Positive Periodic Solutions of Delayed Dynamic Equations with Feedback Control on Time Scales

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Abstract: In this paper, based on the theory of calculus on time scales, by using Avery-Peterson fixed point theorem for cones, some criteria are established for the existence of three positive periodic solutions of delayed dynamic equations with feedback control on time scales of the following form:

$$\begin{cases} \boldsymbol{x}^{\Delta}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t))), \\ \boldsymbol{u}^{\Delta}(t) = B(t)\boldsymbol{u}(t) + D(t)\boldsymbol{x}(\delta_-(\tau_2, t)),\ t \in \mathbb{T}, \end{cases}$$

where $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$ are nonsingular matrix with continuous real-valued functions as their elements, δ_- be a backward shift operator. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Positive periodic solution; Dynamic equations; Feedback control; Time scale.

1 Introduction

Recent years have witnessed increasing interest in ecosystem with feedback controls [1-6]. The reasons for introducing control variables are based on main two points. On one hand, ecosystem in the real world are continuously distributed by unpredictable forces which can results in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables (for more details, one can see [7]). On the other hand, in the literature, it has been proved that, under certain conditions, some species are permanence but some are possible extinction in the competitive system, for example, see [8]. In order to search for certain schemes to ensure all the species coexist, feedback control variables should be introduced to ecosystem.

Compared with advanced in the area of studying the existence of a unique periodic solution [3-6], less progress has been achieved in studying the existence of multiple periodic solutions for higher-dimensional functional differential equations with feedback control, especially systems with the coefficient matrix to be an arbitrary nonsingular $n \times n$ matrix.

In fact, in the natural world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

The theory of calculus on time scales (see [9] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [10] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since his foundational work, one may see [11-17]. Therefore, it is practicable to study that on time scales which can unify the continuous and discrete situations.

Motivated by the above, the main aim of this paper is by employing a multiple fixed point theorem (Avery-Peterson fixed point theorem) for cones to establish the existence of three positive periodic solutions of the following dynamic equations with feedback controls on time scales:

$$\begin{cases} \boldsymbol{x}^{\Delta}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t))), \\ \boldsymbol{u}^{\Delta}(t) = B(t)\boldsymbol{u}(t) + D(t)\boldsymbol{x}(\delta_-(\tau_2, t)),\ t \in \mathbb{T}, \end{cases} \quad (1)$$

where $\mathbb{T}$ is an ω-periodic time scale, $A(t) = (a_{ij}(t))_{n\times n}$ and $B(t) = (b_{ij}(t))_{n\times n}$ are nonsingular matrix with continuous real-valued functions as their elements and $A(t+\omega)=A(t), B(t+\omega)=B(t)$; $D(t)=(d_{ij}(t))_{n\times n}$ with $d_{ij}(t)\in C(\mathbb{T},\mathbb{R}_+)$ and $D(t+\omega)=D(t)$ for all $t\in\mathbb{T}$; $\delta_-(\tau_i,t), i=1,2$ are delay functions with $t\in\mathbb{T}$ and $\tau_i\in[0,\infty)_{\mathbb{T}}, i=1,2$, satisfies $\delta_-(\tau_i,t+\omega)=\delta_-(\tau_i,t)+q\omega, i=1,2$, where δ_- be a backward shift operator on the set $\mathbb{T}^*$, and $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$, $q\in\mathbb{Z}_+$; $\boldsymbol{f}=(f_1,f_2,\cdots,f_n)^T$ is a function defined on $\mathbb{T}\times\mathbb{R}^n\times\mathbb{R}^n$, and satisfies $\boldsymbol{f}(t+\omega,\boldsymbol{x}(t+\omega),\boldsymbol{y})=\boldsymbol{f}(t,\boldsymbol{x}(t),\boldsymbol{y})$, for all $t\in\mathbb{T},\boldsymbol{y}\in\mathbb{R}^n$. Here $\mathbb{R}=(-\infty,+\infty),\mathbb{R}_+=(0,+\infty),\mathbb{R}^n_+=\{(x_1,x_2,\cdots,x_n)^T\in\mathbb{R}^n: x_i>0, i=1,2,\cdots,n\}$.

In this paper, for each $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^T\in\mathbb{R}^n$, the norm of $\boldsymbol{x}$ is defined as $\|\boldsymbol{x}\|=\sup_{t\in[0,\omega]_{\mathbb{T}}}|\boldsymbol{x}(t)|_0$, where $|\boldsymbol{x}(t)|_0=\sum_{i=1}^n|x_i(t)|$, and when it comes to that $\boldsymbol{x}(t)$ is continuous, delta derivative, delta integrable, and so forth; we mean that each element x_i is continuous, delta derivative, delta integrable, and so forth.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary results needed in later sections. In Section 3, we establish our main results for positive periodic solutions by applying Avery-Peterson fixed point theorem. In Section 4, an example is given to illustrate the feasibility and effectiveness of the results.

2 Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ and the graininess $\mu:\mathbb{T}\to\mathbb{R}^+$ are defined, respectively, by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},\rho(t)=\sup\{s\in\mathbb{T}:s<t\},\mu(t)=\sigma(t)-t$.

A point $t\in\mathbb{T}$ is called left-dense if $t>\inf\mathbb{T}$ and $\rho(t)=t$, left-scattered if $\rho(t)<t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, and right-scattered if $\sigma(t)>t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k=\mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k=\mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k=\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f:\mathbb{T}\to\mathbb{R}$ will be denoted by $C(\mathbb{T})=C(\mathbb{T},\mathbb{R})$.

For $y:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon>0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t))-y(s)]-y^\Delta(t)[\sigma(t)-s]\big|<\varepsilon|\sigma(t)-s|$$

for all $s\in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous, if $Y^\Delta(t)=y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s=Y(t)-Y(a).$$

Definition 1. *([19]) A time scale $\mathbb{T}$ is periodic if there exists $p>0$ such that if $t\in\mathbb{T}$, then $t\pm p\in\mathbb{T}$. For $\mathbb{T}\neq\mathbb{R}$, the smallest positive p is called the period of the time scale.*

Definition 2. *([19]) Let $\mathbb{T}\neq\mathbb{R}$ be a periodic time scale with period p. A function $f:\mathbb{T}\to\mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega=np$, $f(t+\omega)=f(t)$ for all $t\in\mathbb{T}$ and ω is the smallest number such that $f(t+\omega)=f(t)$.*

If $\mathbb{T}$ is ω-periodic, then $\sigma(t+\omega)=\sigma(t)+\omega$ with $\mu((t)$ is an ω-periodic function.

Definition 3. *([9]) An $n\times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided*

$$I+\mu((t)A(t)$$

is invertible for all $t\in\mathbb{T}^k$.

Definition 4. *([9]) Let $t_0\in\mathbb{T}$ and assume that A is a regressive $n\times n$-matrix-valued function. The unique matrix-valued solution of the IVP*

$$Y^\Delta=A(t)Y,\ \ Y(t_0)=I,$$

where I denotes as usual the $n\times n$-identity matrix, is called the matrix exponential function(at t_0), and is denoted by $e_A(\cdot,t_0)$.

Lemma 5. *([9]) If A is a regressive $n\times n$-matrix-valued functions on $\mathbb{T}$, then*
(i) $e_0(t,s)\equiv I$ and $e_A(t,t)\equiv I$;
(ii) $e_A(\sigma(t),s)=(I+\mu((t)A(t))e_A(t,s)$;
(iii) $e_A(t,s)=e_A^{-1}(s,t)$;
(iv) $e_A(t,s)e_A(s,r)=e_A(t,r)$.

Lemma 6. *([9]) Let A be a regressive $n\times n$-matrix-valued function on $\mathbb{T}$ and suppose that $\boldsymbol{f}:\mathbb{T}\to\mathbb{R}^n$ is rd-continuous. Let $t_0\in\mathbb{T}$ and*

$$\boldsymbol{y}^\Delta=A(t)\boldsymbol{y}+\boldsymbol{f}(t),\ \ \boldsymbol{y}(t_0)=\boldsymbol{y}_0$$

has a unique solution $\boldsymbol{y}:\mathbb{T}\to\mathbb{R}^n$. *Moreover, the solution is given by*

$$\boldsymbol{y}(t)=e_A(t,t_0)\boldsymbol{y}_0+\int_{t_0}^{t}e_A(t,\sigma(\tau))\boldsymbol{f}(\tau)\Delta\tau.$$

Lemma 7. *Let A be a regressive $n\times n$-matrix-valued function on $\mathbb{T}$, then the function $\boldsymbol{x}(t)$ is an ω-periodic solution of* (1)*, if and only if $\boldsymbol{x}(t)$ is an ω-periodic solution of the following:*

$$\boldsymbol{x}(t)=\int_t^{t+\omega}G(t,s)\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s,$$

where

$$G(t,s)=\big[e_A(0,\omega)-I\big]^{-1}e_A(t,\sigma(s)),\qquad(2)$$

and $\Phi(\boldsymbol{x})$ is defined in (3).

Proof. If $\boldsymbol{u}(t)$ is an ω-periodic solution of the second equation of (1). By using Lemma 6, for $s\in[t,t+\omega]_{\mathbb{T}}$, we have

$$\begin{aligned}\boldsymbol{u}(s)&=e_B(s,t)\boldsymbol{u}(t)\\&\quad+\int_t^s e_B(s,\sigma(\theta))D(\theta)\boldsymbol{x}(\delta_-(\tau_2,\theta))\Delta\theta.\end{aligned}$$

Let $s=t+\omega$ in the above equality, we have

$$\begin{aligned}\boldsymbol{u}(t+\omega)&=e_B(t+\omega,t)\boldsymbol{u}(t)\\&\quad+\int_t^{t+\omega}e_B(t+\omega,\sigma(\theta))D(\theta)\\&\quad\boldsymbol{x}(\delta_-(\tau_2,\theta))\Delta\theta.\end{aligned}$$

Noticing that $\boldsymbol{u}(t+\omega)=\boldsymbol{u}(t)$ and $e_B(t,t+\omega)=e_B(0,\omega)$, then

$$\begin{aligned}\boldsymbol{u}(t)&=\int_t^{t+\omega}\bar{G}(t,s)D(s)\boldsymbol{x}(\delta_-(\tau_2,s))\Delta s\\&:=(\Phi\boldsymbol{x})(t),\end{aligned}\qquad(3)$$

where

$$\bar{G}(t,s)=\big[e_B(0,\omega)-I\big]^{-1}e_B(t,\sigma(s)).\qquad(4)$$

Now we claim that

$$e_A(t+\omega,\sigma(s+\omega))=e_A(t,\sigma(s)).$$

In fact

$$\begin{aligned}e_A(t+\omega,\sigma(s+\omega))&=e_A(t+\omega,\sigma(s)+\omega)\\&=e_A(t,\sigma(s)).\end{aligned}$$

It is clear that $\bar{G}(t,s)=\bar{G}(t+\omega,s+\omega)$ for all $(t,s)\in\mathbb{T}^2$ and $\boldsymbol{u}(t+\omega)=\boldsymbol{u}(t)$ when $\boldsymbol{x}$ is ω-periodic.

Denote $(\Phi\boldsymbol{x})=((\Phi_1\boldsymbol{x}),(\Phi_2\boldsymbol{x}),\cdots,(\Phi_n\boldsymbol{x}))^T$, then any ω-periodic solutions of system (1) is equivalent to that of the following equation

$$\boldsymbol{x}^{\Delta}(t)=A(t)\boldsymbol{x}(t)+\boldsymbol{f}(t,\boldsymbol{x}(t),(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))).$$

Again using Lemma 6, repeating the above process, we have

$$\boldsymbol{x}(t)=\int_t^{t+\omega}G(t,s)\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s,$$

where

$$G(t,s)=\big[e_A(0,\omega)-I\big]^{-1}e_A(t,\sigma(s)).$$

It is also clear that $G(t,s)=G(t+\omega,s+\omega)$ for all $(t,s)\in\mathbb{T}^2$. This completes the proof of Lemma 7. ☐

Definition 8. *Let X be a Banach space and K be a closed nonempty subset of X, K is a cone if:*

(1) $\alpha\boldsymbol{u}+\beta\mathbf{v}\in K$ *for all* $\boldsymbol{u},\mathbf{v}\in K$ *and all* $\alpha,\beta\geq 0$*;*

(2) $\boldsymbol{u},-\boldsymbol{u}\in K$ *imply* $\boldsymbol{u}=0$.

Define $K_r=\{\boldsymbol{x}\in K\big|\|\boldsymbol{x}\|\leq r\}$. Let $\alpha(\boldsymbol{x})$ denote the positive continuous concave functional on K, that is $\alpha:K\to[0,+\infty)$ is continuous and satisfying

$$\alpha(\lambda\boldsymbol{x}+(1-\lambda)y)\geq\lambda\alpha(\boldsymbol{x})+(1-\lambda)\alpha(y)$$

for any $\boldsymbol{x},y\in K$, $0<\lambda<1$, and we denote the set $K(\alpha,a,b)=\{\boldsymbol{x}|\boldsymbol{x}\in K,a\leq\alpha(\boldsymbol{x}),\|\boldsymbol{x}\|\leq b\}$.

Let γ and θ be nonnegative continuous convex functionals on K, α be a non-negative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real numbers a,b,c and d, we define the following convex sets:

$$\begin{aligned}&K(\gamma,d)=\{\boldsymbol{x}\in K|\gamma(\boldsymbol{x})<d\},\\&K(\gamma,\alpha,b,d)=\{\boldsymbol{x}\in K|b\leq\alpha(\boldsymbol{x}),\gamma(\boldsymbol{x})\leq d\},\\&K(\gamma,\theta,\alpha,b,c,d)=\{\boldsymbol{x}\in K|b\leq\alpha(\boldsymbol{x}),\theta(\boldsymbol{x})\leq c,\\&\qquad\qquad\gamma(\boldsymbol{x})\leq d\},\end{aligned}$$

and a closed set $R(\gamma,\psi,a,d)=\{\boldsymbol{x}\in K|\,a\leq\psi(\boldsymbol{x}),\gamma(\boldsymbol{x})\leq d\}$.

The following fixed point theorem due to Avery and Peterson is important in the prove of our main results.

Theorem 9. *([20]) Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying*

$\psi(\rho \boldsymbol{x}) \leq \rho\psi(\boldsymbol{x})$ for $0 \leq \rho \leq 1$, such that for some positive numbers E and d,

$$\alpha(\boldsymbol{x}) \leq \psi(\boldsymbol{x}) \ and \ \|\boldsymbol{x}\| \leq E\gamma(\boldsymbol{x}) \qquad (*)$$

for all $\boldsymbol{x} \in \overline{K(\gamma, d)}$. Suppose $H : \overline{K(\gamma, d)} \to \overline{K(\gamma, d)}$ is completely continuous and there exist positive numbers a, b and c with $a < b$ such that:

1. *$\{\boldsymbol{x} \in K(\gamma, \theta, \alpha, b, c, d) | \alpha(\boldsymbol{x}) > b\} \neq \emptyset$ and $\alpha(H\boldsymbol{x}) > b$ for $\boldsymbol{x} \in K(\gamma, \theta, \alpha, b, c, d)$,*
2. *$\alpha(H\boldsymbol{x}) > b$ for $\boldsymbol{x} \in K(\gamma, \alpha, b, d)$ with $\theta(H\boldsymbol{x}) > c$,*
3. *$0 \overline{\in} R(\gamma, \psi, a, d)$ and $\psi(H\boldsymbol{x}) < a$ for $\boldsymbol{x} \in R(\gamma, \psi, a, d)$ with $\psi(\boldsymbol{x}) = a$.*

Then H has at least three fixed points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \overline{K(\gamma, d)}$ such that:

$$\gamma(\boldsymbol{x}_i) \leq d \ for \ i = 1, 2, 3, \ b < \alpha(\boldsymbol{x}_1),$$

$$a < \psi(\boldsymbol{x}_2), \ with \ \alpha(\boldsymbol{x}_2) < b, \ and \ \psi(\boldsymbol{x}_3) < a.$$

In order to obtain the existence of periodic solutions of system (1), we make the following preparations:

Set

$$X = \big\{\boldsymbol{x}(t) : \boldsymbol{x}(t) \in C(\mathbb{T}, \mathbb{R}^n), \boldsymbol{x}(t + \omega) = \boldsymbol{x}(t)\big\}$$

with the norm defined by $\|\boldsymbol{x}\| = \sup\limits_{t \in [0,\omega]_{\mathbb{T}}} |\boldsymbol{x}(t)|_0$, where $|\boldsymbol{x}(t)|_0 = \sum\limits_{i=1}^{n} |\boldsymbol{x}_i(t)|$, then X is a Banach space.

For convenience, we introduce the following notations:

$$\begin{aligned} G(t,s) &= \big[e_A(0,\omega) - I\big]^{-1} e_A(t, \sigma(s)) \\ &= (G_{ik}(t,s))_{n \times n}, \\ \bar{G}(t,s) &= \big[e_B(0,\omega) - I\big]^{-1} e_B(t, \sigma(s)) \\ &= (\bar{G}_{ik}(t,s))_{n \times n}, \end{aligned}$$

for $t, s \in \mathbb{T}, i, k = 1, 2, \cdots, n$. And

$$A_0 := \min_{1 \leq i,k \leq n} \inf_{s,t \in [0,\omega]_{\mathbb{T}}} |G_{ik}(t,s)|,$$

$$B_0 := \max_{1 \leq i,k \leq n} \sup_{s,t \in [0,\omega]_{\mathbb{T}}} |G_{ik}(t,s)|,$$

$$A_1 := \min_{1 \leq k \leq n} \inf_{s,t \in [0,\omega]_{\mathbb{T}}} |\sum_{i=1}^{n} G_{ik}(t,s)|,$$

$$B_1 := \max_{1 \leq k \leq n} \sup_{s,t \in [0,\omega]_{\mathbb{T}}} |\sum_{i=1}^{n} G_{ik}(t,s)|.$$

Hereafter, we assume that

(P_1) $f_i(t, \xi, \zeta) \geq 0$ for all $(t, \xi, \zeta) \in \mathbb{T} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+, i = 1, 2, \cdots, n$.

(P_2) $\boldsymbol{f}(t, \varphi(t), (\Phi\varphi)(\delta_-(\tau_1, t)))$ is a continuous function of t for each $\varphi \in C(\mathbb{T}, \mathbb{R}^n_+)$, where Φ is defined by (3).

(P_3) For any $L > 0$ and $\varepsilon > 0$, there exists $\nu > 0$, such that

$$\big\{\phi, \psi \in C(\mathbb{T}, \mathbb{R}^n), \|\phi\| \leq L, \|\psi\| \leq L, \|\phi - \psi\| < \nu\big\}$$

imply

$$\begin{aligned} &\big|\boldsymbol{f}(t, \phi(t), \boldsymbol{u}_1(\delta_-(\tau_1, t))) \\ &\quad - \boldsymbol{f}(t, \psi(t), \boldsymbol{u}_2(\delta_-(\tau_1, t)))\big|_0 < \varepsilon, \end{aligned}$$

for all $t \in [0, \omega]_{\mathbb{T}}$, where

$$\begin{aligned} &\boldsymbol{u}_1(\delta_-(\tau_1, t)) \\ &= \int_{\delta_-(\tau_1,t)}^{\delta_-(\tau_1,t)+\omega} \bar{G}(\delta_-(\tau_1, t), s) D(s) \\ &\quad \phi(\delta_-(\tau_2, s))\Delta s \\ &= (\Phi\phi)(\delta_-(\tau_1, t)), \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{u}_2(\delta_-(\tau_1, t)) \\ &= \int_{\delta_-(\tau_1,t)}^{\delta_-(\tau_1,t)+\omega} \bar{G}(\delta_-(\tau_1, t), s) D(s) \\ &\quad \psi(\delta_-(\tau_2, s))\Delta s \\ &= (\Phi\psi)(\delta_-(\tau_1, t)). \end{aligned}$$

(P_4) $A_j > 0, \ B_j > 0, \ j = 0, 1.$

Let

$$\begin{aligned} K = \big\{&\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \in X : x_i \geq \delta\|x_i\|, \\ &t \in [0, \omega]_{\mathbb{T}}, \ i = 1, 2, \cdots, n\big\}, \end{aligned} \qquad (5)$$

where $\delta = \frac{A_0}{B_0} \in (0, 1)$ and A_0, B_0 are defined by the above. Obviously, K is a cone in X.

Define a mapping H by

$$\begin{aligned} (H\boldsymbol{x})(t) = \int_t^{t+\omega} &G(t,s)\boldsymbol{f}(s, \boldsymbol{x}(s), \\ &(\Phi\boldsymbol{x})(\delta_-(\tau_1, s)))\Delta s, \end{aligned} \qquad (6)$$

for all $\boldsymbol{x} \in K, \ t \in \mathbb{T}$, where $G(t, s)$ is defined by (2) and

$$\begin{aligned} (H\boldsymbol{x})(t) = ((&H_1\boldsymbol{x})(t), (H_2\boldsymbol{x})(t), \cdots, \\ &(H_n\boldsymbol{x})(t))^T, \end{aligned} \qquad (7)$$

where

$$(H_i\boldsymbol{x})(t) = \int_t^{t+\omega}\sum_{k=1}^{n} G_{ik}(t,s)f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s,$$

where $i = 1, 2, \cdots, n$.

In the following, we will give some lemmas concerning K and H defined by (5) and (6), respectively.

Lemma 10. *Assume that $(P_1), (P_4)$ hold, then $H : K \to K$ is well defined.*

Proof. For any $\boldsymbol{x} \in K$, it is clear that $H\boldsymbol{x} \in PC(\mathbb{T})$. In view of (6), for $t \in \mathbb{T}$, we obtain

$$\begin{aligned}
&(H\boldsymbol{x})(t+\omega)\\
&= \int_{t+\omega}^{t+2\omega} G(t+\omega,s)\boldsymbol{f}(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s\\
&= \int_t^{t+\omega} G(t+\omega,u+\omega)\boldsymbol{f}(u+\omega,\boldsymbol{x}(u+\omega), (\Phi\boldsymbol{x})(\delta_-(\tau_1,u+\omega)))\Delta u\\
&= \int_t^{t+\omega} G(t,u)\boldsymbol{f}(u,\boldsymbol{x}(u),(\Phi\boldsymbol{x})(\delta_-(\tau_1,u)))\Delta u\\
&= (H\boldsymbol{x})(t).
\end{aligned}$$

That is, $(H\boldsymbol{x})(t+\omega) = (H\boldsymbol{x})(t), t \in \mathbb{T}$. So $H\boldsymbol{x} \in X$.

For any $\boldsymbol{x} \in K, \forall\, t \in [0,\omega]_{\mathbb{T}}$, we have

$$\begin{aligned}
&|H_i\boldsymbol{x}|\\
&= \left|\int_t^{t+\omega}\sum_{k=1}^{n} G_{ik}(t,s)f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s\right|\\
&\le \int_t^{t+\omega}\sum_{k=1}^{n} |G_{ik}(t,s)||f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\\
&\le B_0\Big(\int_t^{t+\omega}\sum_{k=1}^{n}|f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\Big),\ i = 1,2,\cdots,n.
\end{aligned}$$

So

$$\begin{aligned}
\|H_i\boldsymbol{x}\| &= \sup_{t\in[0,\omega]_{\mathbb{T}}} |H_i\boldsymbol{x}|\\
&\le B_0\Big(\int_t^{t+\omega}\sum_{k=1}^{n}|f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\Big),\ i = 1,2,\cdots,n.
\end{aligned}$$

By (P_1) and (P_5), we get

$$\begin{aligned}
&(H_i\boldsymbol{x})(t)\\
&= \int_t^{t+\omega}\sum_{k=1}^{n} G_{ik}(t,s)f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s\\
&= \int_t^{t+\omega}\sum_{k=1}^{n} |G_{ik}(t,s)||f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\\
&\ge A_0\Big(\int_t^{t+\omega}\sum_{k=1}^{n}|f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\Big)\\
&= \frac{A_0}{B_0}B_0\Big(\int_t^{t+\omega}\sum_{k=1}^{n}|f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\Big)\\
&\ge \delta\|H_i\boldsymbol{x}\|,\ i = 1,2,\cdots,n.
\end{aligned}$$

That is, $H\boldsymbol{x} \in K$. This completes the proof. □

Lemma 11. *Assume that $(P_1) - (P_4)$ hold, then $H : K \to K$ is completely continuous.*

Proof. We first show that H is continuous. By (P_3), for any $L > 0$ and $\varepsilon > 0$, there exists a $\nu > 0$ such that

$$\big\{\phi,\psi \in C(\mathbb{T},\mathbb{R}^n), \|\phi\| \le L, \|\psi\| \le L, \|\phi-\psi\| < \nu\big\}$$

imply

$$\sup_{s\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{f}(s,\phi(s),\boldsymbol{u}_1(\delta_-(\tau_1,s))) -\boldsymbol{f}(s,\psi(s),\boldsymbol{u}_2(\delta_-(\tau_1,s)))|_0 < \frac{\varepsilon}{B_1\omega},$$

If $\boldsymbol{x},\boldsymbol{y},\in K$ with $\|\boldsymbol{x}\| \le L, \|\boldsymbol{y}\| \le L, \|\boldsymbol{x}-\boldsymbol{y}\| < \nu$, then

$$\begin{aligned}
&|(H\boldsymbol{x})(t) - (H\boldsymbol{y})(t)|_0\\
&\le \sum_{i=1}^{n}\Big|\int_t^{t+\omega}\sum_{k=1}^{n} G_{ik}(t,s)f_k(s,\boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s\\
&\quad -\int_t^{t+\omega}\sum_{k=1}^{n} G_{ik}(t,s)f_k(s,\boldsymbol{y}(s), (\Phi\boldsymbol{y})(\delta_-(\tau_1,s)))\Delta s\Big|\\
&\le \int_t^{t+\omega}\sum_{k=1}^{n}|\sum_{i=1}^{n} G_{ik}(t,s)|
\end{aligned}$$

$$\begin{aligned}
&|f_k(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\\
&-f_k(s,\boldsymbol{y}(s),(\Phi\boldsymbol{y})(\delta_-(\tau_1,s)))|\Delta s\\
<\;& B_1\bigg(\int_t^{t+\omega}|\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\\
&-\boldsymbol{f}(s,\boldsymbol{y}(s),(\Phi\boldsymbol{y})(\delta_-(\tau_1,s)))|_0\Delta s\bigg)\\
<\;& B_1\bigg(\omega\frac{\varepsilon}{B_1\omega}\bigg)\\
=\;& \varepsilon
\end{aligned}$$

for all $t\in[0,\omega]_{\mathbb{T}}$, which yields $\|H\boldsymbol{x}-H\boldsymbol{y}\| = \sup_{t\in[0,\omega]_{\mathbb{T}}}|(H\boldsymbol{x})(t)-(H\boldsymbol{y})(t)|_0<\varepsilon$, thus H is continuous.

Next, we show that H maps any bounded sets in K into relatively compact sets. Now we first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon=1$, by (P_3), for any $\vartheta>0$, there exists $\nu>0$ such that $\{\boldsymbol{x},\boldsymbol{y}\in C(\mathbb{T},\mathbb{R}^n),\|\boldsymbol{x}\|\le\vartheta,\|\boldsymbol{y}\|\le\vartheta,\|\boldsymbol{x}-\boldsymbol{y}\|<\nu,s\in[0,\omega]_{\mathbb{T}}\}$ imply

$$\begin{aligned}
&|\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\\
&-\boldsymbol{f}(s,\boldsymbol{y}(s),(\Phi\boldsymbol{y})(\delta_-(\tau_1,s)))|_0<1.
\end{aligned}$$

Choose a positive integer N such that $\frac{\vartheta}{N}<\delta$. Let $\boldsymbol{x}\in C(\mathbb{T},\mathbb{R}^n)$ and define $\boldsymbol{x}^k(t)=\frac{\boldsymbol{x}(t)k}{N}, k=0,1,2,\cdots,N$. If $\|\boldsymbol{x}\|<\vartheta$, then

$$\begin{aligned}
\|\boldsymbol{x}^k-\boldsymbol{x}^{k-1}\| &= \sup_{t\in[0,\omega]_{\mathbb{T}}}\left|\frac{\boldsymbol{x}(t)k}{N}-\frac{\boldsymbol{x}(t)(k-1)}{N}\right|_0\\
&\le \|\boldsymbol{x}\|\frac{1}{N}\le\frac{\vartheta}{N}<\delta.
\end{aligned}$$

Thus

$$\begin{aligned}
&|\boldsymbol{f}(s,\boldsymbol{x}^k(s),(\Phi\boldsymbol{x}^k)(\delta_-(\tau_1,s)))\\
&-\boldsymbol{f}(s,\boldsymbol{x}^{k-1}(s),(\Phi\boldsymbol{x}^{k-1})(\delta_-(\tau_1,s)))|_0<1.
\end{aligned}$$

for all $s\in[0,\omega]_{\mathbb{T}}$, these yield

$$\begin{aligned}
&|\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|_0\\
=\;&|\boldsymbol{f}(s,\boldsymbol{x}^N(s),(\Phi\boldsymbol{x}^N)(\delta_-(\tau_1,s)))|_0\\
\le\;&\sum_{k=1}^{N}|\boldsymbol{f}(s,\boldsymbol{x}^k(s),(\Phi\boldsymbol{x}^k)(\delta_-(\tau_1,s)))\\
&-\boldsymbol{f}(s,\boldsymbol{x}^{k-1}(s),(\Phi\boldsymbol{x}^{k-1})(\delta_-(\tau_1,s)))|_0\\
&+|\boldsymbol{f}(s,0,0)|_0\\
<\;&N+|\boldsymbol{f}(s,0,0)|_0=:W. \qquad (8)
\end{aligned}$$

It follows from (7) and (8) that for $t\in[0,\omega]_{\mathbb{T}}$,

$$\begin{aligned}
\|H\boldsymbol{x}\| &= \sup_{t\in[0,\omega]_{\mathbb{T}}}\sum_{i=1}^{n}|(H_i\boldsymbol{x})(t)|\\
&\le \sum_{k=1}^{n}B_1\bigg(\int_0^{\omega}|f_k(s,\boldsymbol{x}(s),\\
&\qquad(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|\Delta s\bigg)\\
&= B_1(|\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|_0\omega)\\
&\le B_1W\omega:=Q.
\end{aligned}$$

Finally, for $t\in\mathbb{T}$, we have

$$(H\boldsymbol{x})^{\Delta}(t)=A(t)(H\boldsymbol{x})(t)+\boldsymbol{f}(t,\boldsymbol{x}(t),(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))).$$

So

$$\begin{aligned}
|(H\boldsymbol{x})^{\Delta}(t)|_0 &\le |A(t)(H\boldsymbol{x})(t)|_0\\
&\quad+|\boldsymbol{f}(t,\boldsymbol{x}(t),(\Phi\boldsymbol{x})(\delta_-(\tau_1,t)))|_0\\
&\le |A|Q+W,
\end{aligned}$$

where $|A|=\max_{1\le i\le n}\sup_{t\in[0,\omega]_{\mathbb{T}}}\sum_{j=1}^{n}|a_{ij}(t)|$.

Hence, $\{H\boldsymbol{x}:\boldsymbol{x}\in K,\|\boldsymbol{x}\|\le\vartheta\}$ is a family of uniformly bounded and equicontinuous functions on $[0,\omega]_{\mathbb{T}}$. By the theorem of Arzela-Ascoli, we know that the function H is completely continuous. □

3 Main Results

Now, we fix $\eta,l\in[0,\omega]_{\mathbb{T}}$, $\eta\le l$, and let the nonnegative continuous concave functional α, the nonnegative continuous convex functions θ,γ and the nonnegative continuous function ψ be defined on the cone K by

$$\alpha(\boldsymbol{x})=\inf_{t\in[\eta,l]_{\mathbb{T}}}|\boldsymbol{x}(t)|_0,\ \psi(\boldsymbol{x})=\theta(\boldsymbol{x})=\sup_{t\in[0,\omega]_{\mathbb{T}}}|\boldsymbol{x}(t)|_0,$$
$$\gamma(\boldsymbol{x})=\sup_{t\in[0,\omega]_{\mathbb{T}}}|(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0,$$

The functions defined above satisfy the following relations

$$\alpha(\boldsymbol{x})\le\psi(\boldsymbol{x})=\theta(\boldsymbol{x}),\ \forall\,\boldsymbol{x}\in K. \qquad (9)$$

Lemma 12. *For $\boldsymbol{x}\in K$, there exists a constant $E>0$ such that*

$$\sup_{t\in[0,\omega]_{\mathbb{T}}}|\boldsymbol{x}(t)|_0\le E\sup_{t\in[0,\omega]_{\mathbb{T}}}|(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0.$$

Proof. For $\boldsymbol{x}\in K$, we have

$$\begin{aligned}
&\sup_{t\in[0,\omega]_{\mathbb{T}}}|(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0\\
=\;&\sup_{t\in[0,\omega]_{\mathbb{T}}}\int_{\delta_-(\tau_1,t)}^{\delta_-(\tau_1,t)+\omega}|\bar{G}(\delta_-(\tau_1,t),s)D(s)\\
&\boldsymbol{x}(\delta_-(\tau_2,s))|_0\Delta s\\
\ge\;&L\delta\|\boldsymbol{x}\|=L\delta\sup_{t\in[0,\omega]_{\mathbb{T}}}|\boldsymbol{x}(t)|_0,
\end{aligned}$$

where $L = \max_{1\le k\le n} \sup_{t\in[0,\omega]_{\mathbb{T}}} |\sum_{i=1}^{n} L_{ik}(t)|$, $(L_{ik}(t))_{n\times n} = \int_0^\omega |\bar{G}(\delta_-(\tau_1,t),s)D(s)|\Delta s$. Setting $E = \frac{1}{L\delta}$. This completes the proof. $\square$

Moreover, for each $\boldsymbol{x} \in K$,

$$\begin{aligned} \|\boldsymbol{x}\| &= \sup_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}(t)|_0 \le \frac{\sup_{t\in[0,\omega]_{\mathbb{T}}} |(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0}{L\delta} \\ &= E\gamma(\boldsymbol{x}). \end{aligned} \tag{10}$$

We also find that $\psi(\rho\boldsymbol{x}) = \rho\psi(\boldsymbol{x})$ for $\forall\ \rho \in [0,1]_{\mathbb{T}}$, $\forall\ \boldsymbol{x} \in K$. Therefore, by (10) the condition $(*)$ of Theorem 9 is satisfied.

To present our main result, we assume there exist constants $a, b, d > 0$ with $a < b < \frac{b}{\delta} < \frac{d}{L}$ such that:

(S_1) $|\boldsymbol{f}(t,\boldsymbol{u},\mathbf{v})|_0 < \frac{d}{B_1 L\omega}$, for $0 \le |\boldsymbol{u}|_0 \le Ed,\ 0 \le |\mathbf{v}|_0 \le d,\ t \in [0,\omega]_{\mathbb{T}}$;

(S_2) $|\boldsymbol{f}(t,\boldsymbol{u},\mathbf{v})|_0 > \frac{b}{A_1\omega}$, for $b \le |\boldsymbol{u}|_0 \le b/\delta,\ 0 \le |\mathbf{v}|_0 \le d,\ t \in [\eta,l]_{\mathbb{T}}$;

(S_3) $|\boldsymbol{f}(t,\boldsymbol{u},\mathbf{v})|_0 < \frac{a}{B_1\omega}$, for $0 \le |\boldsymbol{u}|_0 \le a,\ 0 \le |\mathbf{v}|_0 \le d,\ t \in [0,\omega]_{\mathbb{T}}$.

Theorem 13. *Under assumptions $(S_1)-(S_3)$ and $(P_1)-(P_4)$, then the system (1) has at least three positive ω-periodic solutions $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_3$ satisfying,*

$$\sup_{t\in[0,\omega]_{\mathbb{T}}} |(\Phi\boldsymbol{x}_i)(\delta_-(\tau_1,t))|_0 \le d,\ i = 1,2,3,$$

$$b < \inf_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}_1(t)|_0,$$

$$a < \sup_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}_2(t)|_0,\ with \inf_{t\in[\eta,l]_{\mathbb{T}}} |\boldsymbol{x}_2(t)|_0 < b,\ and$$

$$\sup_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}_3(t)|_0 < a.$$

Proof. For $\boldsymbol{x} \in \overline{K(\gamma,d)}$, there is $\gamma(\boldsymbol{x}) = \sup_{t\in[0,\omega]_{\mathbb{T}}} |(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0 \le d$. From Lemma 12, we have $\sup_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}(t)|_0 \le Ed$, that is $0 \le |\boldsymbol{x}(t)|_0 \le Ed$, for $t \in [0,\omega]_{\mathbb{T}}$. By assumption (S_1), for $\boldsymbol{x} \in K$, there is $H\boldsymbol{x} \in K$, and

$$\begin{aligned} &|(H\boldsymbol{x})(\delta_-(\tau_1,t))|_0 \\ &= \Bigg| \int_{\delta_-(\tau_1,t)}^{\delta_-(\tau_1,t)+\omega} G(\delta_-(\tau_1,t),s)\boldsymbol{f}(s,\boldsymbol{x}(s), \\ &\quad \Phi(\delta_-(\tau_1,s)))\Delta s \Bigg|_0 \\ &\le B_1 \sum_{k=1}^{n} \int_0^\omega |f_k(s,\boldsymbol{x}(s),\Phi(\delta_-(\tau_1,s)))|\Delta s \\ &= B_1 \int_0^\omega |\boldsymbol{f}(s,\boldsymbol{x}(s),\Phi(\delta_-(\tau_1,s)))|_0\Delta s \\ &\le B_1 \int_0^\omega \left(\frac{d}{B_1 L\omega}\right)\Delta s \\ &\le \frac{d}{L}, \end{aligned}$$

then

$$\begin{aligned} &\gamma(H\boldsymbol{x})(t) \\ &= \sup_{t\in[0,\omega]_{\mathbb{T}}} |\Phi(H\boldsymbol{x})(t)|_0 \\ &= \sup_{t\in[0,\omega]_{\mathbb{T}}} \int_0^\omega |\bar{G}(t,s)D(s)(H\boldsymbol{x})(\delta_-(\tau_1,s))|_0\Delta s \\ &\le L\cdot\frac{d}{L} = d. \end{aligned}$$

Therefore, $H : \overline{K(\gamma,d)} \to \overline{K(\gamma,d)}$.

To check condition (1) of Theorem 9, we take $|\tilde{\boldsymbol{x}}|_0 = b/\delta$. It is easy to see that $\tilde{\boldsymbol{x}} \in K(\gamma,\theta,\alpha,b,b/\delta,d)$, and $\alpha(\tilde{\boldsymbol{x}}) = b/\delta > b$, so $\{\boldsymbol{x} \in K(\gamma,\theta,\alpha,b,b/\delta,d) | \alpha(\boldsymbol{x}) > b\} \ne \emptyset$.

Hence, for $\boldsymbol{x} \in K(\gamma,\theta,\alpha,b,b/\delta,d)$, then

$$\inf_{t\in[\eta,l]_{\mathbb{T}}} |\boldsymbol{x}(t)|_0 \ge b,\ \sup_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}(t)|_0 \le b/\delta,$$

$$\sup_{t\in[0,\omega]_{\mathbb{T}}} |(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0 \le d,$$

that is

$$b \le |\boldsymbol{x}(t)|_0 \le b/\delta,\ 0 \le |(\Phi\boldsymbol{x})(\delta_-(\tau_1,t))|_0 \le d,$$

for $t \in [\eta,l]_{\mathbb{T}}$.

Then, by assumption (S_2), we have

$$\begin{aligned} &\alpha(H\boldsymbol{x})(t) \\ &= \inf_{t\in[\eta,l]_{\mathbb{T}}} \Bigg\{ \Bigg| \int_t^{t+\omega} G(t,s)\boldsymbol{f}(s,\boldsymbol{x}(s), \\ &\quad (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s \Bigg|_0 \Bigg\} \\ &\ge \inf_{t\in[0,\omega]_{\mathbb{T}}} \Bigg\{ \Bigg| \int_t^{t+\omega} G(t,s)\boldsymbol{f}(s,\boldsymbol{x}(s), \\ &\quad (\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))\Delta s \Bigg|_0 \Bigg\} \\ &\ge A_1 \int_t^{t+\omega} |\boldsymbol{f}(s,\boldsymbol{x}(s),(\Phi\boldsymbol{x})(\delta_-(\tau_1,s)))|_0\Delta s \\ &> A_1 \int_0^\omega \left(\frac{b}{A_1\omega}\right)\Delta s \\ &= b. \end{aligned}$$

i.e., $\alpha(H\boldsymbol{x}) > b$ for all $\boldsymbol{x} \in K(\gamma,\theta,\alpha,b,b/\delta,d)$. This shows that condition (1) of Theorem 9 is satisfied.

Secondly, by the cone K defined by (5), we have $\alpha(H\boldsymbol{x}) \geq \delta\theta(H\boldsymbol{x}) > \delta(b/\delta) = b$, for all $\boldsymbol{x} \in K(\gamma, \alpha, b, d)$ with $\theta(H\boldsymbol{x}) > b/\delta$. Thus condition (2) of Theorem 9 is satisfied.

Finally, we show that condition (3) of Theorem 9 also holds. Clearly, as $\psi(0) = 0 < a$, there holds $0\overline{\in}R(\gamma, \psi, a, d)$. Suppose that $\boldsymbol{x} \in R(\gamma, \psi, a, d)$ with $\psi(\boldsymbol{x}) = a$, this implies that for $t \in [0, \omega]_{\mathbb{T}}$, there is $\sup_{t\in[0,\omega]_{\mathbb{T}}} |\boldsymbol{x}(t)|_0 = a$, $\sup_{t\in[0,\omega]_{\mathbb{T}}} |(\Phi\boldsymbol{x})(\delta_-(\tau_1, t))|_0 \leq d$. Hence,

$$0 \leq |\boldsymbol{x}(t)|_0 \leq a,\ 0 \leq |(\Phi\boldsymbol{x})(\delta_-(\tau_1, t))|_0 \leq d,$$

for $t \in [0, \omega]_{\mathbb{T}}$. So by assumption (S_3), we have

$$\begin{aligned}\psi(H\boldsymbol{x}) &= \sup_{t\in[0,\omega]_{\mathbb{T}}} |(H\boldsymbol{x})(t)|_0 \\ &\leq B_1 \int_0^\omega |\boldsymbol{f}(s, \boldsymbol{x}(s), (\Phi\boldsymbol{x})(\delta_-(\tau_1, s)))|_0 \Delta s \\ &< B_1 \int_0^\omega \left(\frac{a}{B_1\omega}\right)\Delta s \\ &= a.\end{aligned}$$

So, the condition (3) of Theorem 9 is satisfied.

Therefore, by Theorem 9, we obtain that the operator H has at least three fixed points. This completes the proof. □

4 An Example

Considering the following system with time delays

$$\begin{cases} \boldsymbol{x}^\Delta(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t))), \\ \boldsymbol{u}^\Delta(t) = B(t)\boldsymbol{u}(t) + D(t)\boldsymbol{x}(\delta_-(\tau_2, t)),\ t \in \mathbb{T}, \end{cases} \tag{11}$$

where

$$A(t) = \begin{bmatrix} -1.5 & 1 \\ 1 & -1.5 \end{bmatrix},\ B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$D(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned}&|\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 \\ =& \begin{cases} \frac{|\sin 2\pi t|}{10} + \frac{|\boldsymbol{x}(t)|_0 + |\boldsymbol{u}(\delta_-(\tau_1,t))|_0}{100}, \\ |\boldsymbol{x}|_0 \leq 65,\ 0 \leq |u|_0 \leq 2\times 10^4, \\ 3500 + \frac{|\boldsymbol{x}(t)|_0}{5\times 10^6 + |\sin 2\pi t|} + \frac{|\boldsymbol{u}(\delta_-(\tau_1,t))|_0}{5\times 10^6 + |\cos 2\pi t|}, \\ |\boldsymbol{x}|_0 > 65,\ 0 \leq |u|_0 \leq 2\times 10^4, \end{cases}\end{aligned}$$

where $\delta_-(\tau_1, t)$ is delay function with $t \in \mathbb{T}$.

From the above, we can get

$$\begin{aligned}& e_A(t, t_0) \\ =& e_{-0.5}(t, t_0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ & + e_{-0.5}(t, t_0)\int_{t_0}^t \frac{1}{1 - 2.5\mu((s)}\Delta s \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \\ & e_A(t, \sigma(s)) = e_A(t, s)\big(I + \mu((s)A(s)\big)^{-1}.\end{aligned}$$

Case 1: $\mathbb{T} = \mathbb{R}$, and $\omega = 0.5$,

$$\begin{aligned}& e_A(t, s) = e^{-0.5(t-s)}\begin{bmatrix} 1 - 2(t-s) & (t-s) \\ (t-s) & 1 - 2(t-s) \end{bmatrix}, \\ & e_A(t, \sigma(s)) = e_A(t, s), \\ & G(t, s) = \big(e_A(0, \omega) - I\big)^{-1} e_A(t, s).\end{aligned}$$

By a direct calculation, we can get

$$A_0 = 1.3710,\ B_0 = 3.0793,$$

$$A_1 = 2.7420,\ B_1 = 4.5208,$$

then $\delta = 0.4452$, choose $a = 65$, $b = 70$, $L = 1$, $d = 2 \times 10^4$, and $0 \leq |u|_0 \leq 2 \times 10^4$, then

$$\begin{aligned}& |\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 < \frac{1}{10} + 0.65 \\ & \quad < 28.7560, \text{for } |\boldsymbol{x}|_0 \in [0, 65], \\ & |\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 < 3500 + 8.9848 \times 10^{-2} \\ & \quad < 8848, \text{for } |\boldsymbol{x}|_0 \in [0, 4.4924 \times 10^4], \\ & |\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 > 3500 + \frac{70}{5\times 10^6 + 2} \\ & \quad > 51.0576, \text{for } |\boldsymbol{x}|_0 \in [70, 157.2327].\end{aligned}$$

According to Theorem 13, when $\mathbb{T} = \mathbb{R}$, system (11) exists at least three positive periodic solutions $\hat{\boldsymbol{x}}_1,\ \hat{\boldsymbol{x}}_2,\ \hat{\boldsymbol{x}}_3$, and

$$\sup_{t\in[0,\omega]_{\mathbb{T}}} |\hat{\boldsymbol{x}}_3(t)|_0 < 65 < \sup_{t\in[0,\omega]_{\mathbb{T}}} |\hat{\boldsymbol{x}}_2(t)|_0,$$

$$\inf_{t\in[\eta,l]_{\mathbb{T}}} |\hat{\boldsymbol{x}}_2(t)|_0 < 70 < \inf_{t\in[0,\omega]_{\mathbb{T}}} |\hat{\boldsymbol{x}}_1(t)|_0.$$

Case 2: $\mathbb{T} = \mathbb{Z}$, and $\omega = 0.5$,

$$\begin{aligned}& e_A(t, s) = \left(\frac{1}{2}\right)^{(t-s)}\begin{bmatrix} 1 - \frac{4(t-s)}{3} & \frac{2(t-s)}{3} \\ \frac{2(t-s)}{3} & 1 - \frac{4(t-s)}{3} \end{bmatrix}, \\ & e_A(t, \sigma(s)) = e_A(t, s)\big(I + A\big)^{-1}, \\ & G(t, s) = \big(e_A(0, \omega) - I\big)^{-1} e_A(t, s)(I + A)^{-1}.\end{aligned}$$

By a direct calculation, we can get

$$A_0 = 1.4410,\ B_0 = 3.5916,$$

$$A_1 = 2.4142,\ B_1 = 6.8428,$$

then $\delta = 0.4012$, choose $a = 50,\ b = 80,\ L = 1,\ d = 2 \times 10^4$, and $0 \le |u|_0 \le 2 \times 10^4$, then

$$\begin{aligned}
&|\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 < \frac{1}{10} + 0.5 \\
&\quad < 7.3070, \text{for } |\boldsymbol{x}|_0 \in [0, 50], \\
&|\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 < 3500 + 0.0997 \\
&\quad < 5846, \text{for } |\boldsymbol{x}|_0 \in [0, 4.9850 \times 10^4], \\
&|\boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(\delta_-(\tau_1, t)))|_0 > 3500 + \frac{80}{5 \times 10^6 + 2} \\
&\quad > 66.2745, \text{for } |\boldsymbol{x}|_0 \in [80, 199.4018].
\end{aligned}$$

According to Theorem 13, when $\mathbb{T} = \mathbb{Z}$, system (11) exists at least three positive periodic solutions $\tilde{\boldsymbol{x}}_1,\ \tilde{\boldsymbol{x}}_2,\ \tilde{\boldsymbol{x}}_3$, and

$$\sup_{t \in [0,\omega]_{\mathbb{T}}} |\tilde{\boldsymbol{x}}_3(t)|_0 < 50 < \sup_{t \in [0,\omega]_{\mathbb{T}}} |\tilde{\boldsymbol{x}}_2(t)|_0,$$

$$\inf_{t \in [\eta,l]_{\mathbb{T}}} |\tilde{\boldsymbol{x}}_2(t)|_0 < 80 < \inf_{t \in [0,\omega]_{\mathbb{T}}} |\tilde{\boldsymbol{x}}_1(t)|_0.$$

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