,j}F^{y}_{i+1,j}-h_iF^{xy}_{i+1,j}\right)-2\left(w_{i,j}F^{y}_{i+1,j+1}-h_iF^{xy}_{i+1,j+1}\right)\\+2\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i+1,j}-\frac{h_i}{\hat{h}_j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)\right)\end{array}\right\}$$

$$N_{2,2}=\left\{\begin{array}{l}\left(0.5\widehat{\widehat{w}}_{i,j}+v_{i,j}\right)\left\{\begin{array}{l}2\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i+1,j}-\frac{h_i}{\hat{h}_j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)\right)\\-2\left(w_{i,j}F^{y}_{i+1,j+1}-h_iF^{xy}_{i+1,j+1}\right)\end{array}\right\}\\+3\widehat{\widehat{u}}_{i,j}v_{i,j}\left(w_{i,j}\hat{\Delta}_{i+1,j}-\frac{h_i}{\hat{h}_j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)\right)\\-\hat{w}_{i,j}\left(w_{i,j}F^{y}_{i+1,j}-h_iF^{xy}_{i+1,j}\right)\end{array}\right.$$

$$N_{2,3}=\left\{\begin{array}{l}\left(0.5\widehat{\widehat{w}}_{i,j}+v_{i,j}\right)\left\{\begin{array}{l}2\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i+1,j}-\frac{h_i}{\hat{h}_j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)\right)\\-2\left(w_{i,j}F^{y}_{i+1,j}-h_iF^{xy}_{i+1,j}\right)\end{array}\right\}\\+3\widehat{\widehat{u}}_{i,j}v_{i,j}\left(w_{i,j}\hat{\Delta}_{i+1,j}-\frac{h_i}{\hat{h}_j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)\right)\\-\hat{w}_{i,j}\left(w_{i,j}F^{y}_{i+1,j+1}-h_iF^{xy}_{i+1,j+1}\right)\end{array}\right.$$

$$N_{2,4}=2\hat{v}_{i,j}\left\{\begin{array}{l}\hat{w}_{i,j}\left(w_{i,j}\hat{\Delta}_{i+1,j}-\frac{h_i}{\hat{h}_j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)\right)\\-\left(w_{i,j}F^{y}_{i+1,j}-h_iF^{xy}_{i+1,j}\right)\end{array}\right\}$$

$$N_{2,5}=\hat{v}_{i,j}\left(w_{i,j}F^{y}_{i+1,j+1}-h_iF^{xy}_{i+1,j+1}\right)$$

$$N_3=\sum_{j=0}^{5}(1-\varphi)^{5-j}\varphi^{j}N_{3,j}$$

with,

$$N_{3,0}=v_{i,j}\hat{u}_{i,j}F^{y}_{i+1,j}$$

$$N_{3,1}=v_{i,j}\widehat{\widehat{u}}_{i,j}\left\{\left(2w_{i,j}\hat{\Delta}_{i+1,j}+F^{y}_{i+1,j}\right)-2F^{y}_{i+1,j+1}\right\}$$

$$N_{3,2}=v_{i,j}\left\{\begin{array}{l}\widehat{\widehat{\widehat{w}}}_{i,j}\left(w_{i,j}\widehat{\widehat{\Delta}}_{i+1,j}-F^{y}_{i+1,j+1}\right)+2u_{i,j}\left(w_{i,j}\Delta_{i+1,j}-F^{y}_{i+1,j+1}\right)\\+3\widehat{\widehat{\widehat{u}}}_{i,j}v_{i,j}\hat{\Delta}_{i+1,j}-w_{i,j}F^{y}_{i+1,j}\end{array}\right\}$$

$$N_{3,3}=v_{i,j}\left\{\begin{array}{l}\widehat{\widehat{\widehat{w}}}_{i,j}\left(w_{i,j}\widehat{\widehat{\Delta}}_{i+1,j}-F^{y}_{i+1,j+1}\right)+2v_{i,j}\left(w_{i,j}\Delta_{i+1,j}-F^{y}_{i+1,j}\right)\\+3\widehat{\widehat{\widehat{u}}}_{i,j}v_{i,j}\hat{\Delta}_{i+1,j}-w_{i,j}F^{y}_{i+1,j}\end{array}\right\}$$

$$N_{3,4}=v_{i,j}\widehat{\widehat{v}}_{i,j}\left\{\left(2w_{i,j}\hat{\Delta}_{i+1,j}+F^{y}_{i+1,j+1}\right)+2F^{y}_{i+1,j}\right\}$$

$$N_{3,5}=v_{i,j}\hat{u}_{i,j}F^{y}_{i+1,j}$$

Firstly, from equation (15), the rational function $S_x(x,y)>0$ if

$$\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^{i}M_i>0 \text{ and } (q_{i,j}(\theta))^2\hat{q}_{i,j}(\varphi)>0.$$

We have $(q_{i,j}(\theta))^2\hat{q}_{i,j}(\varphi)>0$ if equation (16) is satisfied and $\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^{i}M_i>0$ if $M_i>0 \;\; \forall i$.

$M_0>0$ if $M_{0,j}, \quad j=0,1,2,3$

We have $M_{0,0}>0$ and $M_{0,3}>0$ if equations (14) and (15) are satisfied.

$M_{0,1}>0$ if

$$\hat{w}_{i,j}>\frac{-\hat{h}_jF^{xy}_{i,j}}{F^{x}_{i,j}}$$

$M_{0,2}>0$ if

$$\hat{w}_{i,j}>\frac{\hat{h}_jF^{xy}_{i,j+1}}{F^{x}_{i,j+1}}$$

Hence $M_0>0$ if

$$\hat{w}_{i,j}>max\left\{\frac{-\widehat{\widehat{h}}_jF^{xy}_{i,j}}{F^{x}_{i,j}},\frac{h_jF^{xy}_{i,j+1}}{F^{x}_{i,j+1}}\right\}\tag{20}$$

$M_1>0$ if

$$w_{i,j}>max\left\{\frac{F^{x}_{i+1,j}}{\Delta_{i,j}},\frac{F^{x}_{i+1,j+1}}{\Delta_{i,j+1}},-\frac{F^{x}_{i,j}}{2\Delta_{i,j}},-\frac{F^{x}_{i,j+1}}{2\Delta_{i,j+1}}\right\}\tag{21}$$

$$\hat{w}_{i,j}>max\left\{-\frac{h_jF^{xy}_{i+1,j}}{F^{x}_{i+1,j}},\frac{h_jF^{xy}_{i+1,j+1}}{F^{x}_{i+1,j+1}}\right\}\tag{22}$$

$M_2>0$ if

$$w_{i,j}>max\left\{\frac{F^{x}_{i,j}}{\Delta_{i,j}},\frac{F^{x}_{i+1,j}}{\Delta_{i,j}},\frac{F^{x}_{i,j+1}}{\Delta_{i,j+1}},\frac{F^{x}_{i+1,j+1}}{\Delta_{i,j+1}}\right\}\tag{23}$$

$$\hat{w}_{i,j}>max\left\{-\frac{h_jF^{xy}_{i+1,j}}{F^{x}_{i+1,j}},-\frac{h_jF^{xy}_{i,j}}{F^{x}_{i,j}},\frac{h_jF^{xy}_{i+1,j+1}}{F^{x}_{i+1,j+1}},\frac{h_jF^{xy}_{i,j+1}}{F^{x}_{i,j+1}}\right\}\tag{24}$$

$M_3>0$ if

$$w_{i,j}>max\left\{\frac{F^{x}_{i,j}}{\Delta_{i,j}},\frac{F^{x}_{i+1,j}}{\Delta_{i,j}},\frac{F^{x}_{i,j+1}}{\Delta_{i,j+1}},\frac{F^{x}_{i+1,j+1}}{\Delta_{i,j+1}}\right\}\tag{25}$$

$$\hat{w}_{i,j}>max\left\{-\frac{h_jF^{xy}_{i+1,j}}{F^{x}_{i+1,j}},-\frac{h_jF^{xy}_{i,j}}{F^{x}_{i,j}},\frac{h_jF^{xy}_{i+1,j+1}}{F^{x}_{i+1,j+1}},\frac{h_jF^{xy}_{i,j+1}}{F^{x}_{i,j+1}}\right\}\tag{26}$$

$M_4>0$ if

$$w_{i,j}>max\left\{\frac{F^{x}_{i,j}}{\Delta_{i,j}},-\frac{F^{x}_{i+1,j}}{2\Delta_{i,j}},\frac{F^{x}_{i,j+1}}{\Delta_{i,j+1}},-\frac{F^{x}_{i+1,j+1}}{2\Delta_{i,j+1}}\right\}\tag{27}$$

$$\hat{w}_{i,j}>max\left\{-\frac{h_jF^{xy}_{i,j}}{F^{x}_{i,j}},\frac{h_jF^{xy}_{i,j+1}}{F^{x}_{i,j+1}}\right\}\tag{28}$$

Lastly, $M_5 > 0$ if

$$\hat{w}_{i,j} > max\left\{-\frac{h_j F^{xy}_{i+1,j}}{F^{x}_{i+1,j}}, \frac{h_j F^{xy}_{i+1,j+1}}{F^{x}_{i+1,j+1}}\right\} \qquad (29)$$

After reducing the similar terms in above expressions of $w_{i,j}$ and $\hat{w}_{i,j}$ from equations (20)-(29), we have,

$$w_{i,j} > max\left\{\begin{array}{l}\frac{F^x_{i,j}}{\Delta_{i,j}}, -\frac{F^x_{i,j}}{2\Delta_{i,j}}, \frac{F^x_{i+1,j}}{\Delta_{i,j}}, -\frac{F^x_{i+1,j}}{2\Delta_{i,j}}, \\ \frac{F^x_{i,j+1}}{\Delta_{i,j+1}}, -\frac{F^x_{i,j+1}}{2\Delta_{i,j+1}}, \frac{F^x_{i+1,j+1}}{\Delta_{i,j+1}}, -\frac{F^x_{i+1,j+1}}{2\Delta_{i,j+1}},\end{array}\right\} \qquad (30)$$

$$\hat{w}_{i,j} > max\left\{-\frac{h_j F^{xy}_{i,j}}{F^x_{i,j}}, \frac{h_j F^{xy}_{i,j+1}}{F^x_{i,j+1}}, -\frac{h_j F^{xy}_{i+1,j}}{F^x_{i+1,j}}, \frac{h_j F^{xy}_{i+1,j+1}}{F^x_{i+1,j+1}}\right\} \qquad (31)$$

Secondly, from equation (17), the second rational function $S_y(x,y) > 0$ if

Both $\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i N_i > 0$ and $q_{i,j}(\theta)\left(\hat{q}_{i,j}(\varphi)\right)^2 > 0$.

We have $q_{i,j}(\theta)\left(\hat{q}_{i,j}(\varphi)\right)^2 > 0$ if equation (16) is satisfied.

$\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i N_i > 0$ if $N_i > 0, \;\; i = 0,1,2,3$.

$N_0 > 0$ if $N_{0,j} > 0, \;\; j = 0,1,...,5$

We have $N_{0,0}$ >0 and $N_{0,5} > 0$ if equations (14) and (15) are satisfied.

$N_{0,1} > 0$ if

$$\hat{w}_{i,j} > -\frac{F^y_{i,j}}{2\hat{\Delta}_{i,j}}$$

$N_{0,2} > 0$ and $N_{0,3} > 0$ if

$$\hat{w}_{i,j} > max\left\{\frac{F^y_{i,j}}{\hat{\hat{\Delta}}_{i,j}}, \frac{F^y_{i,j+1}}{\Delta_{i,j}}\right\}$$

$N_{0,4} > 0$ if

$$\hat{w}_{i,j} > -\frac{F^y_{i,j+1}}{2\hat{\Delta}_{i,j}}$$

Hence, $N_0 > 0$ if

$$\hat{w}_{i,j} > max\left\{\frac{F^y_{i,j}}{\widetilde{\Delta}_{i,j}}, -\frac{F^y_{i,j}}{2\Delta_{i,j}}, \frac{F^y_{i,j+1}}{\Delta_{i,j}}, -\frac{F^y_{i,j+1}}{2\Delta_{i,j}}\right\} \qquad (32)$$

$N_1 > 0$ if

$$w_{i,j} > max\left\{-\frac{h_i F^{xy}_{i,j}}{F^y_{i,j}}, -\frac{h_i F^{xy}_{i,j+1}}{F^y_{i,j+1}}\right\} \qquad (33)$$

$N_2 > 0$ if

$$w_{i,j} > max\left\{\frac{h_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}\right\} \qquad (34)$$

Lastly, $N_3 > 0$ if

$$\hat{w}_{i,j} > max\left\{\frac{F^y_{i+1,j}}{\widetilde{\Delta}_{i+1,j}}, -\frac{F^y_{i+1,j}}{2\Delta_{i+1,j}}, \frac{F^y_{i+1,j+1}}{\Delta_{i+1,j}}, -\frac{F^y_{i+1,j+1}}{2\Delta_{i+1,j}}\right\} \qquad (35)$$

Combining the above expressions of $w_{i,j}$ and $\hat{w}_{i,j}$, from equations (32)-(35), we get

$$w_{i,j} > max\left\{\frac{h_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}, -\frac{h_i F^{xy}_{i,j}}{F^y_{i,j}}, -\frac{h_i F^{xy}_{i,j+1}}{F^y_{i,j+1}}\right\} \qquad (36)$$

$$\hat{w}_{i,j} > max\left\{\begin{array}{l}\frac{F^y_{i,j}}{\widetilde{\Delta}_{i,j}}, -\frac{F^y_{i,j}}{2\Delta_{i,j}}, \frac{F^y_{i,j+1}}{\Delta_{i,j}}, -\frac{F^y_{i,j+1}}{2\Delta_{i,j}}, \\ \frac{F^y_{i+1,j}}{\widetilde{\Delta}_{i+1,j}}, -\frac{F^y_{i+1,j}}{2\Delta_{i+1,j}}, \frac{F^y_{i+1,j+1}}{\Delta_{i+1,j}}, -\frac{F^y_{i+1,j+1}}{2\Delta_{i+1,j}}\end{array}\right\} \qquad (37)$$

Joining the equations (30) and (36), we have

$$w_{i,j} > max\left\{\begin{array}{l}\frac{F^x_{i,j}}{\Delta_{i,j}}, -\frac{F^x_{i,j}}{2\Delta_{i,j}}, \frac{F^x_{i+1,j}}{\Delta_{i,j}}, -\frac{F^x_{i+1,j}}{2\Delta_{i,j}}, \\ \frac{F^x_{i,j+1}}{\Delta_{i,j+1}}, -\frac{F^x_{i,j+1}}{2\Delta_{i,j+1}}, \frac{F^x_{i+1,j+1}}{\Delta_{i,j+1}}, -\frac{F^x_{i+1,j+1}}{2\Delta_{i,j+1}}, \\ \frac{h_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}, -\frac{h_i F^{xy}_{i,j}}{F^y_{i,j}}, -\frac{h_i F^{xy}_{i,j+1}}{F^y_{i,j+1}}\end{array}\right\} \qquad (38)$$

The above result can be rewritten as:

$$w_{i,j} = l_{i,j} + max\left\{\begin{array}{l}\frac{F^x_{i,j}}{\Delta_{i,j}}, -\frac{F^x_{i,j}}{2\Delta_{i,j}}, \frac{F^x_{i+1,j}}{\Delta_{i,j}}, -\frac{F^x_{i+1,j}}{2\Delta_{i,j}}, \\ \frac{F^x_{i,j+1}}{\Delta_{i,j+1}}, -\frac{F^x_{i,j+1}}{2\Delta_{i,j+1}}, \frac{F^x_{i+1,j+1}}{\Delta_{i,j+1}}, -\frac{F^x_{i+1,j+1}}{2\Delta_{i,j+1}}, \\ \frac{h_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}, -\frac{h_i F^{xy}_{i,j}}{F^y_{i,j}}, -\frac{h_i F^{xy}_{i,j+1}}{F^y_{i,j+1}}\end{array}\right\}, \;\; l_{i,j} > 0$$

Combining the equations (31) and (37), we have

$$\hat{w}_{i,j} > max\left\{\begin{array}{l}\frac{F^y_{i,j}}{\widetilde{\Delta}_{i,j}}, -\frac{F^y_{i,j}}{2\Delta_{i,j}}, \frac{F^y_{i,j+1}}{\Delta_{i,j}}, -\frac{F^y_{i,j+1}}{2\Delta_{i,j}}, \\ \frac{F^y_{i+1,j}}{\widetilde{\Delta}_{i+1,j}}, -\frac{F^y_{i+1,j}}{2\Delta_{i+1,j}}, \frac{F^y_{i+1,j+1}}{\Delta_{i+1,j}}, -\frac{F^y_{i+1,j+1}}{2\Delta_{i+1,j}}, \\ -\frac{h_j F^{xy}_{i,j}}{F^x_{i,j}}, \frac{h_j F^{xy}_{i,j+1}}{F^x_{i,j+1}}, -\frac{h_j F^{xy}_{i+1,j}}{F^x_{i+1,j}}, \frac{h_j F^{xy}_{i+1,j+1}}{F^x_{i+1,j+1}}\end{array}\right\} \qquad (39)$$

Rewriting the above result as:

$$\hat{w}_{i,j} = m_{i,j} + max\left\{\begin{array}{l} \frac{F^y_{i,j}}{\widetilde{\Delta}_{i,j}}, -\frac{F^y_{i,j}}{2\Delta_{i,j}}, \frac{F^y_{i,j+1}}{\Delta_{i,j}}, -\frac{F^y_{i,j+1}}{2\Delta_{i,j}}, \\ \frac{F^y_{i+1,j}}{\widetilde{\Delta}_{i+1,j}}, -\frac{F^y_{i+1,j}}{2\Delta_{i+1,j}}, \frac{F^y_{i+1,j+1}}{\Delta_{i+1,j}}, -\frac{F^y_{i+1,j+1}}{2\Delta_{i+1,j}}, \\ -\frac{h_j F^{xy}_{i,j}}{F^x_{i,j}}, \frac{h_j F^{xy}_{i,j+1}}{F^x_{i,j+1}}, -\frac{h_j F^{xy}_{i+1,j}}{F^x_{i+1,j}}, \frac{h_j F^{xy}_{i+1,j+1}}{F^x_{i+1,j+1}} \end{array}\right\}, \; m_{i,j} > 0$$

□

Table.1: 3D monotone data

y/x	1	2	3	4	5	6	7	8
1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
2	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
3	4	4.1	4.2	4.3	4.4	4.5	4.6	4.7
4	6	6.1	6.2	6.3	6.4	6.5	6.6	6.7
5	8	8.1	8.2	8.3	8.4	8.5	8.6	8.7
6	65	65.1	65.2	65.3	65.4	65.5	65.6	65.7
7	70	70.1	70.2	70.3	70.4	70.5	70.6	70.7
8	140	140.1	140.2	140.3	140.4	140.5	140.6	140.7

Table. 2: Monotone surface data

y/x	1	2	3	4	5	6	7
1	0.1491	0.1492	0.1493	0.1494	0.1495	0.1496	0.1497
2	0.1691	0.1692	0.1693	0.1694	0.1695	0.1696	0.1697
3	0.2098	0.2099	0.2100	0.2101	0.2102	0.2103	0.2104
4	0.9437	0.9438	0.9439	0.9440	0.9441	0.9442	0.9443
5	0.9986	0.9987	0.9988	0.9989	0.9990	0.9991	0.9992
6	0.9994	0.9995	0.9996	0.9997	0.9998	0.9999	1.0000
7	1.0001	1.0002	1.0003	1.0004	1.0005	1.0006	1.0007

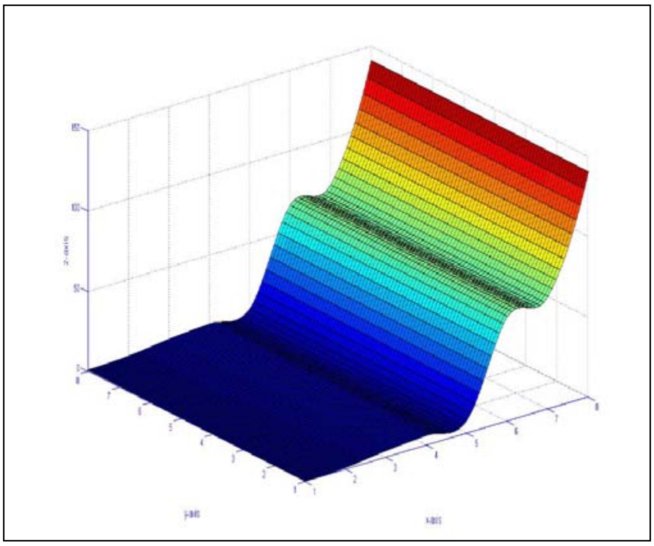

Fig.1: Bi-cubic Hermite Spline

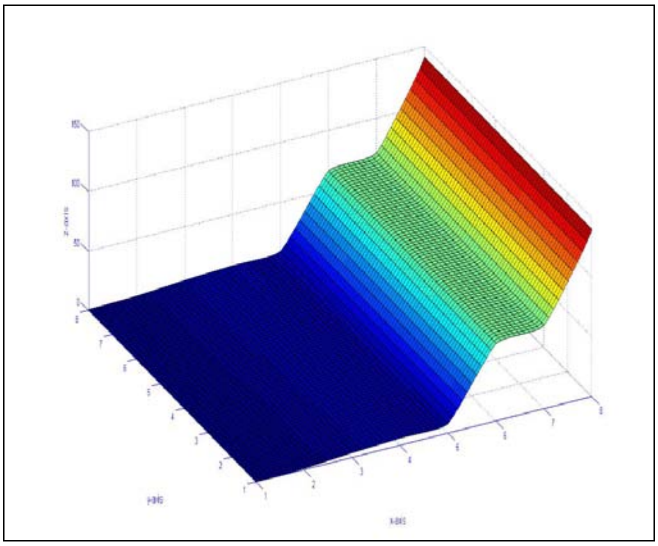

Fig.4: Shape-preserving rational bi-cubic surface interpolation

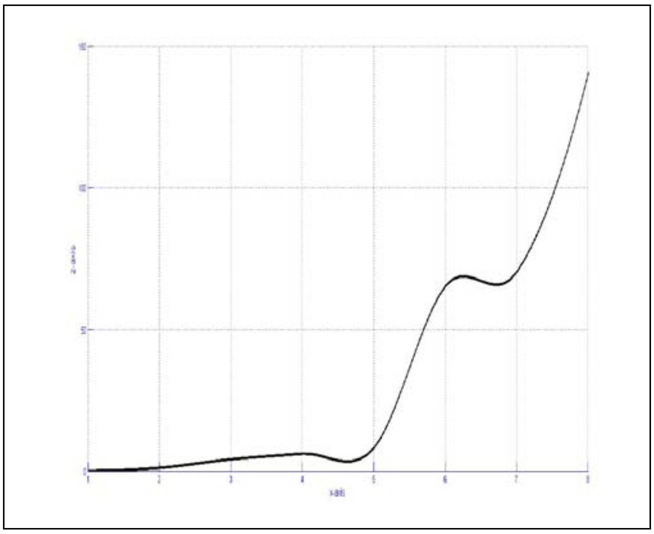

Fig.2: xz view of Figure.1

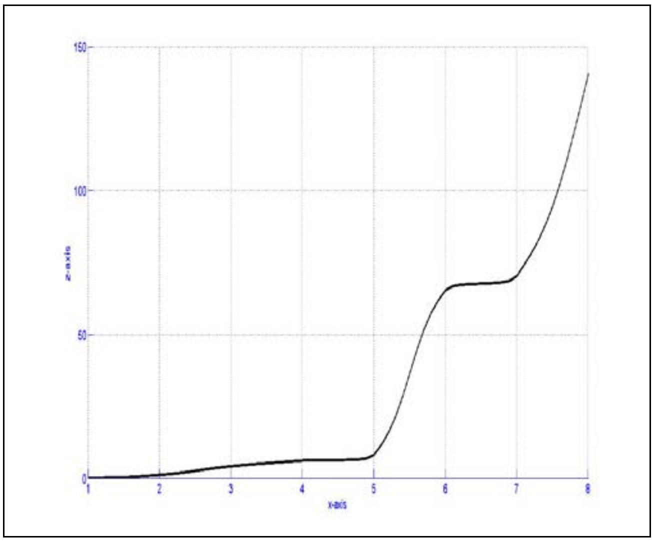

Fig.5: xz view of Figure.4

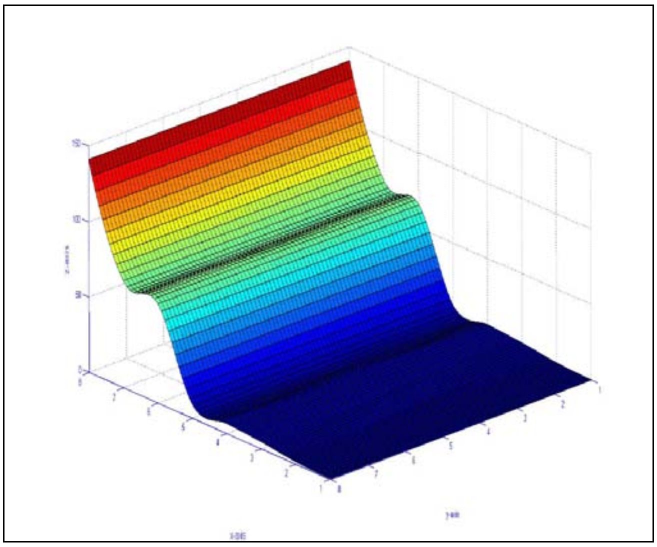

Fig.3: Different view of Figure.1

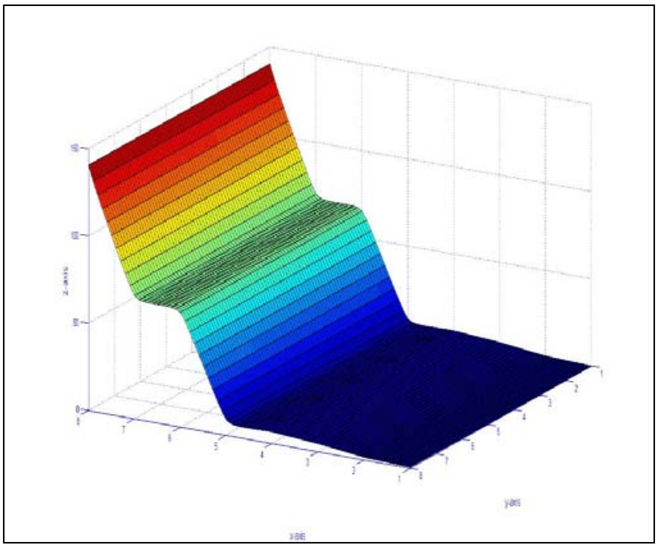

Fig.6: Different view of Figure.4

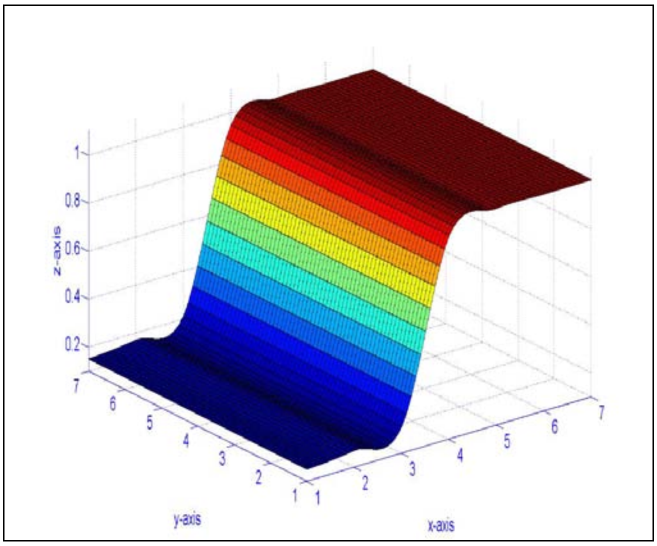

Fig.7: Bi-cubic Hermite Spline

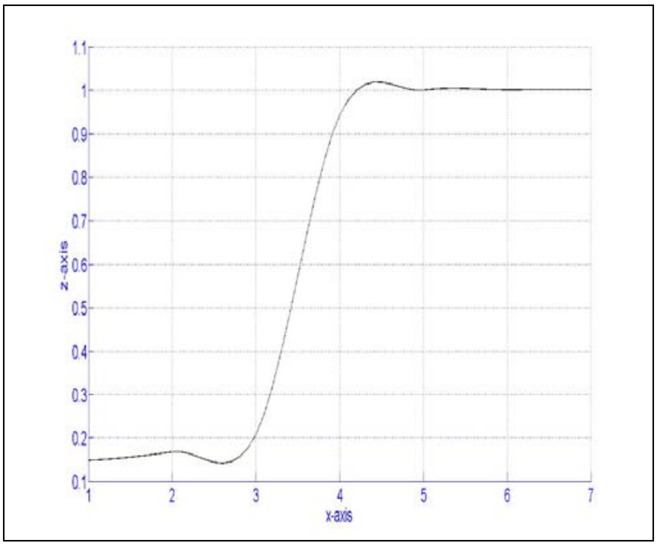

Fig.8: xz view of Figure.7

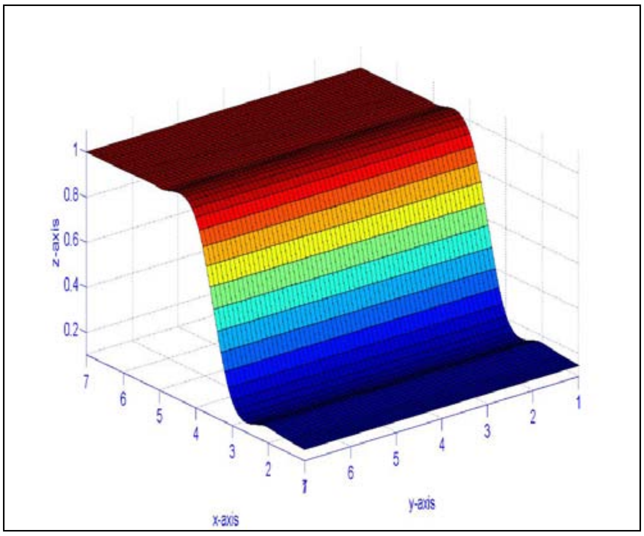

Fig.9: Different view of Figure.7

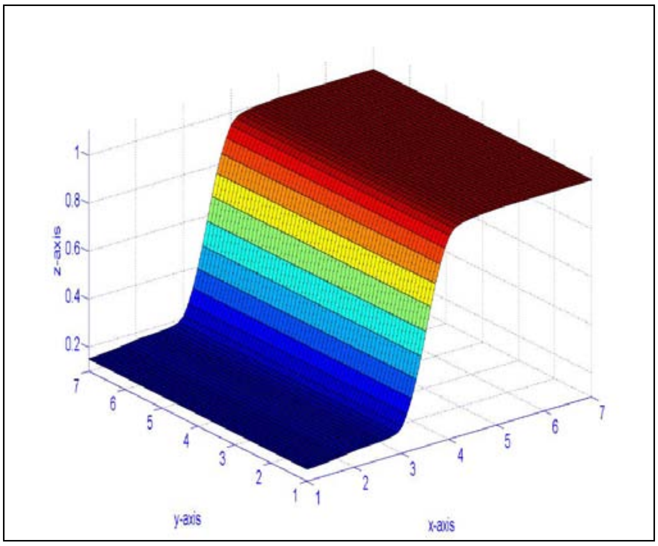

Fig.10: Shape-preserving rational bi-cubic surface interpolation with $u_{i,j}=1.5, \hat{u}_{i,j}=1.5, v_{i,j}=1.5, v_{i,j}=1.5$

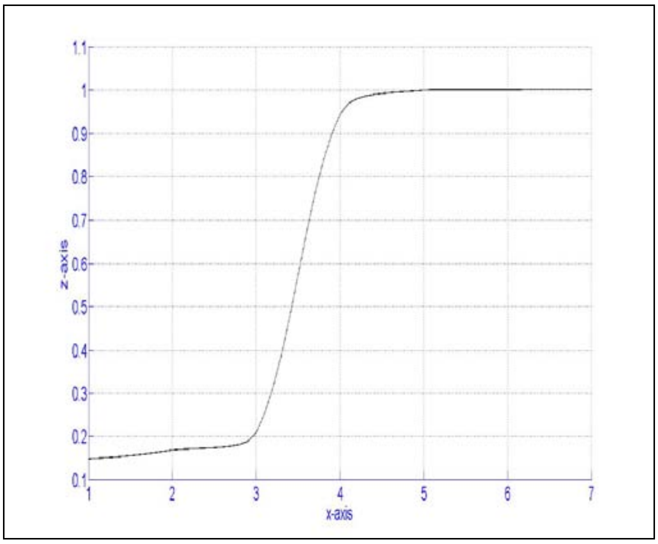

Fig.11: xz view of Figure.10

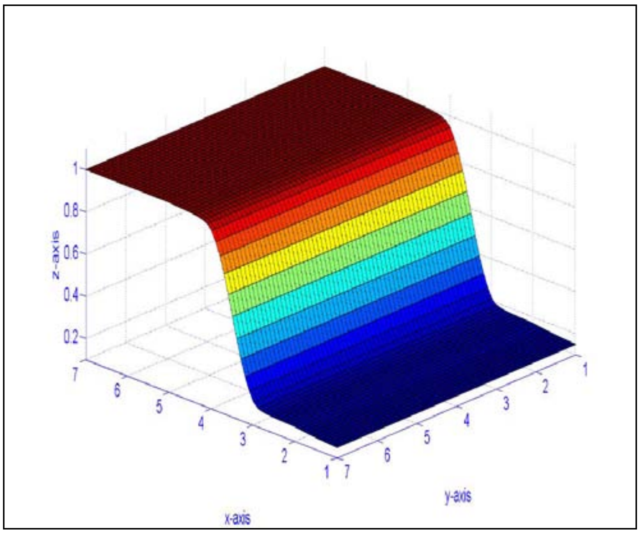

Fig.12: Different view of Figure.

6. Numerical Examples

In this section, a numerical demonstration of monotonicity preserving scheme given in section 5 is presented.

Example.1

A monotone surface data set taken in Table 1. Figure.1 is drawn by bi-cubic Hermite spline that does not preserve the monotonicity of monotone surface data. Figure.2 and Figure.3 are representing different view of Figure.1; we remark that these figures do not preserve the shape of data. To overcome this flaw, Figure.4 is generated by the monotonicty preserving rational bi-cubic surface scheme developed in section 5 with the values of free parameters set as: $u_{i,j} = 1.5$, $\hat{u}_{i,j} = 1.5$, $v_{i,j} = 1.5$ and $\hat{v}_{i,j} = 1.5$ to preserve the monotonicity of monotone data. Figure.5 and Figure.6 represent different view of Figure.4; it is clearly shown that these figures preserve monotonicity everywhere. A prominent difference in the smoothness with a visually pleasant view can be seen in these figures due to the liberty bestowed to the designer on the values of shape parameters.

Example.2

The bi-cubic Hermite spline scheme has been used to draw the Figure.7 using the monotone surface data set taken in Table 2. A more clear view of non-monotonicity can be seen in Figure.8 and Figure.9 which are xz and different view of Figure.7 respectively. On the other hand, the efficiency of the monotonicity preserving rational bi-cubic scheme developed in section 5 can be seen in Figure.10. A more comprehensible view of monotonicity can be seen in Figure.11 and Figure.12 which are xz and different view of Figure.10 respectively. A remarkable difference in the smoothness with a shape preserving pleasant graphical view is visible in the figures drawn by the newly developed scheme due to the freedom granted to the designer on the values of shape parameters.

7 Concluding Remark

In this paper, we have extended a C^1 piecewise rational cubic function to rational bi-cubic function with six free parameters in each rectangular patch to preserve the monotonicity of 3D monotone data. The free parameters are arranged in such a way that two of them are constrained parameters (not depend on the other parameter like [11]) to preserve the shape of 3D monotone data while the remaining are left free for designer 's choice for the refinement of monotone surface as desired. The developed surface scheme has been tested through different numerical examples and it is shown that the scheme is not only local and computationally economical but also visually pleasant.

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