A General Iterative Algorithm for System of Equilibrium Problems and Infinitely many Strict Pseudo-contractions

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Abstract: In this paper, an explicit iterative scheme is proposed for finding a common element of the set of fixed point of infinitely many strict pseudo-contractive mappings and the set of solutions of system of equilibrium problems by the general iterative method. In the setting of real Hilbert spaces, strong convergence theorem is proved. Our results improve and extend the corresponding results reported by many others.

Key–Words: Strict pseudo-contraction; System of equilibrium problems; Fixed point; Strong convergence; General iterative algorithm.

1 Introduction

Let H be a real Hilbert space with inner product $\langle , \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H.

Let $\{F_k\}$ be a countable family of bifunctions from $C \times C$ to R, where R is the set of real numbers. Combettes and Hirstoaga [1] considered the following system of equilibrium problems which is to find $x \in C$ such that:

$$F_k(x, y) \geq 0, \ \forall k \in \Gamma \text{ and } \forall y \in C, \tag{1}$$

where Γ is an arbitrary index set. If Γ is a singleton, then problem (1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) \geq 0, \forall y \in C. \tag{2}$$

The solution set of (2) is denoted by $EP(F)$.

A mapping S of C is said to be a $\kappa-$strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2$$

for all $x, y \in C$; see [2]. We denote the set of fixed points of S by $F(S)$ i.e.,

$$F(S) = \{x \in C : Sx = x\}.$$

Note that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mapping S on C such that

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. That is, S is nonexpansive if and only if S is a $0-$strict pseudo-contraction.

Numerous problems in physics, optimization and economics reduce to finding a solution of the problem (1). Some methods have been proposed to solve the system of equilibrium problems. See, for instance, [3-10].

In 2006, Marino and Xu [11] introduced the following general iterative method and proved that under certain appropriate conditions, the algorithm converges strongly. To be more precise, they proved the following theorem.

Theorem 1 *Let x_n be generated by algorithm*

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n)$$

with the sequence $\{\alpha_n\}$ of parameters satisfying conditions (C1)–(C3):

(C1) $\alpha_n \to 0$;

(C2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(C3) either $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1.$

Then x_n converges strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \leq 0, \forall z \in Fix(T) \tag{3}$$

where T is a nonexpansive mapping, A is a strongly positive bounded linear operator and f is a contraction.

The variational inequality (3) is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-h(x),$$

where h is a potential function for $\gamma f(x)$ (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

In recent years, variational inequality problems have been extended to study a large variety problems arising in structural analysis, economics, engineering sciences and so on, for example, see [3, 7, 12, 13] and the references therein.

On the other hand, Yamada [14] proposed the following hybrid iterative method for solving the variational inequality

$$x_{n+1}=Tx_n-\mu\lambda_nF(Tx_n),n\geq 0,$$

where F is $k-$Lipschitzian and $\eta-$strongly monotone operator with $k>0,\eta>0,0<\mu<2\eta/k^2$. He proved that if λ_n satisfying appropriate conditions, then $\{x_n\}$ generated by the above algorithm converges strongly to the unique solution of variational inequality

$$\langle F\tilde{x},x-\tilde{x}\rangle\geq 0,x\in Fix(T).$$

Recently, Tian [15] revealed the interior connection of the Yamada's algorithm and viscosity iterative algorithm, then proposed a more general iterative algorithm combining a $L-$ Lipschitzian and $\eta-$strong monotone operator. They obtained the following result in a real Hilbert space.

Theorem 2 *Let x_n be generated by algorithm*

$$x_{n+1}=(I-\alpha_n\mu A)Tx_n+\alpha_n\gamma f(x_n)$$

with the sequence $\{\alpha_n\}$ of parameters satisfying conditions (C1)–(C3):

(C1) $\alpha_n\to 0$;

(C2) $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$;

(C3) either $\sum\limits_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ *or* $\lim\limits_{n\to\infty}\frac{\alpha_{n+1}}{\alpha_n}=1$.

Then x_n converges strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle(\mu A-\gamma f)\tilde{x},\tilde{x}-z\rangle\leq 0,\ \ \forall z\in Fix(T).$$

More recent, He, Liu and Cho [6] considered an explicit method for system of equilibrium problems and infinite family nonexpansive mappings. They introduced an explicit scheme as follows:

$$\begin{aligned}&x_{n+1}=\alpha_n\gamma f(x_n)\\&\quad+(I-\alpha_nA)W_nT_{r_{M,n}}^{F_M}T_{r_{M-1,n}}^{F_{M-1}}...T_{r_{2,n}}^{F_2}T_{r_{1,n}}^{F_1}x_n,\\&\forall n\in N.\end{aligned}$$

Under the appropriate conditions, the sequence $\{x_n\}$ converges strongly to

$$x^*\in F=\bigcap_{i=1}^{\infty}F(S_i)\bigcap\bigcap_{k=1}^{M}EP(F_k)\neq\emptyset$$

which satisfies the variational inequality

$$\langle(A-\gamma f)x^*,x-x^*\rangle\geq 0,\forall x\in F,$$

where f is a contraction and A is a strongly positive bounded linear operator.

In this paper, motivated and inspired by the above facts, we introduce a new iterative scheme and obtain strong convergence theorem for finding a common element of the set of fixed points of a infinite family of strict pseudo-contractions and the set of solutions of the system of equilibrium problems (1) by the general iterative algorithm. Our results improve and extend the corresponding results given by Wang [17], He [6], Tian [15] and many others. Furthermore, we give an example which support our main theorem in the last section.

2 Preliminaries

Throughout this paper, the notations $\rightharpoonup$ denotes weak convergence and $\to$ denotes strong convergence.

We at first introduce some lemmas that are used in proofs of the main results later.

Lemma 3 *Let H be a real Hilbert space. There hold the following identities:*
(i) $\|x-y\|^2=\|x\|^2-\|y\|^2-2\langle x-y,y\rangle,\forall x,y\in H.$
(ii) $\forall t\in[0,1],\forall x,y\in H,$

$$\|tx+(1-t)y\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2.$$

Lemma 4 *[16] Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1}\leq(1-\gamma_n)\alpha_n+\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum\limits_{n=1}^{\infty}\gamma_n=\infty$;

(ii) $\limsup\limits_{n\to\infty}\frac{\delta_n}{\gamma_n}\leq 0$ *or* $\sum\limits_{n=1}^{\infty}|\delta_n|<\infty$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Recall that given a nonempty closed convex subset C of a real Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C. As we know, $y = P_C x$ if and only if there holds the relation:

$$\langle x - y, y - z\rangle \geq 0 \ for\ all\ \ z \in C. \tag{4}$$

Lemma 5 *[17] Let $A : H \to H$ be a L-Lipschitzian and $\eta-$strongly monotone operator on a Hilbert space H with $L > 0, \eta > 0, 0 < \mu < 2\eta/L^2$ and $0 < t < 1$. Then $S = (I - t\mu A) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = \frac{1}{2}\mu(2\eta - \mu L^2)$.*

Lemma 6 *[2] Let $S : C \to C$ be a $\kappa-$strict pseudo-contraction. Define $T : C \to C$ by*

$$Tx = \lambda x + (1 - \lambda)Sx, \forall x \in C.$$

Then, as $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.

Lemma 7 *[15] Let H be a Hilbert space and $f : H \to H$ be a contraction with coefficient $0 < \rho < 1$, and $A : H \to H$ an $L-$Lipschitzian continuous operator and $\eta-$strongly monotone with $L > 0, \eta > 0$. Then for $0 < \gamma < \mu\eta/\rho$,*

$$\langle x - y, (\mu A - \gamma f)x - (\mu A - \gamma f)y\rangle \\ \geq (\mu\eta - \gamma\rho)\|x - y\|^2, x, y \in H.$$

That is, $\mu A - \gamma f$ is strongly monotone with coefficient $\mu\eta - \gamma\rho$.

Let $\{S_n\}$ be a sequence of κ_n-strict pseudo-contractions. Define $S_n^{'} = \theta_n I + (1 - \theta_n)S_n$, $\theta_n \in [\kappa_n, 1)$. Then, by Lemma 6, $S_n^{'}$ is nonexpansive. In this paper, we consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = t_n S_n^{'} U_{n,n+1} + (1 - t_n)I, \\ U_{n,n-1} = t_{n-1} S_{n-1}^{'} U_{n,n} + (1 - t_{n-1})I, \\ \cdots, \\ U_{n,i} = t_i S_i^{'} U_{n,i+1} + (1 - t_i)I, \\ \cdots, \\ U_{n,2} = t_2 S_2^{'} U_{n,3} + (1 - t_2)I, \\ W_n = U_{n,1} = t_1 S_1^{'} U_{n,2} + (1 - t_1)I, \end{cases} \tag{5}$$

where $t_1, t_2, \cdots$ are real numbers such that $0 \leq t_n < 1$. Such a mapping W_n is called a $W-$mapping generated by $S_1^{'}, S_2^{'}, \cdots$ and $t_1, t_2, \cdots$. It is easy to see W_n is nonexpansive.

Lemma 8 *[18] Let C be a nonempty closed convex subset of a strictly convex Banach space E, let $S_1^{'}, S_2^{'}, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(S_i^{'}) \neq \emptyset$ and $t_1, t_2, \cdots$ be real numbers such that $0 < t_i \leq b < 1$, for every $i = 1, 2, \cdots$. Then, for any $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 8, one can define the mapping W from C into itself as follows:

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, x \in C.$$

Lemma 9 *[18] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1^{'}, S_2^{'}, \cdots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(S_i^{'}) \neq \emptyset$ and $t_1, t_2, \cdots$ be real numbers such that $0 < t_i \leq b < 1, \forall i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Lemma 10 *[19] Let C be a nonempty closed convex subset of a Hilbert space $H, \{S_i^{'} : C \to C\}$ be a family of infinite nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i^{'}) \neq \emptyset$, $t_1, t_2, \cdots$ be real numbers such that $0 < t_i \leq b < 1$, for every $i = 1, 2, \cdots$. Then $F(W) = \cap_{i=1}^{\infty} F(S_i^{'})$.*

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0, \forall x \in C$;

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) For each $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontionuous for each $x \in C$.

We recall some lemmas which will be needed in the rest of this paper.

Lemma 11 *[20] Let C be a nonempty closed convex subset of H, let F be bifunction from $C \times C$ to R satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\ y \in C.$$

Lemma 12 *[1] For $r > 0,\ x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C | F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$$

for all $x \in H$. Then, the following statements hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 13 *[21] Let C, H, F and $T_r x$ be as in Lemma 11. Then the following holds:*

$$\|T_s x - T_t x\|^2 \leq \frac{s - t}{s}\langle T_s x - T_t x, T_s x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 14 *[22] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and $\{\beta_n\}$ be a sequence of real numbers such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ for all $n = 0, 1, 2, \ldots$ Suppose that $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all $n = 0, 1, 2, \ldots$ and $\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 15 *[17] Let H be a Hilbert space and C be a nonempty closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

3 Main result and its proof

The rest of this paper, we always assume that f is a contraction mapping from H into itself with coefficient $\rho \in (0, 1)$, and A is a $L-$ Lipschitzian continuous operator and $\eta-$strongly monotone on H with $L > 0, \eta > 0$. Assume that $0 < \mu < 2\eta/L^2$ and

$$0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\rho = \tau/\rho.$$

Denote

$$\Theta_n^k = T_{r_{k,n}}^{F_k} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$$

for every $k \in \{1, 2, \ldots, M\}$ and $\Theta_n^0 = I$ for all $n \in N$. Define a mapping

$$V_n = \beta_n I + (1 - \beta_n)W_n\Theta_n^M.$$

Since both W_n and $T_{r_{k,n}}^{F_k}$ are nonexpansive, it is easy to check that V_n is also nonexpansive. Consider the following mapping G_n on H defined by

$$G_n x = \alpha_n\gamma f(x) + (I - \alpha_n\mu A)V_n x,$$

for all $x \in H, n \in N$, where $\alpha_n \in (0, 1)$. By Lemma 3 (ii), Lemma 5 and Lemma 12, we have

$$\begin{aligned}
&\|G_n x - G_n y\| \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| + (1 - \alpha_n\tau)\|V_n x - V_n y\|) \\
&\leq \alpha_n\gamma\rho\|x - y\| + (1 - \alpha_n\tau)\|x - y\| \\
&= (1 - \alpha_n(\tau - \gamma\rho))\|x - y\|.
\end{aligned}$$

Since $0 < 1 - \alpha_n(\tau - \gamma\rho) < 1$, it follows that G_n is a contraction mapping. By the Banach contraction principle, G_n has a unique fixed pointed $x_n^f \in H$ such that

$$x_n^f = \alpha_n\gamma f(x_n^f) + (I - \alpha_n\mu A)V_n x_n^f.$$

For simplicity, we will write x_n for x_n^f provided without confusion. Next we prove the sequences $\{x_n\}$ converges strongly to a point

$$x^* \in \Omega = \cap_{i=1}^{\infty} F(S_i) \cap \cap_{k=1}^{M} EP(F_k)$$

which solves the variational inequality

$$\langle(\gamma f - \mu A)x^*, p - x^*\rangle \leq 0, \quad \forall p \in \Omega. \tag{6}$$

Equivalently, $x^* = P_\Omega(I - \mu A + \gamma f)x^*$.

Theorem 16 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F_k, k \in \{1, 2, \ldots M\}$, be bifunctions from $C \times C$ to R which satisfies conditions (A1)–(A4). Let $S_i : C \to C$ be a family κ_i-strict pseudo-contractions for some $0 \leq \kappa_i < 1$. Assume the set $\Omega = \cap_{i=1}^{\infty} F(S_i) \cap \cap_{k=1}^{M} EP(F_k) \neq \emptyset$. Let f be a contraction mapping on H with $\rho \in (0, 1)$ and let A be a $L-$Lipschitzian continuous operator and $\eta-$strongly monotone with $L > 0, \eta > 0$, $0 < \mu < 2\eta/L^2$ and $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\rho = \tau/\rho$. For every $n \in N$, let W_n be the mapping generated by $S_i^{'}$ and $0 < t_i \leq b < 1$. Given $x_1 \in H$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases}
u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \ldots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\
y_n = \beta_n x_n + (1 - \beta_n)W_n u_n, \\
x_{n+1} = \alpha_n\gamma f(x_n) + (I - \mu\alpha_n A)y_n,
\end{cases} \tag{7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_{k,n}\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 \le \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{r_{k,n}\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for* $k \in \{1, 2, ...M\}$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Omega$*, which solves the variational inequality (6).*

Proof: The proof is divided into several steps.

Step 1. $\{x_n\}$ is bounded.

Take $p \in \Omega$, since for each $k \in \{1, 2, \ldots, M\}$, $T^{F_k}_{r_{k,n}}$ is nonexpansive, $p = T^{F_k}_{r_{k,n}} p$ and $u_n = \Theta^M_n x_n$, we have

$$\|u_n - p\| = \|\Theta^M_n x_n - \Theta^M_n p\| \le \|x_n - p\| \tag{8}$$

for all $n \in N$.

Since W_n is nonexpansive, by Lemma 10 and (8), we get

$$\begin{aligned} \|y_n - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\|W_n u_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{9}$$

Take any $p \in \Omega$, from (9), we have

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|\alpha_n \gamma f(x_n) + (I - \mu\alpha_n A)y_n - p\| \\ &\le \|\alpha_n(\gamma f(x_n) - \mu Ap)\| \\ &\quad + \|(I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)p\| \\ &\le \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - \mu Ap)\|) \\ &\quad + (1 - \alpha_n\tau)\|y_n - p\| \\ &\le \alpha_n\rho\gamma\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu Ap)\| \\ &\quad + (1 - \alpha_n\tau)\|u_n - p\| \\ &= (1 - \alpha_n(\tau - \rho\gamma))\|x_n - p\| \\ &\quad + \alpha_n(\tau - \rho\gamma)\tfrac{\|\gamma f(p) - \mu Ap)\|}{\tau - \rho\gamma} \\ &\le \max\{\|x_n - p\|, \tfrac{\|\gamma f(p) - \mu Ap)\|}{\tau - \rho\gamma}\}. \end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{\|\gamma f(p) - \mu Ap)\|}{\tau - \rho\gamma}\}, n \ge 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. It follows from the Lipschitz continuity of A that $\{Ax_n\}$ and $\{Au_n\}$ are also bounded. From the nonexpansivity of f and W_n, it follows that $\{f(x_n)\}$ and $\{W_n x_n\}$ are also bounded.

Step 2.

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{10}$$

and

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{11}$$

Observe that

$$\begin{aligned} &\|u_{n+1} - u_n\| \\ &= \|\Theta^M_{n+1} x_{n+1} - \Theta^M_n x_n\| \\ &\le \|\Theta^M_{n+1} x_{n+1} - \Theta^M_{n+1} x_n\| + \|\Theta^M_{n+1} x_n - \Theta^M_n x_n\| \\ &\le \|x_{n+1} - x_n\| + \|\Theta^M_{n+1} x_n - \Theta^M_n x_n\| \\ &\le \|x_{n+1} - x_n\| \\ &\quad + \|T^{F_M}_{r_{M,n+1}} \Theta^{M-1}_{n+1} x_n - T^{F_M}_{r_{M,n}} \Theta^{M-1}_{n+1} x_n\| \\ &\quad + \|T^{F_M}_{r_{M,n}} \Theta^{M-1}_{n+1} x_n - T^{F_M}_{r_{M,n}} T^{F_{M-1}}_{r_{M-1,n}} \Theta^{M-2}_{n+1} x_n\| + \cdots \\ &\quad + \|T^{F_M}_{r_{M,n}} T^{F_{M-1}}_{r_{M-1,n}} ... \Theta^2_{n+1} x_n - T^{F_M}_{r_{M,n}} ... T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n+1}} x_n\| \\ &\quad + \|T^{F_M}_{r_{M,n}} ... T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n+1}} x_n - u_n\| \\ &\le \|x_{n+1} - x_n\| \\ &\quad + \|T^{F_M}_{r_{M,n+1}} \Theta^{M-1}_{n+1} x_n - T^{F_M}_{r_{M,n}} \Theta^{M-1}_{n+1} x_n\| \\ &\quad + \|\Theta^{M-1}_{n+1} x_n - T^{F_{M-1}}_{r_{M-1,n}} \Theta^{M-2}_{n+1} x_n\| + \cdots \\ &\quad + \|T^{F_2}_{r_{2,n+1}} T^{F_1}_{r_{1,n+1}} x_n - T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n+1}} x_n\| \\ &\quad + \|T^{F_1}_{r_{1,n+1}} x_n - T^{F_1}_{r_{1,n}} x_n\|. \end{aligned} \tag{12}$$

From (5), we have

$$\begin{aligned} &\|W_{n+1} u_n - W_n u_n\| \\ &= \|t_1 S_1' U_{n+1,2} u_n - t_1 S_1' U_{n,2} u_n\| \\ &\le t_1\|U_{n+1,2} u_n - U_{n,2} u_n\| \\ &= t_1\|t_2 S_2' U_{n+1,3} u_n - t_2 S_2' U_{n,3} u_n\| \\ &\le t_1 t_2\|U_{n+1,3} u_n - U_{n,3} u_n\| \\ &\le \cdots \\ &\le \textstyle\prod_{i=1}^n t_i\|U_{n+1,n+1} u_n - U_{n,n+1} u_n\| \\ &\le M_1 \textstyle\prod_{i=1}^n t_i, \end{aligned} \tag{13}$$

where $M_1 = \sup_n\{\|U_{n+1,n+1} u_n - U_{n,n+1} u_n\|\}$.

Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$, then

$$\begin{aligned} z_n &= \tfrac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \tfrac{\alpha_n \gamma f(x_n) + (I - \mu\alpha_n A) y_n - \beta_n x_n}{1 - \beta_n}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &z_{n+1} - z_n \\ &= \tfrac{\alpha_{n+1}\gamma f(x_{n+1}) + (I - \mu\alpha_{n+1} A) y_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \tfrac{\alpha_n \gamma f(x_n) + (I - \mu\alpha_n A) y_n - \beta_n x_n}{1 - \beta_n} \\ &= \tfrac{\alpha_{n+1}(\gamma f(x_{n+1}) - \mu A y_{n+1})}{1 - \beta_{n+1}} + \tfrac{y_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \tfrac{\alpha_n(\gamma f(x_n) - \mu A y_n)}{1 - \beta_n} - \tfrac{y_n - \beta_n x_n}{1 - \beta_n} \\ &= \tfrac{\alpha_{n+1}(\gamma f(x_{n+1}) - \mu A y_{n+1})}{1 - \beta_{n+1}} \\ &\quad + \tfrac{\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) W_{n+1} u_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \tfrac{\alpha_n(\gamma f(x_n) - \mu A y_n)}{1 - \beta_n} - \tfrac{\beta_n x_n + (1 - \beta_n) W_n u_n - \beta_n x_n}{1 - \beta_n} \end{aligned}$$

It follows from (12), (13) and the above result that

$$\begin{aligned}&\|z_{n+1}-z_n\|\\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|\mu Ay_{n+1}\|)\\ &\quad+\tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\|+\|\mu Ay_n\|)\\ &\quad+\|W_{n+1}u_{n+1}-W_nu_n\|\\ &\le (\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}+\tfrac{\alpha_n}{1-\beta_n})M_2+\|W_{n+1}u_{n+1}-W_{n+1}u_n\|\\ &\quad+\|W_{n+1}u_n-W_nu_n\|\\ &\le (\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}+\tfrac{\alpha_n}{1-\beta_n})M_2+\|u_{n+1}-u_n\|\\ &\quad+\|W_{n+1}u_n-W_nu_n\|\\ &\le \|x_{n+1}-x_n\|+(\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}+\tfrac{\alpha_n}{1-\beta_n})M_2\\ &\quad+M_1\textstyle\prod_{i=1}^n t_i\\ &\quad+\|T^{F_M}_{r_{M,n+1}}\Theta^{M-1}_{n+1}x_n-T^{F_M}_{r_{M,n}}\Theta^{M-1}_{n+1}x_n\|\\ &\quad+\|\Theta^{M-1}_{n+1}x_n-T^{F_{M-1}}_{r_{M-1,n}}\Theta^{M-2}_{n+1}x_n\|+\cdots\\ &\quad+\|T^{F_2}_{r_{2,n+1}}T^{F_1}_{r_{1,n+1}}x_n-T^{F_2}_{r_{2,n}}T^{F_1}_{r_{1,n+1}}x_n\|\\ &\quad+\|T^{F_1}_{r_{1,n+1}}x_n-T^{F_1}_{r_{1,n}}x_n\|.\end{aligned}$$

Let $M_2=\sup_n\{\|\gamma f(x_n)\|+\|\mu Ay_n\|\}$. We have

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\ &\le \|T^{F_M}_{r_{M,n+1}}\Theta^{M-1}_{n+1}x_n-T^{F_M}_{r_{M,n}}\Theta^{M-1}_{n+1}x_n\|\\ &\quad+\|\Theta^{M-1}_{n+1}x_n-T^{F_{M-1}}_{r_{M-1,n}}...T^{F_2}_{r_{2,n+1}}T^{F_1}_{r_{1,n+1}}x_n\|\\ &\quad+\cdots\\ &\quad+\|T^{F_2}_{r_{2,n+1}}T^{F_1}_{r_{1,n+1}}x_n-T^{F_2}_{r_{2,n}}T^{F_1}_{r_{1,n+1}}x_n\|\\ &\quad+\|T^{F_1}_{r_{1,n+1}}x_n-T^{F_1}_{r_{1,n}}x_n\|\\ &\quad+(\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}+\tfrac{\alpha_n}{1-\beta_n})M_2+M_1\textstyle\prod_{i=1}^n t_i.\end{aligned}$$

From condition (i), (iii), $0<t_n\le b<1$, and Lemma 13, we obtain

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

By Lemma 14, we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. Thus

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0.$$

By Lemma 13, (10) and (12), we obtain

$$\|u_{n+1}-u_n\|\to 0.$$

Step 3.

$$\|x_n-Wx_n\|\to 0. \tag{14}$$

Observe that

$$\begin{aligned}\|x_n-W_nx_n\| &\le \|x_n-W_nu_n\|+\|W_nu_n-W_nx_n\|\\ &\le \|x_n-W_nu_n\|+\|u_n-x_n\|,\end{aligned}$$

and

$$\begin{aligned}&\|x_n-W_nu_n\|\\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-W_nu_n\|\\ &= \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|\\ &\quad+\beta_n(\|u_n-x_n\|+\|x_n-W_nu_n\|).\end{aligned}$$

From condition (i) and (20), we obtain

$$\begin{aligned}&(1-\beta_n)\|x_n-W_nu_n\|\\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\beta_n\|u_n-x_n\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-\mu Ay_n\|+\\ &\quad\beta_n\|u_n-x_n\|.\end{aligned}$$

In order to prove that $\|u_n-x_n\|\to 0$, we will show that

$$\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|\to 0, k\in\{1,2,\cdots,M\}. \tag{15}$$

Indeed, for $p\in\Omega$, it follows from the firmly nonexpansivity of $T^{F_k}_{r_{k,n}}$ that for each $k\in\{1,2,\cdots,M\}$, we have

$$\begin{aligned}&\|\Theta^k_nx_n-p\|^2=\|T^{F_k}_{r_{k,n}}\Theta^{k-1}_nx_n-T^{F_k}_{r_{k,n}}p\|^2\\ &\le \langle\Theta^k_nx_n-p,\Theta^{k-1}_nx_n-p\rangle\\ &=\tfrac12(\|\Theta^k_nx_n-p\|^2+\|\Theta^{k-1}_nx_n-p\|^2\\ &-\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2).\end{aligned}$$

Thus we get

$$\begin{aligned}&\|\Theta^k_nx_n-p\|^2\\ &\le \|\Theta^{k-1}_nx_n-p\|^2-\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2,\\ &k=1,2,\ldots,M,\end{aligned}$$

which implies that for each $k\in\{1,2,\ldots,M\}$,

$$\begin{aligned}&\|\Theta^k_nx_n-p\|^2\\ &\le \|\Theta^0_nx_n-p\|^2-\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2\\ &\quad-\|\Theta^{k-1}_nx_n-\Theta^{k-2}_nx_n\|^2-\cdots\\ &\quad-\|\Theta^2_nx_n-\Theta^1_nx_n\|^2-\|\Theta^1_nx_n-\Theta^0_nx_n\|^2\\ &\le \|x_n-p\|^2-\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2.\end{aligned}$$

Therefore, by the convexity of $\|\cdot\|^2$ and Lemma 12, we get

$$\begin{aligned}&\|y_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|\Theta^k_nx_n-p\|^2\\ &\le \|x_n-p\|^2-(1-\beta_n)\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2.\end{aligned}$$

Further we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\le \alpha_n\|\gamma f(x_n)-\mu Ap\|^2+(1-\alpha_n\tau)\|y_n-p\|^2\\ &\le \alpha_n\|\gamma f(x_n)-\mu Ap\|^2+(1-\alpha_n\tau)(\|x_n-p\|^2\\ &\quad-(1-\beta_n)\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2)\\ &\le \alpha_n\|\gamma f(x_n)-\mu Ap\|^2+\|x_n-p\|^2\\ &\quad-(1-\beta_n)\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&(1-\beta_n)\|\Theta^k_nx_n-\Theta^{k-1}_nx_n\|^2\\ &\le \alpha_n\|\gamma f(x_n)-\mu Ap\|^2+\|x_n-p\|^2\\ &\quad-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|\gamma f(x_n)-\mu Ap\|^2\\ &\quad+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|).\end{aligned}$$

From condition (i), (ii), we get from (10) that (15) holds. Thus we have

$$\begin{aligned}&\|u_n - x_n\|\\ &\leq \|u_n - \Theta_n^{M-1}x_n\| + \|\Theta_n^{M-1}x_n - \Theta_n^{M-2}x_n\|\\ &\quad + \cdots + \|\Theta_n^1 x_n - x_n\| \to 0.\end{aligned} \tag{16}$$

Furthermore we have $\|x_n - W_n u_n\| \to 0$. So we have

$$\|x_n - W_n x_n\| \to 0. \tag{17}$$

On the other hand, we have

$$\begin{aligned}&\|x_n - Wx_n\|\\ &\leq \|x_n - W_n x_n\| + \|W_n x_n - Wx_n\|\\ &\leq \|x_n - W_n x_n\| + \sup_{x_n \in C}\|W_n x_n - Wx_n\|.\end{aligned}$$

Combining (17), the last inequality and Lemma 9, we obtain (14).

Step 4.

$$\limsup_{n\to\infty}\langle(\gamma f - \mu A)x^*, x_n - x^*\rangle \leq 0, \tag{18}$$

where $x^* = P_\Omega(I - \mu A + \gamma f)x^*$ is a unique solution of the variational inequality (6).

Indeed, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\gamma f - \mu A)x^*, x_n - x^*\rangle\\ &= \lim_{j\to\infty}\langle(\gamma f - \mu A)x^*, x_{n_j} - x^*\rangle.\end{aligned}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which converges weakly to q. Without loss of generality, we can assume $x_{n_j} \rightharpoonup q$. From (14), we obtain $Wx_{n_j} \rightharpoonup q$.

Next we will show that $q \in \Omega$. By Lemma 6, Lemma 10 and Lemma 15, we have $q \in F(W) = \cap_{i=1}^{\infty}F(S_i') = \cap_{i=1}^{\infty}F(S_i)$.

We need to show that $q \in \cap_{i=1}^{M}EP(F_k)$. By Lemma 11, we have that for each $k = 1, 2, \ldots, M$,

$$\begin{aligned}&F_k(\Theta_n^k x_n, y) + \tfrac{1}{r_{k,n}}\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1}x_n\rangle\\ &\geq 0, \forall y \in C.\end{aligned}$$

From (A2), we get

$$\begin{aligned}&\tfrac{1}{r_{k,n}}\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1}x_n\rangle \geq F_k(y, \Theta_n^k x_n),\\ &\forall y \in C.\end{aligned}$$

Hence,

$$\begin{aligned}&\langle y - \Theta_{n_j}^k x_{n_j}, \tfrac{\Theta_{n_j}^k x_{n_j} - \Theta_{n_j}^{k-1}x_{n_j}}{r_{k,n_j}}\rangle \geq F_k(y, \Theta_{n_j}^k x_{n_j}),\\ &\forall y \in C.\end{aligned}$$

From (15), we obtain that

$$\Theta_{n_j}^k x_{n_j} \rightharpoonup q, \text{ as } j \to \infty$$

for each $k = 1, 2, \ldots, M$ (especially, $u_{n_j} = \Theta_{n_j}^M x_{n_j}$). Together with (15) and (A4) we have, for each $k = 1, 2, \ldots, M$, that

$$0 \geq F_k(y, q), \quad \forall y \in C.$$

Now, for any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, we obtain that $y_t \in C$ and hence $F_k(y_t, q) \leq 0$. So, we have

$$\begin{aligned}&0 = F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, q)\\ &\leq tF_k(y_t, y).\end{aligned}$$

Dividing by t, we get, for each $k = 1, 2, \ldots, M$, that

$$F_k(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \to 0$ and from (A3), we get

$$F_k(q, y) \geq 0. \quad \forall y \in C \text{ and } q \in EP(F_k)$$

for each $k = 1, 2, \ldots, M$, i.e., $q \in \cap_{k=1}^{M}EP(F_k)$. Therefore, we have $q \in \Omega$.

Since $x^* = P_\Omega(I - \mu A + \gamma f)x^*$, it follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\gamma f - \mu A)x^*, x_n - x^*\rangle\\ &= \lim_{j\to\infty}\langle(\gamma f - \mu A)x^*, x_{n_j} - x^*\rangle\\ &= \langle(\gamma f - \mu A)x^*, q - x^*\rangle \leq 0.\end{aligned}$$

Step 5.

$$x_n \to x^*. \tag{19}$$

Since

$$\begin{aligned}&\langle(\gamma f - \mu A)x^*, x_{n+1} - x^*\rangle\\ &= \langle(\gamma f - \mu A)x^*, x_{n+1} - x_n\rangle\\ &\quad + \langle(\gamma f - \mu A)x^*, x_n - x^*\rangle\\ &\leq \|(\gamma f - \mu A)x^*\|\|x_{n+1} - x_n\|\\ &\quad + \langle(\gamma f - \mu A)x^*, x_n - x^*\rangle.\end{aligned}$$

It follows from (10) and (18) that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu A)x^*, x_{n+1} - x^*\rangle \leq 0.$$

Thus we get

$$\begin{aligned}&\|x_{n+1} - x^*\|^2\\ &= \|\alpha_n\gamma f(x_n) + (I - \mu\alpha_n A)y_n - x^*\|^2\\ &= \|(I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)x^*\\ &\quad + \alpha_n(\gamma f(x_n) - \mu Ax^*)\|^2\\ &\leq \|(I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)x^*\|^2\\ &\quad + 2\alpha_n\langle\gamma f(x_n) - \mu Ax^*, x_{n+1} - x^*\rangle\\ &\leq (1 - \alpha_n\tau)^2\|y_n - x^*\|^2\\ &\quad + 2\alpha_n\langle\gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle +\\ &\quad 2\alpha_n\langle(\gamma f - \mu A)x^*, x_{n+1} - x^*\rangle\\ &\leq (1 - \alpha_n\tau)^2\|x_n - x^*\|^2\\ &\quad + \alpha_n\rho\gamma(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2)\\ &\quad + 2\alpha_n\langle(\gamma f - \mu A)x^*, x_{n+1} - x^*\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \tfrac{(1-\alpha_n\tau)^2+\alpha_n\rho\gamma}{1-\alpha_n\rho\gamma}\|x_n - x^*\|^2 \\
&\ \ +\tfrac{2\alpha_n}{1-\alpha_n\rho\gamma}\langle(\gamma f - \mu A)x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \tfrac{2\alpha_n(\tau-\rho\gamma)}{1-\alpha_n\rho\gamma})\|x_n - x^*\|^2 \\
&\ \ +\tfrac{2\alpha_n}{1-\alpha_n\rho\gamma}\langle(\gamma f - \mu A)x^*, x_{n+1} - x^*\rangle + \\
&\ \ \tfrac{(\alpha_n\tau)^2}{1-\alpha_n\rho\gamma}M_3,
\end{aligned}$$

where $M_3 = \sup_n \|x_n - x^*\|^2, n \geq 1$. It is easily to see that $\gamma_n = \frac{2\alpha_n(\tau-\rho\gamma)}{1-\alpha_n\rho\gamma}$. Hence by Lemma 4, the sequence $\{x_n\}$ converges strongly to x^*.

Putting $\beta_n \equiv 0$ in Theorem 16, we can draw the desired conclusion immediately.

Corollary 17 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F_k, k \in \{1, 2, ...M\}$, be bifunctions from $C \times C$ to R which satisfies conditions (A1)-(A4). Let $S_i : C \to C$ be a family κ_i-strict pseudo-contractions for some $0 \leq \kappa_i < 1$. Assume the set $\Omega = \cap_{i=1}^{\infty} F(S_i) \cap \cap_{k=1}^{M} EP(F_k) \neq \emptyset$. Let f be a contraction of H into itself with $\rho \in (0, 1)$ and let A be a $L-$Lipschitzian continuous operator and $\eta-$strongly monotone with $L > 0, \eta > 0$, $0 < \mu < 2\eta/L^2$ and $0 < \gamma < \mu(\eta - \frac{\mu L^2}{2})/\rho = \tau/\rho$. For every $n \in N$, let W_n be the mapping generated by S_i' and $0 < t_i \leq b < 1$. Given $x_1 \in H$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{cases} u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n A)W_n u_n, \end{cases} \tag{20}$$

where $\{\alpha_n\}$ and $\{r_{k,n}\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\{r_{k,n}\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_{k,n} > 0$ and $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ for $k \in \{1, 2, ...M\}$.

Then, $\{x_n\}$ converge strongly to $x^* \in \Omega$, which solves the variational inequality (6).

Remark 18 *If $F_k \equiv 0, k \in \{1, 2, \ldots, M\}$, then Theorem 16 reduces to Theorem 3.1 of Wang [17].*

4 Numerical result

In this section, in order to demonstrate the effectiveness, realization and convergence of the algorithm in Theorem 16, we consider the following simple example.

Let R^2 be the two dimensional Euclidean space with usual inner product and norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ $(\forall\, x = (x_1, x_2)^\top \in R^2)$. For convenience, we consider the following simple example:

Example Let $H = R^2$, $C = [-1, 1] \times [-1, 1]$, $S_n = I$, $t_n = a \in (0, 1), n \in N$. $F_k \equiv 0, \forall x, y \in C, r_{n,k} = 1, k \in \{1, 2, \ldots, M\}$. $A = I, f(x) = (\frac{1}{4}x_1, -\frac{1}{4}x_2)^\top, \forall x \in H$, with contraction coefficient $\rho = \frac{1}{4}$. Take $\alpha_n = \frac{1}{n}$ for every $n \in N$, $\mu = 1$ and $\gamma = 1$. Then $\{x^{(n)}\}$ is the sequence generated by

$$x^{(n+1)} = ((1 - \frac{3}{4n})x_1^{(n)}, (1 - \frac{5}{4n})x_2^{(n)})^\top. \tag{21}$$

and $\{x_n\} \to \mathbf{0} = (0, 0)^\top$ as $n \to \infty$, which solves the variational inequality $\langle(f - I)\mathbf{0}, p - \mathbf{0}\rangle \leq 0, \forall p \in C$. By the definition of f and γ, it is easy to get $h(x) = \frac{1}{8}x_1^2 - \frac{1}{8}x_2^2 + q, \forall x \in R^2$. Hence $\mathbf{0}$ is also the unique solution of the minimization problem

$$\min_{x\in C} \frac{3}{8}x_1^2 + \frac{5}{8}x_2^2 - q.$$

Proof: The proof is divided into three steps.

Step 1. Show that

$$T_{r_{k,n}}^{F_k}(x) = P_C x, \forall x \in H, k \in \{1, 2, \ldots, M\}.$$

Indeed, since $F_k(x, y) = 0, \forall x, y \in C$, $k \in \{1, 2, \ldots, M\}$, from the definition of $T_{r_{k,n}}^{F_k}$ in Lemma 11, we get

$$T_{r_{k,n}}^{F_k}(x) = \{z \in C : \langle y - z, z - x\rangle \geq 0, \forall\, y \in C.\}$$

By the equivalent property (4) of the nearest projection P_C from H to C, the conclusion is obtained.

When we take $x \in C$, $T_{r_{k,n}}^{F_k} x = P_C x = Ix, k \in \{1, 2, \ldots, M\}$. By the condition (iii) of Lemma 12, we get

$$\cap_{k=1}^{M} EP(F_k) = C. \tag{22}$$

Step 2. Show that $W_n = I$.

If $S_n = I$, $t_n = a \in (0, 1), n \in N$, by the definition of $W-$mapping, we get $W_n = I$. The detailed proof can be found in the reference [6].

Step 3. Show that $x^{(n)} \to \mathbf{0}$.

From (21), we get

$$\|x^{(n+1)}\| \leq (1 - \frac{3}{4n})\|x^{(n)}\|, \forall n \geq 1.$$

Since $S_n = I, n \in N$, we obtain

$$\bigcap_{i=1}^{\infty} F(S_i) = H.$$

Combining with (22), we have

$$\Omega = C = [-1,1] \times [-1,1].$$

By Lemma 4, it is easy to get $x_n \to \mathbf{0}$. $\mathbf{0}$ is the unique solution of the minimization problem

$$\begin{aligned}&\min_{x\in C} \tfrac{1}{2}\langle Ax, x\rangle - h(x)\\ &= \min_{x\in C} \tfrac{1}{2}\|x\|^2 - (\tfrac{1}{8}x_1^2 - \tfrac{1}{8}x_2^2 + q)\\ &= \min_{x\in C} \tfrac{1}{2}(x_1^2 + x_2^2) - (\tfrac{1}{8}x_1^2 - \tfrac{1}{8}x_2^2 + q)\\ &= \min_{x\in C} \tfrac{3}{8}x_1^2 + \tfrac{5}{8}x_2^2 - q.\end{aligned}$$

Now we turn to realizing (21) for approximating a fixed point of T. Take the initial guess $x^{(1)} = (0.1, -0.2)^\top$, $x^{(1)} = (0.01, 0.01)^\top$ and $x^{(1)} = (0.005, 0.005)^\top$, respectively. All the numerical experiment results are given in Table 1, Table 2 and Table 3.

TABLE 1. $x^{(1)} = (0.1, -0.2)^\top$(initial guess)

n(iterative number)	$x^{(n)}$	errors(n)
8.	(0.0057,0.0031).	6.5×10^{-3}
23.	(0.0026,0.0008).	2.7×10^{-3}
81.	(0.0010,0.0002).	1.0×10^{-3}
295.	(0.00005,0).	5.417×10^{-5}

TABLE 2. $x^{(1)} = (0.01, 0.01)^\top$(initial guess)

n(iterative number)	$x^{(n)}$	errors(n)
3.	(0.0012,-0.0005).	1.32×10^{-3}
7.	(0.0006,-0.00018).	6.58×10^{-4}
22.	(0.00027,-0.00004).	2.74×10^{-4}
155.	(0.000038,0).	3.88×10^{-5}

TABLE 3. $x^{(1)} = (0.005, 0.005)^\top$(initial guess)

n(iterative number)	$x^{(n)}$	errors(n)
2.	(0.00078,-0.00047).	9.12×10^{-4}
5.	(0.0004,-0.00014).	4.28×10^{-4}
15.	(0.00018,-0.00003).	6.91×10^{-5}
105.	(0.000026,0).	2.66×10^{-5}

From the above numerical results, we can see that the initial value is more close to the fixed point, the convergence is more quickly.

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