On an extension of Camina′s theorem on conjugacy class sizes

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Abstract: Let G be a finite group. We extend Alan Camina′s theorem on conjugacy class sizes which asserts that if the conjugacy class sizes of G are exactly $\{1, p^a, q^b, p^aq^b\}$, where p and q are two distinct primes and a and b are integers, then G is nilpotent. We show that when the set of conjugacy class sizes of all elements of primary and biprimary orders of G is $\{1, p^a, q^b, p^aq^b\}$, where p and q are two distinct primes and a and b are integers, then G is nilpotent.

Key–Words: Conjugacy class sizes; Nilpotent groups; Solvable groups; Sylow p-subgroup; Finite groups.

1 Introduction

In this paper, any group is a finite group. We say that a group element has primary or biprimary order respectively if its order is divisible by at most one or two primes. We will denote by x^G the conjugacy class of x in G and (following Baer [1]) we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G(in some other papers, $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$ is called conjugacy class size or length of x in G, for example, [2, 3]). We will often refer to the index of an element, this is just the size of the conjugacy class containing the element. The benefit of this definition is entirely linguistic. Given an element g in some group G we can talk about the index of g rather than talking about the size of the conjugacy class containing g. So if we are referring to elements we use the term index but if we are talking about conjugacy classes we refer to size. The rest of our notation and terminology are standard. The reader may refer to ref.[4].

Here, we consider the influence of the sizes of conjugacy classes on the structure of finite groups. Over the last 30 years there have been many papers on this topic and it would seem to be a good idea to try to bring some of the key results together in one place. This is especially relevant as some authors seem unaware of others writing in the field as well as some of the older results which seem to get reproved quite regularly. It is hoped that in writing this, less time will be spent in reproving old results, enabling more progress to be made on some of the more interesting problems. How much information can one expect to obtain from the sizes of conjugacy classes? Sylow in 1872 examined what happened if there was information about the sizes of all conjugacy classes, whereas in 1904 Burnside showed that strong results could be obtained if there was particular information about the size of just one conjugacy class. Landau in 1903 bounded the order of the group in terms of the number of conjugacy classes whilst in 1919 Miller gave a detailed analysis of groups with very few conjugacy classes. Very little then seems to have been done until 1953 when both Baer and Itô published papers on this topic but with different conditions on the sizes. By looking at these early results it can been seen that much will depend on how much information is given and it is important to be explicit. For example if one knows that there is only one conjugacy class size then the group is abelian, but this can be any abelian group. However if you know the collection of conjugacy class sizes, that is the multiplicities, then the order of the group is also known. However it would still not be possible to identify the group. Some authors have considered the situation where the multiplicities of the conjugacy class sizes are used if the size is not 1. This is particularly true when the authors have been studying aspects of the problem related to graphs. Again if we only demand information about the sizes of conjugacy classes and not their multiplicities, the group G and $G \times P$ will have the same set whenever P is an abelian group. So we can only state results modulo a direct abelian factor.

It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist several results studying the solvability or the nilpotence of a group under some arithmetical conditions on its conjugacy class sizes. In [1], R. Baer proves that a group G is solvable if its elements of prime power order have also prime power index. N. Itô shows in [5] that if the sizes of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. In [6], Li Shirong prove that if the finite group G has exactly two conjugacy class lengths of elements of prime power order of G, then G is solvable. There exist other deeper results. For instance, in [7], Itô shows that if the conjugacy class sizes of G are $\{1, n, m\}$, then G is solvable. In [8], Yakov Berkovich and Lev Kazarin prove that suppose that indices of all elements of primary or biprimay orders of a non-abelian group G are powers of primes, then one and only one of the following holds: (a) $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ is non-abelian and A is abelian. (b) $G = F \times A$, where A is abelian, F is a nonnilpotent biprimary Hall subgroup of G with abelian Sylow subgroups. On the other hand, A.R. Camina proves in [2] that if the conjugacy class sizes of G are $\{1, p^a, q^b, p^a q^b\}$, where p and q are two distinct primes and a and b are integers, then G is nilpotent. Notice that the hypotheses of Caminas theorem imply the solvability of G just by using Burnside′s $p^a q^b$ theorem.

In this paper, we will replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to generalize the above Alan Camina′s theorem. We put our emphasis on conjugacy class sizes of all elements of prime-power or biprimary orders of G and obtain the following main result: Let G be a group. Assume that the conjugacy class sizes of primary and biprimary orders of G are exactly $\{1, p^a, q^b, p^a q^b\}$, where p and q are two distinct primes and a and b are positive integers, then G is nilpotent. In addition, we analyze a new case of groups having three conjugacy class sizes of primary and biprimary orders of G and generalize the result of Camina. Our theorem determine the structure of those groups whose conjugacy class sizes of primary and biprimary orders of G are $\{1, p^a, p^a q^b\}$, where p and q are coprime. In the proof we have not used the solvability obtained by Itô. We have preferred to avoid it by using more elementary technique at the cost of making the proof longer. The main result is: Let G be a group. If the set of conjugacy class sizes of all elements of primary and biprimary orders of G is $\{1, p^a, p^a q^b\}$ with $(p, q) = 1$, then $G \cong H \times K$, where K is abelian and H contains a normal subgroup of index p, $M \times P_1$, where M is an abelian q-subgroup and P_1 is an abelian p-subgroup, neither being central in G, and $M \times P_1$ is the set of all elements of H of index p^a. Finally, $p^a = p$ and P/P_1 acts fixed-point-free on M and $\Phi(P) \le Z(P)$, where $P \in \mathrm{Syl}_p(G)$.

In order to prove the above main results, we will first prove a supplementary result which is also an extension of Camina′s theorem. We will replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes of elements of primary and biprimary orders to generalize the Alan Camina′s theorem. Such an extension is the following: Let G be a group such that p^a is the highest power of the prime p which divides the index of an element of primary and biprimary order of G. Assume that there is a p-element in G whose index is precisely p^a. Then G has a normal p-complement.

2 Basic definitions and preliminary results

In this section, we introduce the basic definitions and some elementary results that are used time and time again in this paper.

Definition 1 *Let G be a finite group and let $x \in G$. The index of x in G is given by $[G : C_G(x)]$ and is denoted by $Ind_G(x)$.*

We will make use of the classic Thompson′s $A \times B-$Lemma.

Lemma 2 *[9, Chap. 5, Theorem 3.4] Let $A \times B$ be a group of automorphisms of the p-group P with A a $p'-$group and B a p-group. If A acts trivially on $C_P(B)$, then $A = 1$.*

Lemma 3 *[3, Lemma 1.1] Let $N \trianglelefteq G, x \in N$, and $y \in G$. Then*
(i)$|x^N| \mid |x^G|$.
(ii)$|(yN)^{G/N}| \mid |y^G|$.

In order to prove our main theorem, we need the following important lemma.

Lemma 4 *[10, Theorem 5] Let G be a finite group and p a prime divisor of $|G|$. Then there is in G no $p'-$element of prime power order whose index is divisible by p if and only if $G = P \times H$, where P is a Sylow p-subgroup of G and H has order prime to p.*

Lemma 5 *Let G be a group. A prime p does not divide any conjugacy class size of any element of prime power order of G if and only if G has a central Sylow p-subgroup.*

Proof: By Lemma 4 we know that $G = P \times H$, where P is a Sylow p-subgroup of G and H has order prime to p. In the following we only need to prove P is abelain. For any element $y \in P$, then p does not divide $|y^G| = |G : C_G(y)|$ according to the hypotheses.

Thus $P \leq C_G(y)$, P is abelian. Our proof is complete now. □

The Lemma 5 can be seen as a generalization of [11, Theorem 33.4]:Let G be a group. A prime p does not divide any conjugacy class size of G if and only if G has a central Sylow p-subgroup.

Lemma 6 *[12, Lemma 6] Suppose that the three smallest non-trivial indices of elements of a group G are $a < b < c$, with $(a, b) = 1$ and $a^2 < c$. Then the set $\{g \in G : |g^G| = 1$ or $a\}$ is a normal subgroup of G.*

Lemma 7 *(Wielandt)[1, Lemma 6] $O_p(G)$ contains every element in G whose order and index are powers of p.*

In the following Lemma, we characterize those groups whose conjugacy class size of any non-central p'−element of prime power order of G is a power of the prime p and determine the structure of those groups whose conjugacy class size of every p-element of G is a p-number.

Lemma 8 *Let G be a finite group.*

(a) The conjugacy class size of any p'−element of prime power order of G is a p-number if and only if G has abelian p-complememnt.

(b) The conjugacy class size of every p-element of G is a p-number if and only if $G = P \times H$, where $P \in Syl_p(G)$ and H is a p-complement of G.

Proof: (a)We show first that G is solvable in both direction of the Lemma. Suppose first that any p'−element of prime power order G is a p-number. Our assumption is inherited by normal subgroup and quotient groups by Lemma 3, and hence, arguing by induction on $|G|$ we may assume that G is simple nonabelian. However, Burnside′s Theorem (15.2 of [11] for instance]) asserts that a simple group cannot possess a conjugacy class of prime power length and thus, the claim follows. Conversely, suppose that G has an abelian p-complement. Then G can be written as the product of two nilpotent subgroups, that is, an abelian p-complement and a Sylow p-subgroup of G. By Kegel-Wielandt′s Theorem (VI.4.3 of [13]), it follows that G is solvable too.

Suppose now that every conjugacy class size of any p'−element of prime power order of G is a p-number if and work by induction on $|G|$ to show that G has abelian p-complement. We assume first that $O_p(G) \neq 1$. By induction, $G/O_p(G)$ has abelian p-complement and trivially so does G. Thus, we can assume that $O_p(G) = 1$ and consequently $O_{p'}(G) \neq 1$. Let x be a non-central p'−element of prime power order of G. As G is solvable and $|G : C_G(x)|$ is a p-power, we can get that there exists a p-complement of G, say, H, such that $x \in H \subseteq C_G(x)$. Observe that $1 \neq F(G) \subseteq O_{p'}(G) \subseteq H$, where $F(G)$ is the Fitting subgroup. This implies that $x \in C_G(H) \subseteq C_G(F(G)) \subseteq F(G) \subseteq O_{p'}(G)$. Therefore, any non-central p'−element of prime power order of G belongs to $O_{p'}(G)$(and any central p'−element of prime power order of G belongs to $O_{p'}(G)$ too). Since H can be generated by elements of prime power order, we have that $H \trianglelefteq G$, that is, G has normal p-complement. Moreover, notice that this complement is abelian.

The converse direction is trivial, just noticing that since G is solvable, then any two p-complements of G are conjugated, whence all of them are abelian.

(b) See[15, Lemma 3]. □

Lemma 9 *([21, Lemma 1 (c)]) Let G be a π-separable group. If $x \in G$ and $|x^G|$ is a π-number, then $x \in O_{\pi,\pi'}(G)$.*

In order to give the structure of a finite group with two conjugacy class sizes, we need the following important lemma.

Lemma 10 *Let G be a group. Then the following two conditions are equivalent:*

(i) 1 and $m > 1$ are the only lengths of conjugacy classes of p'-elements of primary and biprimary orders of G;

(ii) 1 and $m > 1$ are the only lengths of conjugacy classes of p'-elements of G.

Proof: (i)⟹(ii)

Let a be any q-element of index m and b be any r-element of $C_G(a)$, where $q \neq p$ and $r \neq p$. Notice that

$$C_G(ab) = C_G(a) \bigcap C_G(b) \subseteq C_G(a)$$

and since m is the largest conjugacy class size of p'-elements of primary and biprimary orders of G, then $C_G(ab) = C_G(a)$ and hence $C_G(a) \subseteq C_G(b)$. This implies that $b \in Z(C_G(a))$.

Now let x be any non-central p'-element of G and write $x = x_1 x_2 \cdots x_s$, $s \geq 3$, where the order of each x_i is a power of a prime p_i ($p_i \neq p$, $i = 1, 2, \cdots, s$) and the x_i commute pairwise. As x is a non-central p'-element of G, we know that at least one of the x_i such that x_i is non-central. Without loss of generality, we can assume that x_1 is non-central. Now

$$\begin{aligned} C_G(x) &= C_G(x_1 x_2 \cdots x_s) \\ &= C_G(x_1) \bigcap C_G(x_2 \cdots x_s) \\ &= C_G(x_1) \bigcap C_G(x_2) \bigcap \cdots \bigcap C_G(x_s) \\ &\subseteq C_G(x_1), \end{aligned}$$

and by the previous argument we may conclude that have that $x_i \in Z(C_G(x_1))$ for $i = 2, \cdots, s$. Hence we get that $C_G(x_1) \leq C_G(x_i), i = 2, \cdots, s$. Thus

$$\begin{aligned} C_G(x) &= C_G(x_1x_2\cdots x_s) \\ &= C_G(x_1)\bigcap C_G(x_2\cdots x_s) \\ &= C_G(x_1)\bigcap C_G(x_2)\bigcap\cdots\bigcap C_G(x_s) \\ &= C_G(x_1). \end{aligned}$$

It follows that the conjugacy class size of x is equal to the conjugacy class size of x_1, that is, m. □

Lemma 11 *[23, Theorem A] Let G be a finite p-solvable group. If the set of p-regular conjugacy class sizes of G has exactly two elements, for some prime p, then G has Abelian p-complement or $G = PQ \times A$, with $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and $A \subseteq Z(G)$, with q a prime distinct from p. As a consequence, if $\{1, m\}$ are the p-regular conjugacy class sizes of G, then $m = p^aq^b$. In particular, if $b = 0$ then G has abelian p-complement and if $a = 0$ then $G = P\times Q\times A$ with $A \subseteq Z(G)$.*

Lemma 12 *[24, Theorem A] Let N be a p-solvable normal subgroup of a group G such that N contains a noncentral Sylow $r(\neq p)$-subgroup R of G. If $|x^G| = 1$ or m for every p-regular element x of N whose order is divisible by at most two distinct primes, then the p-complements of N are nilpotent.*

Lemma 13 *[24, Corollary 1] Let N be a p-solvable normal subgroup of a group G such that N contains a noncentral Sylow $r(\neq p)$-subgroup R of G. If $|x^G| = 1$ or m for every p-regular element x of N whose order is divisible by at most two distinct primes, then one and only one of the following statements holds:*

(1) If $r|m$, then $N = N_pR \times N_{\{p,r\}'}$, where $N_{\{p,r\}'} \leq Z(G)$ and R is non-abelian;

(2) If $r \nmid m$, then N has abelian p-complements.

3 Main results

Theorem 14 *Let G be a group such that p^a is the highest power of the prime p which divides the index of an element of primary and biprimary order of G. Assume that there is a p-element in G whose index is precisely p^a. Then G has a normal p-complement.*

Proof: By the hypothesis we let x be a p-element of index p^a. It is easy to know (see, for example, [1]) that the normal closure of x will be a p-group, say H. Let $Z = C_G(H)$. Now $[G : C_G(x)] = p^a$, and so if $y \in C_G(x)$ and y has prime power order prime to p, $[C_G(x) : C_G(xy)]$ is prime to p. For otherwise p^{a+1} would divide the index of xy, contrary to the hypothesis. However, $C_G(xy) = C_G(x)\bigcap C_G(y)$, as x and y has coprime order and $[x, y] = 1$. Since H is normal in G, $H \cap C_G(x) \leq H \cap C_G(y)$ or $C_H(x) \leq C_H(y)$. We can now use Lemma 2.1 to deduce that $C_H(y) = H$.

So H centralizes every element of prime power order prime to p in $C_G(x)$. We can conclude that H centralizes every element of order prime to p in $C_G(x)$. In fact, for any element of order prime to p in $C_G(x)$, we can write $z = z_1z_2\dots z_s$, where z_i is a power of a prime distinct from to p and the z_i commute pairwise, and $z_i \in C_G(x)$. By the above paragraph it follows that H centralizes every element of order prime to p in $C_G(x)$.

As $[G : C_G(x)] = p^a$, we can deduce that $[G : Z]$ is a power of p. Now let w be any $p'-$element of prime power order in Z. By the previous argument, $[C_G(x) : C_G(w) \cap C_G(x)]$ is prime to p, but, as Z is a normal subgroup of $C_G(x)$, we have that $[Z : C_Z(w)]$ is prime to p by Lemma 3. Thus every $p'-$element of prime power order in Z has index in Z prime to p and so, by lemma 4 we have that $Z = K \times P_1$, where K has order prime to p and P_1 is the Sylow p-subgroup of Z. As $[G : Z]$ is a power of p, K is a normal p-complement of G.

Our proof is complete now. □

Remark 15 *In Theorem 14 we note that as H is a normal p-subgroup of G, H is contained in all Sylow p-subgroups of G and so Z contains all p-elements of index prime to p. But clearly any p-element in Z has index a power of p and so, if P is a Sylow p-subgroup of G, $Z(P) = Z(G)\cap P$. Further, if v is an element of index prime to p, then $v = st$, where s is a p'-element of G and t is in $Z(G)$. For, if $v = st$, where s is a p'-element and t is a p-element or $t = 1$, t has index prime to p and so is in $Z(G)$.*

Theorem 16 *Let G be a group. Assume that the set of conjugacy class sizes of all elements of primary and biprimary orders of G is exactly $\{1, p^a, q^b, p^aq^b\}$, where p and q are two distinct primes and a and b are positive integers, then G is nilpotent.*

Proof: The proof has been divided into several steps.

Step 1. We may assume that G is a $\{p, q\}-$group.

If the order of G is divisible by a prime r, $r \neq p, q$, then by Lemma 5 we have that $G = R \times H$, where R is an abelian r-group and H is an r'-group. So we need only consider H. Thus we can assume that the order of G is divisible only by prime p and q.

Step 2. If there is a p-element in G whose index is precisely p^a, then the theorem is proved. Consequently, if there is a q-element in G whose index is precisely q^b, then the theorem is proved too.

Assume that we have a p-element of index p^a. Then by Theorem 3.1 we know that G has a normal p-complement and any element of index prime to p is a product of a q-element and an element in the center of G by Remark 15 and Step 1. So there is a q-element of index q^b, and thus G has a normal q-complement by Theorem 14, and so G is nilpotent.

This argument will work just well if there is a q-element of index q^b.

Step 3. we may assume that if x is a p-element of G, then $\text{Ind}_G(x) = 1, q^b$ or p^aq^b; if x is a q-element of G, then $\text{Ind}_G(x) = 1, p^a$ or p^aq^b.

By Step 1 and Step 2 it is obvious.

Step 4. We may assume that there exist some q-element of index p^a. Consequently, there exist some p-element of index q^b.

By the hypothesis and Step 2 it is enough to consider to the decomposition of any element of index p^a as a product of a p-element by a q-element. In the same way we can prove the second part of this Step.

Step 5. If x is a p-element of index p^aq^b, then $C_G(x) = P_x \times V_x$ with P_x a p-group and V_x an abelian q-group such that $V_x \nsubseteq Z(G)$. If y is a q-element of index p^aq^b, then $C_G(y) = P_y \times V_y$ with P_y an abelian p-group such that $P_y \nsubseteq Z(G)$ and V_y a q-group.

Let x be a p-element of index p^aq^b and let y be any q-element of $C_G(x)$. Notice that $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ and since p^aq^b is the largest conjugacy class size of G, then $C_G(xy) = C_G(x)$, so $C_G(x) \subseteq C_G(y)$. This implies that $y \in Z(C_G(x))$, so we can write $C_G(x) = P_x \times V_x$ with P_x a p-group and V_x an abelian q-group. It remains to show that V_x cannot be central in G.

Suppose that $V_x \subseteq Z(G)$, and notice that then $V_x = Z(G)_q$ and $|G : Z(G)|_q = q^b$. Choose z a non-central p-element of G, which must have index q^b or p^aq^b by Step 2. In every case, notice that $Z(G)_q$ is a p-complement of $C_G(z)$. This implies that if we choose any non-central q-element w of G, then any p-element of $C_G(w)$ must be central in G. Thus $Z(G)_p$ is a Sylow p-subgroup of $C_G(w)$. Since w has index p^a or p^aq^b, then $|G : Z(G)|_p = p^a$. This yields $|G : Z(G)| = |G : Z(G)|_p|G : Z(G)|_q = p^aq^b$, which contradicts the existence in G of elements of index p^aq^b. Thus, the first assertion of the step is proved.

The second part of this step can be proved by reasoning in a similar way with a q-element of index p^aq^b.

Step 6. If $p^a > q^b$, then the set $L_p := \{x : x$ is p-element and $|x^G| = 1$ or $q^b\}$ is an abelian normal p-subgroup of G. If $p^a < q^b$, then the set $L_q := \{x : x$ is q-element and $|x^G| = 1$ or $p^a\}$ is an abelian normal q-subgroup of G.

It is enough to apply Lemma 6 to obtain that if $p^a > q^b$ then the set $W := \{x : |x^G| = 1$ or $q^b\}$ is a normal subgroup of G. Analogously, if $p^a < q^b$, then the set $W := \{x : |x^G| = 1$ or $p^a\}$ is a normal subgroup of G. Now, if x is any element of index q^b and factorize $x = x_px_q$, with x_p and x_q a p-element and a q-element, respectively, it follows that x_q must be central by Step 2, whence $x \in L_p \times Z(G)_q$. Therefore, $W = L_p \times Z(G)_q$ and L_p is also a normal p-subgroup of G. The argument for L_q is similar.

Finally, we see for instance that L_p is abelian, as the argument for L_q is the same. If we take any $y \in L_p$ then $|L_p : C_{L_p}(y)|$ divides $(|L_p|, q^b) = 1$. Consequently, L_p is abelian.

Step 7. If $p^a > q^b$, then L_p is an abelian normal Sylow p-subgroup of G; If $p^a < q^b$, then L_q is an abelian normal Sylow q-subgroup of G.

At first we assume that $p^a > q^b$. In order to prove that L_p is a Sylow p-subgroup of G it is enough to show, by taking into account Step 2, that there are no p-elements of index p^aq^b. Suppose that z is a p-element of index p^aq^b and by Step 5, write $C_G(z) = P_z \times V_z$, with V_z a noncentral abelian q-group and P_z a p-group. If $t \in V_z$, it is clear that $C_G(z) \subseteq C_G(t)$, so in particular $C_{L_p}(z) \subseteq C_{L_p}(t)$. By applying Lemma 2.1, we get $t \in M := C_G(L_p)$ and therefore, $V_z \subseteq M$. On the other hand, by Step 2, we know that t has index p^a or p^aq^b, so $|C_G(t) : C_G(z)|$ must be equal to 1 or q^b. This proves that $L_p \subseteq C_G(z)$ and we conclude that L_p centralizes every p-element of index p^aq^b. But on the other hand, any p-element of index q^b trivially centralizes L_p as it is abelian. Therefore, we conclude that any p-element of G lies in M, whence $|G : M|$ is a q-number. Furthermore, since $L_p \subseteq M \subseteq C_G(k)$ for any k noncentral element of L_p, which has index q^b, then q^b must divide $|G : M|$. Now, if we consider the equality $|G : M||M : V_z| = |G : C_G(z)||C_G(z) : V_z|$, then all the properties remarked above imply that V_z is a p-complement of M.

Let x be a p-element of G, which we know lies in M. If x has index 1 or q^b, then it certainly follows that $x \in Z(M)$. If x has index p^aq^b, then by Step 5, we write $C_G(x) = P_x \times V_x$ with V_x a non-central abelian q-group and P_x a p-group. As we have seen above, V_x is a p-complement of M, and in particular $V_x \subseteq C_M(x)$ and $|M : C_M(x)|$ is a p-number. Therefore, we have shown that the index of any p-element of M is a p-number. Thus, by applying Lemma 8(b), we can factor $M = P \times T$, where $P \in Syl_p(G)$ and T is a q-group, which must be equal to V_z. In particular, P is normal in G. But now, if we choose some non-central $y \in V_z$, then $P \subseteq C_G(y)$, which contradicts Step 2.

The second part of this step can be proved by reasoning in a similar way.

Step 8. If $p^a > q^b$, then the p-complements of G are abelian; If $p^a < q^b$, then the q-complements of G are abelian.

We firstly consider $p^a > q^b$. Let H be a p-complement of G and assume that it is not abelian. By Lemma 8(a) and Step 2, there exist q-elements in H of index $p^a q^b$. Let w be any such element. By Step 5, we write $C_G(w) = P_w \times V_w$ with P_w an abelian p-group such that $P_w \nsubseteq Z(G)$ and V_w a q-group. We will prove that V_w is abelian too. We may choose a non-central p-element $u \in C_G(w)$, which certainly satisfies $C_G(w) \subseteq C_G(u)$. By Step 7, we know that $|u^G| = q^b$, so $|C_G(u) : C_G(w)| = p^a$. Therefore, V_w is a Sylow q-subgroup of $C_G(u)$. On the other hand, if v is a q-element of $C_G(u)$, then $|C_G(u) : C_G(uv)| = |C_G(u) : C_G(u) \cap C_G(v)|$ is a power of p. Thus, by Lemma 8(b), $C_G(u)$ has abelian Sylow q-subgroups. So V_w is abelian as we wanted to show and consequently, $C_G(w)$ is abelian too.

If $Z(H) = Z(G)_q$, then there would not be q-elements of index p^a, and this yields a contradiction with Step 4. Thus there exist non-central elements in $Z(H)$. For any such element, say y, note that $y \in C_G(w)$ and as $C_G(w)$ is abelian, we have $C_G(w) \subseteq C_G(y) = C_{L_p}(y)H$. Moreover, since $L_p \trianglelefteq G$, we have $C_{L_p}(y) \trianglelefteq C_G(y)$. Since $H \subseteq C_G(y)$ and L_p is abelian, it follows that $T := C_{L_p}(y) \trianglelefteq G$. Furthermore, as $|C_G(y) : C_G(w)| = q^b$, it follows that T is the Sylow p-subgroup of $C_G(w)$, so $T = P_w$ and in particular, T is not central in G. Notice that we have also proved that T centralizes any q-element in H of index $p^a q^b$ and any element in $Z(H)$.

Now, if we take $v \in H$ of index p^a, then there exists some $g \in G$ such that $H^g \subseteq C_G(v)$, whence $v^{g^{-1}} \in Z(H)$. By the above paragraph, $T \in C_G(v^{g^{-1}})$ and as T is normal in G, we get that T also centralizes v. Then $T \subseteq C_G(H)$ and as L_p is abelian, we conclude that $T \subseteq Z(G)$, a contradiction.

Step 9. (Conclusion). Assume first that $p^a > q^b$. We claim that $|G : Z(G)|_q = q^b$ by the hypothesis. Let z be an element of index $p^a q^b$ and write $z = z_p z_q$, with z_p and z_q the p-part and q-part of z, respectively. If z_p is not contained in $Z(G)$, then by Step 7, $|z_p^G| = q^b$ and $Z(G)_q$ is a Sylow q-subgroup of $C_G(z)$, so $z_q \in Z(G)$, which is a contradiction since z has index $p^a q^b$. If $z_p \in Z(G)$, then $|z_q^G| = p^a q^b$, by Step 8, this is a contradiction too.

If $p^a < q^b$, the proof is the same.

The proof of the theorem is now complete. □

Remark 17 *The most obvious connection with character theory is that the number of conjugacy classes is the same as the number of irreducible characters. A number of authors, including those of this article, have seen a connection between character degrees and conjugacy class sizes and have searched for analogous results. Perhaps the first obvious difference is that the order of the group is given by the sum of the sizes of the conjugacy classes, but the sum of the squares of the degrees of the irreducible characters. This might help to explain the following dichotomies. As previously mentioned in Section 2, if p is coprime to all indices of elements of G then the Sylow p-subgroup of G is an abelian direct factor of G. However, Itô proved that if G has a normal abelian Sylow p-subgroup then p is coprime to all character degrees of G. Also, in [17] Cossey and Wang noted that if all indices are square-free then G is soluble. However, if all irreducible character degrees are square-free G need not be soluble, the smallest example we know is Alt(7), see for example in [18]. Another example is given by the two different conclusions drawn when* $\{1, p^a, q^b, p^a q^b\}$ *is either the set of character degrees or the set of conjugacy class sizes and p and q are distinct primes. In the conjugacy class case we can conclude that* $G = P \times Q$ *where P is the Sylow p-subgroup and Q the Sylow q-subgroup in [2]. This conclusion does not hold in the character case in [19].*

Theorem 18 *Let G be a group. If the set of conjugacy class sizes of all elements of primary and biprimary orders of G is* $\{1, p^a, p^a q^b\}$ *with* $(p, q) = 1$*, then* $G \cong H \times K$*, where K is abelian and H contains a normal subgroup of index p,* $M \times P_1$*, where M is an abelian q-subgroup and* P_1 *is an abelian p-subgroup, neither being central in G, and* $M \times P_1$ *is the set of all elements of H of index* p^a*. Finally,* $p^a = p$ *and* P/P_1 *acts fixed-point-free on M and* $\Phi(P) \leq Z(P)$*, where* $P \in Syl_p(G)$.

Proof: The proof has been divided into several steps.

Step 1. The q-Hall subgroup M of G is normal and abelian, Further $M = [M, P] \times C_M(P)$, where $P \in \mathrm{Syl}_p(G)$.

We consider two cases.

Case a. At first we assume that there is no p-element of index p^a. Then assume further that $N = O_p(G)$ is not contained in the center of G. Now let x be a p-element of index $p^a q^b$. We can conclude that $C_G(x) = U \times V$, where U is a p-group and V is an abelian q-group. In fact, let y be any q-element of $C_G(x)$. Notice that $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ and since $p^a q^b$ is the largest conjugacy class size of G, then $C_G(xy) = C_G(x)$, so $C_G(x) \subseteq$

$C_G(y)$. This implies that $y \in Z(C_G(x))$, so we can write $C_G(x) = U \times V$ with U a p-group and V an abelian q-group. Also, by use of Thompson′s Theorem [9, Theorem 5.3.2], we have that $V \leq C_G(N)$. As $N \nleq Z(G)$, we get that $q^b | [G : C_G(N)]$. But $q^b | [G : C_G(x)]$ exactly and so V is a q-Hall subgroup of $C_G(N)$. Since $C_G(N)$ is a nilpotent group, we have that V is normal in G. Let y be any q-element of G not in V, and let P be a Sylow p-subgroup of G such that $[P : P \bigcap C_G(y)] = p^a$. Now assume that w is in P and w is in $C_G(y)$, then we have that $[G : C_G(w)] \neq p^a q^b$, otherwise $y \in V$. So $P \bigcap C_G(y) = P \bigcap Z(G)$ as G has no p-element of index p^a. Then $[P : P \bigcap Z(G)] = p^a$. But if $z \in P$ but is not in $P \bigcap Z(G)$, p^a cannot divide $[G : C_G(z)]$, which leads to a contradiction. Hence we have to assume that $N = O_p(G) \leq Z(G)$.

As there exist elements in G of index p^a, they must be q-elements. In fact, if x is any element of index p^a and $x = x_p x_q$, with x_p a p-element and x_q a q-element, respectively. Then we have $C_G(x) = C_G(x_p) \bigcap C_G(x_q) \subseteq C_G(x_q)$. Hence $[G : C_G(x_q)] | [G : C_G(x)] = p^a$. By hypotheses we have that $[G : C_G(x_q)] = p^a$. Thus $O_{p,q}(G)$ contains all q-elements of index p^a by Lemma 9 and so $O_{p,q}(G) \nleq Z(G)$. Further as $O_p(G) \leq Z(G)$, we have that $O_{p,q}(G) = O_p(G) \times O_q(G)$. Put $O_q(G) = L$ and notice that $L \nleq Z(G)$.

Now we suppose that we have a q-element y of G such that $[G : C_G(y)] = p^a q^b$. Then by using arguments applied earlier, we can deduce that $C_G(y) = H \times K$, where K is a p-group and $K \leq C_G(L)$. As G is solvable by Itô′s Theorem. We have that $C_G(O_{p,q}(G)) \leq O_{p,q}(G)$. Furthermore $O_{p,q}(G) = O_p(G) \times L$ and $O_p(G) \leq Z(G)$. We get that $C_G(L) \leq O_p(G) \times L$. So $K = O_p(G)$, which is central. Now if P is any a Sylow p-subgroup of G, $|P/K| = p^a$ and this leads to a contradiction. Hence every q-element of G has index p^a and is in L and L is abelian.

Case b. Now we can consider the case where there exist p-elements of index p^a. Then by Theorem 14 we have know that G has a normal p-complement, M, say, and that $O_p(G)$ consists of all the p-elements of index p^a and 1 by Lemma 7. If there exists an element $x \in M$ such that $[G : C_G(x)] = p^a q^b$, we can get again deduce that $C_G(x) = X \times Y$ with Y a p-group and $Y \leq C_G(M)$. As M is a q-Hall subgroup of G, we have that $Y = O_p(G)$. Let P be an arbitrary Sylow p-subgroup of G, then we get that $[P : Y] = p^a$. If $u \in P$ but u is not in Y, then $C_M(u) = M \bigcap Z(G)$ since Y is the Sylow p-subgroup of the centralizer of any non-central element of M. Then $[M : M \bigcap Z(G)] = q^b$, which contradicts the existence of q-element of index $p^a q^b$.

Hence we now get that in either case the q-Hall subgroup M of G is normal and abelian. Then we can use [9, Theorem 5.2.3] and deduce that $M = [M, P] \times C_M(P)$, where $P \in \mathrm{Syl}_p(G)$.

Step 2. Let $P_1 = O_p(G)$, then P/P_1 acts fixed-point-free on M and so $|P/P_1| = p^a$ and $|M| = q^b$, Also P/P_1 is cyclic of order p^a or generalized quaternion of order p^a.

By Step 1 we know that $M = [M, P] \times C_M(P)$ and $C_M(P)$ is a direct factor we can assume that $M = [M, P]$. Now $O_p(G) = C_P(M)$ which is normal in G. Let $P_1 = O_p(G)$. If we consider the action of P/P_1 on M, we know that P/P_1 acts half-transitively as every q-element has p^a conjugates and now by [22], we have that P/P_1 acts fixed-point-free or M is an elementary abelian q-group and P/P_1 acts semi-regularity (and hence fixed-point-free) or irreducibly on M. In the last case $Z(P/P_1)$ acts fixed-point-free on M, that is, $|M| = q^b$, but then P/P_1 acts fixed-point-free on M.

Therefore in all cases P/P_1 acts fixed-point-free and so $|P/P_1| = p^a$ and $|M| = q^b$. Also P/P_1 is cyclic of order p^a or generalized quaternion of order p^a.

Step 3. $p^a = p$ and $\Phi(P) \leq Z(P)$, where $P \in \mathrm{Syl}_p(G)$.

We will show next that the case where $P_1 \leq Z(P)$ does not occur. Notice that as $P_1 \geq Z(P)$ in any case, we have that $P_1 = Z(P)$ in this situation. If P/P_1 is cyclic, then P would be abelian and then we would have no elements of index $p^a q^b$. if P/P_1 is generalized quaternion, then there exists $x \in P$ such that $[P : \langle x \rangle P_1] = 2$ and so, as $\langle x \rangle P_1$ is abelian, $[P : C_P(x)] = 2$ and so the conjugacy class lengths of G are $\{1, 2, 2q^b\}$. But now $P \cong G/Q$ is a p-group whose conjugacy class lengths are $\{1, 2\}$ and so has class 2, which contradicts the assertion that $P/Z(P) \cong$ quaternion group.

Hence we get that there exist elements in P_1 of index p^a and elements in M of index p^a. Let $x \in P_1$, $y \in M$ such that $[G : C_G(x)] = [G : C_G(y)] = p^a$. Then clearly $C_G(xy) = C_G(x) \bigcap C_G(y)$ and xy has index p^a. So $C_G(x) = C_G(y) = M \times P_1$. This is true for all $x \in P_1$ such that $[G : C_G(x)] = p^a$ and so P_1 is abelian and for $x \in P_1$, $[G : C_G(x)] \neq 1$, we have that $C_G(x) = M \times P_1$. That is , if u is p-element but u is not in P_1, then $C_G(u) \bigcap P_1 = P_1 \bigcap Z(G)$. Let $\overline{P} \cong G/Q$ and denote by – the homomorphism from G to G/Q. Then $\overline{P}$ is a p-group whose conjugacy class lengths are $\{1, p^a\}$, Further, $\overline{P}$ has a normal abelian subgroup $\overline{P_1}$ of index p^a with $\overline{P}/\overline{P_1}$ cyclic or generalized quaternion.

Let $\overline{x} \in \overline{P_1}$, we have that $C_{\overline{P}}(\overline{x}) = Z(\overline{P})$ or $\overline{x} \in Z(\overline{P})$. Let $\overline{u}$, $\overline{v} \in \overline{P} \backslash \overline{P_1}$ and assume that

$C_{\overline{P}}(\overline{u}) \bigcap C_{\overline{P}}(\overline{v}) > Z(\overline{P})$. Let $\overline{w}$ be not in $Z(\overline{P})$ such that $[\overline{w}, \overline{u}] = [\overline{w}, \overline{v}] = 1$. Now using the structure of $\overline{P}/\overline{P_1}$ we can find an integer n and an element $\overline{z}$ of P_1 such that either $\overline{w} = \overline{u}^n\overline{z}$ or $\overline{u} = \overline{w}^n\overline{z}$. But as $[\overline{u}, \overline{w}] = 1$, $[\overline{z}, \overline{u}] = 1$ and so $\overline{z} \in Z(\overline{P})$. Then $C_{\overline{P}}(\overline{u}) = C_{\overline{P}}(\overline{v})$ and similarly $C_{\overline{P}}(\overline{w}) = C_{\overline{P}}(\overline{v})$. Hence if $C_{\overline{P}}(\overline{u}) \bigcap C_{\overline{P}}(\overline{v}) \geq Z(\overline{P})$, then we can conclude that $C_{\overline{P}}(\overline{u}) = C_{\overline{P}}(\overline{v})$. Thus we have a partition for $\overline{P}/Z(\overline{P})$. Also, as $|C_{\overline{P}}(\overline{x})| = |C_{\overline{P}}(\overline{y})|$ for any two non-central elements, this is an abelian partition, and so by [5, 1.3], $\overline{P}/Z(\overline{P})$ has just one normal component, $\overline{P_1}/Z(\overline{P})$, which must contain all elements of order greater than p, or $\overline{P}/Z(\overline{P})$ has exponent p. In either case we have to conclude that $a = 1$. Then $\Phi(\overline{P}) \leq Z(\overline{P})$ and so $\overline{P}$ is a p-group of class 2 and that concludes the proof of the theorem. □

Remark 19 *We remark that the following example show that the situation as described in Theorem 3.1 can occur. Let P be a p-group of exponent and order p^3 generated by x, y. Let $P_1 = \langle x, [x, y]\rangle$ and let $M = \langle v\rangle$, a cyclic group of order q where $p|(q-1)$. We construct the split extension $MP = G$, where P is mapped into Aut(M) by mapping x to the identity automorphism and by mapping y to the automorphism of M of order p. Then this split extension $MP = G$ is the required example.*

Theorem 20 *Let G be a finite p-solvable group. If $\{1, m\}$ are the conjugacy class sizes of p-regular elements of primary and biprimary orders of G, for some prime p, then G has Abelian p-complement or $G = PQ \times A$, with $P \in Syl_p(G)$, $Q \in Syl_q(G)$ and $A \subseteq Z(G)$, with q a prime distinct from p. As a consequence, if $\{1, m\}$ are the conjugacy class sizes of p-regular elements of primary and biprimary orders of G, then $m = p^a q^b$. In particular, if $b = 0$ then G has abelian p-complement and if $a = 0$ then $G = P \times Q \times A$ with $A \subseteq Z(G)$.*

Proof: By Lemma 10 and 11, we can get that Theorem 20 holds. □

At last, we use conjugacy class sizes to study the structure of the normal subgroup N of G. It is well known that if N is a normal subgroup of a group G, then it is clear that N is a set union of some conjugacy classes of a group G. Hence, it is natural to explore the structure of the normal subgroup N of G if the sizes of the G-conjugacy class of N are given. In this aspect, Riese et al. [25, 26] described the structure of the normal subgroup N when N is a set union of four or three conjugacy classes of a group G, respectively.

We now consider the following question.

Question *Let G be a finite group and N a p-solvable normal subgroup of G. If $|x^G| = 1$ or m for every p-regular element x of primary and biprimary orders in N, whether the p-complements of N are nilpotent or not?*

Our answer to the above question is the following theorem.

Theorem 21 *Let N be a p-solvable normal subgroup of a group G such that N contains a noncentral Sylow $r(\neq p)$-subgroup R of G. If $|x^G| = 1$ or m for every p-regular element x of primary and biprimary orders of N whose order is divisible by at most two distinct primes, then the p-complements of N are nilpotent.*

Proof By Lemma 10 and 12, we can get that Theorem 21 holds. □

Corollary 22 *Let N be a p-solvable normal subgroup of a finite group G such that N contains a noncentral Sylow $r(\neq p)$-subgroup R of G. If $|x^G| = 1$ or m for every p-regular element x of primary and biprimary orders of N whose order is divisible by at most two distinct primes, then one and only one of the following statements holds:*

(1) If $r|m$, then $N = N_pR \times N_{\{p,r\}'}$, where $N_{\{p,r\}'} \leq Z(G)$ and R is non-abelian;

(2) If $r \nmid m$, then N has abelian p-complements.

Proof: By Lemma 10 and 13, we can get that Corollary 22 holds. □

Corollary 23 *Let G be a finite p-solvable group. If $|x^G| = 1$ or m for every p-regular element x of primary and biprimary orders of G whose order is divisible by at most two distinct primes, then the p-complements of G are nilpotent.*

Proof: By Lemma 10 and Corollary 2 in [24], we can get that Corollary 23 holds. □

Corollary 24 *Let G be a finite p-solvable group. If $|x^G| = 1$ or m for every p-regular element x of primary and biprimary orders of G, then the p-complements of G are nilpotent.*

Proof: By Lemma 10 and Corollary 3 in [24], we can get that Corollary 24 holds. □

Corollary 25 *Let G be a finite group. If $|x^G| = 1$ or m for every p-regular element x of primary and biprimary orders of G, then G is nilpotent.*

Proof: By Lemma 10 and Corollary 4 in [24], we can get that Corollary 25 holds. □

Comments: In our paper, we proved that a group G is nilpotent when its conjugacy class sizes of primary and biprimary orders of G are exactly $\{1, p^a, q^b, p^a q^b\}$, where p and q are two distinct primes and a and b are integers. We conjecture that a group G is nilpotent when its conjugacy class sizes of primary and biprimary orders of G are exactly $\{1, m, n, mn\}$, where m and n are two distinct integers.

4 Conclusion

The results explained in the previous sections show that the method that we replace conditions for all conjugacy classes by conditions referring to only some of the classes in order to investigate the structure of a finite group is very useful. Results of this type are interesting since they can be used to simplify the proofs of new or known properties related to conjugacy classes. In addition, according to the parallel property of conjugacy class sizes and character degrees, we may consider using the character degrees to characterize the structure of finite groups. As an application, we can investigate the structure of a finite group when its character degrees of G are exactly $\{1, p^a, q^b, p^a q^b\}$, where p and q are two distinct primes and a and b are integers.

Acknowledgements: The research was supported financially by the NNSF-China (10771132) and the Research Grant of Tianjin Polytechnic University..

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