# Transient Temperature Solutions of a Cylindrical Fin with Lateral Heat Loss

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*Abstract:* Analytical temperature solutions to the transient heat conduction for a two dimensional cylindrical fin with lateral heat source together with various convective effects on lateral surface is obtained by the method of superposition and separation variables. The temperature distributions are generalized for a linear combination of the product of Bessel function, Fourier series and exponential type for nine different cases. Relevant connections with some other closely-related recent works are also indicated.

*Key–Words:* Bessel function, Fourier series, Heat conduction, Temperature distribution, Separation variables, Superposition.

## 1 Introduction

Analytical solutions of different linear heat conduction equations are meaningful in heat transfer theory. In addition, they are very useful to computational heat transfer to verify numerical analysis and to develop numerical schemes, grid generation methods. A systematic procedure for determining the separation of variables for a given partial differential equation can be found in [1-2]. However, a cylindrical heat problem involved two dimensions with lateral heat source together with various convective effects in present study is presumably not solved in the existing literature on this subject. The superposition and the separation method are used in this study to get the analytical solutions of the temperature distribution. The separation variables method is applied in this study. The partial differential equations are transferred into ordinary differential equations by separating the independent variables involved in the problem. The temperature distributions of fins under transient condition are important for proper prediction and control of the fin performance. Closed-form analytical solution for the transient temperature distribution would provide continuous physical insight which is much better than discrete numbers from a numerical computation. Analytical solutions play an important role in heat transfer analysis. Many analytical solutions with constant coefficient equations played a key role in the early development of heat conduction. The main purpose of this study is to investigate the analytical transient solutions by using the method of separation of variables. A group of theory, being a systematic procedure of determining the separation variables can be found in [3-10]. The methods of superposition are widely used [9-15]. Sawandtong et.al. [11-12] established a blow-up solution and the blow-up set of such a solution of the degenerate parabolic problem and generalize the degenerate parabolic problem into the general form. Arruda [13] empirically choice the period and number of frequency lines of the Fourier series in the regressive discrete Fourier series and studied numerical instability. It showed that the regressive discrete Fourier series were useful for data smoothing, extrapolation, and computation of derivatives from noisy signals. Wen and Khonsari [14] present the Fourier series as the working function involving a body subjected to oscillating heat flux on one of its boundaries. Meshii and Watanabe [15] indicated a surface axisymmetric circumferential crack inside a hollow cylinder shows tendency toward crack arrest. The transient temperature distribution of the cylinder under thermal striping was analytically obtained by Bessel function. Gordeliy, Crouch, and Mogilevskaya [16] depicted the semi-analytical temperature solution of a truncated Fourier series for the transient heat conduction in a medium with two circular cavities. V. Jovanovic [17] investigated the problem of the transverse vibration of a beam with a viscous boundary and generalized Fourier series solution. H. M. Srivastava, K. Y. Kung and K-J Wang [10] present analytic solutions of a two-dimensional rectangular heat equation with a heat source. The superposition and the separation method are used in this study to get the analytical solutions for nine different cases.

### 1.1 Subsection

Fins provide a considerable increase in the surface area available for heat transfer rate. The transient behavior is related to many natural phenomena and industrial applications, such as the cooling of electronic components, power generators, semiconductors, cylinders of air-cooled aircraft, automatic control mechanism and many other equipments in which heat is generated and must be disposed. It is important to understand the temperature distribution of fins under transient or unsteady thermal condition for proper prediction and control of the fin performance. Analytic solutions are particularly useful, since they may be used in an on-line computer. A number of restrictive assumptions are introduced before studying the transient analysis of two dimensional cylinder fin, some of which are due to Kung [3]. The fundamental assumptions are as follows: a. The pin fin is equipped with homogeneous material properties and thickness. b. The convective heat transfer coefficient, the ambient temperature and the fin properties are assumed to be constant. c. The temperature distribution in the fin is axial symmetric. d. There is no heat source in the inner part of the fin, e. Thermal radiation effects are negligible.

The present research investigates the transient analysis of two- dimensional cylindrical fin with finite length. The method of separable variables is used in this study to get an analytical solution for the transient two-dimensional heat conduction. Moreover, besides theoretical meaning, analytical solutions can also be used to check the accuracy, convergence and effectiveness of several numerical computation methods and their difference schemes, grid generation ways. The analytical solutions are therefore very useful even for the rapidly developing computational heat conduction. One can apply Fourier's law and energy conservation law to form a set of dimensionless governing differential equation, the initial condition and the boundary conditions as following.

$$\frac{\partial v(r,z,t)}{\partial t} = \frac{\partial^2 v(r,z,t)}{\partial z^2} + \frac{\partial^2 v(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,z,t)}{\partial r} - v(r,z,t) \quad (1)$$

$$t = 0, \quad v(r,z,0) = 0 \quad (2)$$

$$t > 0, r = 0, \quad v(0,z,t) \to \text{finite} \quad (3)$$

$$r = 1, \quad v_r(1,z,t) + Bi_r v(1,z,t) = 0, \quad (4)$$

$$z = 0, \quad -v_z(r,0,t) + Bi v(r,0,t) = Bi + q, \quad (5)$$

$$z = L, \quad v_z(r,L,t) + Bi_L v(r,L,t) = 0. \quad (6)$$

Where $v(r,z,t)$ denotes the temperature, $L$ fin length; $\mathrm{B}i$ , $\mathrm{Bi_L}$ , $\mathrm{Bi_r}$ are the base Biot number; tip Biot number and lateral Boit number respectively. The governing partial differential equations are converted to ordinary differential equations by reducing the number of independent variables. One can decompose the equation (1) into two factors $v(r,z,t)$=$e^{-t}u(r,z,t)$. After a routine calculation, and we can arrive the situation stated in [9], so we solve the transformed problem- the homogeneous problem, and solve it by separation of variables, and then decompose the heat source into the relating eigenfunctions. Upon separation of variables in the r-direction, the Fourier equation is formed, and the corresponding solution and eigen-values are derived accordingly. Similarly, the differential equation in the z-coordinate with non-homogeneous boundary conditions can be transformed into one with homogeneous boundary conditions and thus solved by eigenfunction expansions (see, for details, [3-9]).

And the differential equation in the time space can also solved by the familiar integrating factor method for the first order differential equations. The solutions of the revised partial differential equation can be assured, and then multiply the solution by $e^t$ , thereafter the complete solution of the temperature distribution and heat transfer rate can be constructed. Note that a Bessel's equation satisfied a cylindrical coordinate in $r$-direction can be made orthogonal, one can separate the temperature distribution as follows:

$$u(r,z,t) = \sum_{n=1}^{\infty} u_n(z,t) J_0(\beta_n r), \quad (7)$$

where $\beta_n$ are the positive characteristic values of the transcendent equation (8),

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, (wherer = 1), \quad (8)$$

Because the boundary conditions in equations (3) and (4) are satisfied automatically, the partial differential equation with initial and boundary conditions can be simplified as:

$$\frac{\partial u_n(z,t)}{\partial t} = \frac{\partial^2 u_n(z,t)}{\partial z^2} - \beta_n u_n(z,t), \tag{9}$$

I.C.

$$u_n(z,0) = 0, \tag{10}$$

B. Cs. $t > 0$

$$z = 0, \quad u_{nz}(0,t) + Bi_r u_n(0,z,t) = g_n, \tag{11}$$

$$z = L, \quad u_{nz}(L,t) + B_{iL} u_n(L,t) = 0, \tag{12}$$

where

$$g_n = \frac{2\beta_n(B_i + q)e^t J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}. \tag{13}$$

Revised the boundary conditions to a homogeneous one, which lead to the required coefficients are

$$A = \frac{g_n}{(Bi + Bi_L + LBiBi_L)L},$$
$$B = \frac{(LBi_L + 1)g_n}{(Bi + Bi_L + LBiBi_L)L} \tag{14}$$

The differential equation (9) and initial condition and boundary condition could be written as

$$\overline{u}_n(z,0) = -[zA + (L-z)B], \tag{15}$$

$$\frac{\partial \overline{u}_n(z,t)}{\partial t} = \frac{\partial^2 \overline{u}_n(z,t)}{\partial z^2} - \beta_n^2[zA + (L-z)B + \overline{u}_n(z,t)], \tag{16}$$

And the boundary conditions are B.Cs. $t > 0$

$$z = 0, \quad -\overline{u}_{nz}(0,t) + Bi\overline{u}_n(0,t) = 0, \tag{17}$$

$$z = L, \quad \overline{u}_{nz}(L,t) + Bi_L \overline{u}_n(L,t) = 0. \tag{18}$$

Then let the dependent variable written as

$$\overline{u}_n(z,t) = \sum_{m=1}^{\infty} u_{nm}(t)(\cos\alpha_m z + \frac{Bi}{\alpha_m}\sin\alpha_m z) \tag{19}$$

where $\alpha_m$ are the characteristic values of the transcendent equation (20),

$$\tan\alpha_m L = \frac{\alpha_m(Bi + Bi_L)}{\alpha_m - BiBi_L}. \tag{20}$$

Substitute Equation (19) into equation (15), initial condition and do necessary calculation. Thus only time variable is left and the ordinary differential equation could be written as

$$\frac{\partial u_{nm}(t)}{\partial t} = -(\beta_n^2 + \alpha_m^2)u_{nm}(t) + \beta_n^2 C_{nm}, \tag{21}$$

with

$$u_{nm}(0) = C_{nm}. \tag{22}$$

In the last equation,$C_{nm}$ are the values of the equation (23),

$$\begin{aligned} C_{nm} &= \frac{E}{F}, \\ E &= -2[(A - B - ABiL)\alpha_m\cos\alpha_m L \\ &\quad +(A\alpha_m^2 L + ABi - BBi)\sin\alpha_m L \\ &\quad -(A - B - BBiL)\alpha_m], \\ F &= (\alpha_m^2 - Bi^2)[\cos\alpha_m L\sin\alpha_m L \\ &\quad -2Bi\alpha_m\cos^2\alpha_m L \\ &\quad +(\alpha_m^2 L + Bi^2 L + 2Bi)\alpha_m]. \end{aligned} \tag{23}$$

The analytic solution of equation (21) combined with the initial condition of equation (22) is shown as

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2 + \beta_n^2)t}]. \tag{24}$$

By combining the relating solutions, we get the analytic temperature solution for the partial differential equation (1) together with the relating I.C. and B.Cs.

## 2 Results and Discussion

The analytical temperature profiles for above governing equation with the initial condition and the boundary conditions obtained by the method of separation variables and superposition are shown below. The temperature of lateral surface is natural convection, and heat flux can dissipate through circumferential and tip surface. The temperature profile involved nine different boundary conditions are presented.

*Case 1:* $Bi =$ constant, $Bi_L =$constant.

While $Bi_L$ constant, heat convection condition on the tip of the fin, the larger $Bi_L$ will result in the faster heat dissipation though the fin. The analytical temperature is

$$v(r,z,t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{zA + (L-z)B + \sum_{m=1}^{\infty} u_{nm}(t)(\cos\alpha_m z + \frac{Bi}{\alpha_m}\sin\alpha_m z)\}. \tag{25}$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2 + \beta_n^2)t}]. \tag{26}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n(B_i+q)e^t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \quad (27)$$

$$A = \frac{g_n}{(Bi+Bi_L+LBiBi_L)L},$$
$$B = \frac{(LBi_L+1)g_n}{(Bi+Bi_L+LBiBi_L)L}, \quad (28)$$

where

$$\begin{aligned} C_{nm} &= \frac{E}{F}, \\ E &= -2[(A-B-ABiL)\alpha_m\cos\alpha_m L \\ &\quad +(A\alpha_m^2 L+ABi-BBi)\sin\alpha_m L \\ &\quad -(A-B-BBiL)\alpha_m], \\ F &= (\alpha_m^2-Bi^2)[\cos\alpha_m L\sin\alpha_m L \\ &\quad -2Bi\alpha_m\cos^2\alpha_m L \\ &\quad +(\alpha_m^2 L+Bi^2 L+2Bi)\alpha_m]. \end{aligned} \quad (29)$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (30) and (31), respectively.

$$\tan\alpha_m L = \frac{\alpha_m(Bi+Bi_L)}{\alpha_m^2-BiBi_L}, \quad (30)$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0, \quad (\text{ where } r=1). \quad (31)$$

*Case 2:* $Bi=0$, $Bi_L$ =0.

As $Bi = 0$, the fin roots is constraint to constant heat flux and a constant heat flux conduct into fin through fin roots. While $Bi_L$ =0, the tip of the fin is adiabatic, heat cannot dissipate though the fin and the lateral surface also adiabatic. All energy will be stored in the fin, and the boundary conditions are revised to be as following:

$$z=0, \quad -v_z(r,0,t)=q, \quad (32)$$

$$z=L, \quad v_z(r,L,t)=0 \quad (33)$$

The analytical temperature profile is

$$v(r,z,t)=\sum_{n=1}^{\infty}J_0(\beta_n r)e^{-t}\{[Lz-\frac{z^2}{2}]B+C_{n0} + \sum_{m=1}^{\infty}u_{nm}(t)\cos\alpha_m z\} \quad (34)$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \quad (35)$$

The relating coefficients are shown as following:

$$g_n=\frac{2qe^t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \quad (36)$$

$$\begin{cases} C_{n0}=\frac{-BL^2}{3}, & m=0 \\ C_{nm}=\frac{2BL^2}{(m\pi)^2}, & m>0 \end{cases} \quad (37)$$

$$B=-\frac{g_n}{L}.$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (38) and (39), respectively.

$$\alpha_m=\frac{m\pi}{L};\ m,\ n=1,2,3,..., \quad (38)$$

$$\beta_n\frac{dJ_0(\beta_n r)}{dr}+Bi_r J_0(\beta_n)=0, \quad (\text{ where } r=1). \quad (39)$$

*Case 3:* $Bi=0$, $Bi_L\to\infty$.

While $Bi_L\to\infty$, the tip of the fin is isothermal to environment, the lateral surface also isothermal and heat dissipates fast though the fin. The temperature of the fin, left sitting on the base temperature, will fall until it reaches the surrounding temperature, and the boundary conditions are revised to be as following:

$$z=0, \quad -v_z(r,0,t)=q, \quad (40)$$

$$z=L, \quad v(r,L,t)=0. \quad (41)$$

The analytical temperature profile is

$$v(r,z,t)=\sum_{n=1}^{\infty}J_0(\beta_n r)e^{-t}\{(L-z)B + \sum_{m=1}^{\infty}u_{nm}(t)\cos\alpha_m z\} \quad (42)$$

where

$$u_{nm}(t)=\frac{C_{nm}}{\beta_n^2+\alpha_m^2}[\beta_n^2+\alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \quad (43)$$

The relating coefficients are shown as following:

$$g_n=\frac{2\beta_n qe^t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \quad (44)$$

$$C_{nm}=\frac{-8BL}{[(2m-1)\pi]^2}. \quad (45)$$

$$B=g_n. \quad (46)$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (47) and (48), respectively.

$$\alpha_m=\frac{(2m-1)\pi}{2L};\ n,\ m=1,2,3,..., \quad (47)$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1) \tag{48}$$

*Case 4:* $Bi = 0$, $Bi_L$ =constant.

While $Bi_L$ =constant, heat convection condition on the tip of the fin, the larger $Bi_L =$ will result in the faster heat dissipation though the fin tip surface, and the boundary conditions are revised to be as following:

$$z = 0, -v_z(r, 0, t) = q, \tag{49}$$

$$z = L, v_z(r, L, t) + Bi_L v(r, L, t) = 0. \tag{50}$$

The analytical temperature profile is

$$v(r, z, t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{zA + (L - z)B + \sum_{m=1}^{\infty} u_{nm}(t) \cos \alpha_m z\} \tag{51}$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \tag{52}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n q e^t J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}, \tag{53}$$

$$C_{nm} = \frac{-2[(A - B)(\cos \alpha_m L - 1) + A\alpha_m L \sin \alpha_m L]}{\alpha_m(\cos \alpha_m L \sin \alpha_m L + \alpha_m L)}, \tag{54}$$

$$A = \frac{g_n}{LBi_L}, \quad B = \frac{(LBi_L + 1)g_n}{LBi_L} \tag{55}$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (56) and (57), respectively.

$$\alpha_m \tan \alpha_m L = Bi_L, \tag{56}$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1). \tag{57}$$

*Case 5:* $Bi \to \infty$ , $Bi_L$ =0.

As $Bi \to \infty$, the fin root is constraint to isothermal conductivity, the root interface temperature is kept constant. While $Bi_L$ =0 , the tip of the fin is adiabatic, heat cannot dissipate though the fin and the lateral surface also isothermal. All energy will be dissipated by lateral surface. The boundary conditions are revised to

$$z = 0, \quad v(r, 0, t) = 1, \tag{58}$$

$$z = L, \quad v_z(r, L, t) = 0. \tag{59}$$

The analytical temperature profile is

$$v(r, z, t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{zA + (L - z)B + \sum_{m=1}^{\infty} u_{nm}(t) \sin(\alpha_m z)\} \tag{60}$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \tag{61}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n e^t J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}, \tag{62}$$

$$C_{nm} = \frac{4L[2(A - B)(-1)^m - (2m - 1)B\pi]}{[(2m - 1)\pi]^2}, \tag{63}$$

$$A = \frac{g_n}{L}, \quad B = \frac{g_n}{L} \tag{64}$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (65) and (66), respectively.

$$\alpha_m = \frac{(2m - 1)\pi}{2L}, \ m = 1, 2, 3, ..., \tag{65}$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1). \tag{66}$$

*Case 6:* $Bi \to \infty$ , $Bi_L \to \infty$.

While $Bi_L \to \infty$, the tip and root of the fin are isothermal, heat dissipates fast though the tip of the fin. The boundary conditions are

$$z = 0, \quad v(r, 0, t) = 1, \tag{67}$$

$$z = L, \quad v(r, L, t) = 0. \tag{68}$$

The analytical temperature profile is

$$v(r, z, t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{(L - z)B + \sum_{m=1}^{\infty} u_{nm}(t) \sin(\alpha_m z)\} \tag{69}$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \tag{70}$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n e^t J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}, \tag{71}$$

$$C_{nm} = \frac{-2BL}{m\pi}, \quad (72)$$

$$B = \frac{g_n}{L} \quad (73)$$

while $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (74) and (75), respectively.

$$\alpha_m = \frac{m\pi}{L}, \; m = 1, 2, 3, ..., \quad (74)$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1). \quad (75)$$

*Case 7:* $Bi \to \infty$ , $Bi_L$ =constant.

While $Bi_L$ is constant, While constant, heat convection condition on the tip of the fin, the larger $Bi_L$ will result in the faster heat dissipation though the fin. The analytical solution is

$$v(r,z,t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{zA + (L-z)B + \sum_{m=1}^{\infty} u_{nm}(t)\sin(\alpha_m z)\} \quad (76)$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \quad (77)$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n e^t J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}, \quad (78)$$

$$C_{nm} = \frac{2[(A-B)\sin\alpha_m L - A\alpha_m L\cos\alpha_m L + B\alpha_m L]}{\alpha_m(\cos\alpha_m L\sin\alpha_m L - \alpha_m L)}, \quad (79)$$

$$A = \frac{g_n}{(LBi_L + 1)L}, \; B = \frac{g_n}{L}. \quad (80)$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (81) and (82), respectively.

$$\alpha_m \cot\alpha_m L = -Bi_L, \quad (81)$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1). \quad (82)$$

*Case 8:* $Bi$ =constant , $Bi_L = 0$.

As $Bi$ keep constant, it exist heat convection effects between the root of the fin and heat source. The root of the fin approaches to isothermal as the value of $Bi$ is large enough, but it come near to constant heat flux for infinitesimal value. While $Bi_L$ =0 , the tip of the fin is adiabatic, heat cannot dissipate though the fin and the lateral surface also adiabatic. All heat energy will be reserved in the fin. The analytical solution is

$$v(r,z,t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{zA + (L-z)B + \sum_{m=1}^{\infty} u_{nm}(t)\frac{\cos\alpha_m(L-z)}{\cos\alpha_m L})\} \quad (83)$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \quad (84)$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n(Bi+q)e^t J_1(\beta_n)}{(Bi_r^2 + \beta_n^2)J_0^2(\beta_n)}, \quad (85)$$

$$C_{nm} = \frac{2(A-B)(\cos\alpha_m L - 1)\cos\alpha_m L}{\alpha_m(\cos\alpha_m L\sin\alpha_m L + \alpha_m L)} - \frac{2B\alpha_m L\sin\alpha_m L\cos\alpha_m L}{\alpha_m(\cos\alpha_m L\sin\alpha_m L + \alpha_m L)}, \quad (86)$$

$$A = \frac{g_n}{LBi}, \; B = \frac{g_n}{LBi}. \quad (87)$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (88) and (89), respectively.

$$\alpha_m \tan\alpha_m L = Bi, \quad (88)$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1). \quad (89)$$

*Case 9:* $Bi$ =constant , $Bi_L \to \infty$.

While $Bi_L \to \infty$, the tip of the fin is isothermal to environment, the lateral surface also adiabatic and heat dissipates fast though the fin tip. The boundary conditions are when $z = 0$,

$$-v_z(r,0,t) + Biu(r,0,t) = Bi + q, \quad (90)$$

and $z = L$,

$$v(r,L,t) = 0. \quad (91)$$

The analytical solution is

$$v(r,z,t) = \sum_{n=1}^{\infty} J_0(\beta_n r)e^{-t}\{(L-z)B + \sum_{m=1}^{\infty} u_{nm}(t)\frac{\sin\alpha_m(L-z)}{\sin\alpha_m L}\}, \quad (92)$$

where

$$u_{nm}(t) = \frac{C_{nm}}{\beta_n^2 + \alpha_m^2}[\beta_n^2 + \alpha_m^2 e^{-(\alpha_m^2+\beta_n^2)t}]. \quad (93)$$

The relating coefficients are shown as following:

$$g_n = \frac{2\beta_n(Bi+q)e^t J_1(\beta_n)}{(Bi_r^2+\beta_n^2)J_0^2(\beta_n)}, \tag{94}$$

$$C_{nm} = \frac{-2B[\cos^2\alpha_m L + \alpha_m L\cos\alpha_m L\sin\alpha_m L - 1]}{\alpha_m(\cos\alpha_m L\sin\alpha_m L - \alpha_m L)}, \tag{95}$$

and

$$B = \frac{g_n}{LBi+1}. \tag{96}$$

While $\alpha_m$ and $\beta_n$ are the positive roots of the characteristic equations (88) and (89), respectively.

$$\alpha_m \tan\alpha_m L = Bi, \tag{97}$$

$$\beta_n \frac{dJ_0(\beta_n r)}{dr} + Bi_r J_0(\beta_n) = 0, \quad (\text{ where } r = 1). \tag{98}$$

# 3 Conclusion

The principle of superposition and separable variables are applied to the transient heat conduction in a cylindrical fin subjected to convective lateral surface to provide a simplified formulation that can be used to identify the temperature distribution. The temperature distributions are formed in a Fourier Bessel series and exponential type and are given by nine different cases.

# 4 Nomenclature

$Bi$ base Biot number
$Bi_L$ tip Biot number
$Bi_r$ lateral Boit number
$J_0$ First kind Bessel function of zero order
$\beta_n$ Characteristic values of Bessel function
$L$ pin fin length
$m, n$ positive integral values
$t$ dimensionless time
$u(r, z, t)$ dimensionless transient temperature
$\alpha_m; \beta_n$ eigenvalues

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