An Update on Supereulerian Graphs

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Abstract: A graph is *supereulerian* if it has a spanning Eulerian subgraph. Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell ([2]) in 1997 proposed the supereulerian problem, which seeks a characterization of graphs that have spanning Eulerian subgraphs, and they indicated that this problem would be very difficult. Pulleyblank ([71]) later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been lots of researches on this topic. Catlin ([7]) in 1992 presented the first survey on supereulerian graphs. This paper is intended as an update of Catlin's survey article and will focus on the developments in the study of supereulerian graphs and the related problems over the past 20 years.

Key–Words: Eulerian graphs, Supereulerian graphs, Collapsible graphs, Catlin's reduction method, Line graphs, Claw-free graphs

1 Introduction

We follow Bondy and Murty [3] for terminology and notation not defined here, and consider only loopless finite graphs. As in [3], $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and $\Delta(G)$ denote the connectivity, edge-connectivity, minimum degree and maximum degree of a graph G, respectively. A graph is *nontrivial* if it has at least one non-loop edge. A cycle of order n is denoted by C_n. For a graph G, denote $O(G) = \{$odd-degree vertices of $G\}$ and define $D_i(G) = \{v \in V(G) \mid d(v) = i\}$.

A graph with $O(G) = \emptyset$ is called an *even graph*. A graph is *Eulerian* if it is connected and even. A graph G is *supereulerian* if G has a spanning Eulerian subgraph. Thus a graph G is supereulerian if G has a spanning closed trail. By definition, K_1 is supereulerian.

The supereulerian graph problem, raised by Boesch, Suffel, and Tindell [2], seeks to characterize supereulerian graphs. Pulleyblank [71] showed that determining whether a graph is supereulerian, even when restricted to planar graphs, is NP-complete. For more literature on supereulerian graphs, see Catlin's survey [7] and its supplement by Chen and Lai [19].

The supereulerian graph problem is also motivated by the study of Hamiltonian problems of graphs. A graph G is *Hamiltonian* if G has a spanning cycle. For integers $a, b > 0$, an $[a, b]$-*factor* F of G is a spanning subgraph of G such that for any $v \in V(F)$, $a \le d_F(v) \le b$. Thus a graph G is Hamiltonian if and only if G has a connected $[2, 2]$-factor; and is supereulerian if and only if G has a connected even $[2, \Delta(G)]$-factor. For a non-Hamiltonian graph G, and an even number k with $2 \le k \le \Delta(G)$, if G has a connected even $[2, k]$-factor, then the smaller k is, the closer G is to being Hamiltonian.

In this paper, we will survey the more recent developments of the research on supereulerian graphs and the related problems. As shown in [7] and [19], supereulerian graphs and Eulerian subgraphs with certain properties have many applications to other areas, including Hamiltonian properties of line graphs and claw-free graphs. Limited by the length of this survey, this paper will focus on surveying the results mainly on supereulerian graphs, and will not include the results on the applications of supereulerian graphs.

2 Catlin's Reduction Method and Collapsible Graphs

Catlin [5] discovered the collapsible graphs and invented a very useful reduction method using collapsible graphs. Catlin's method is used in many of the researches on Eulerian subgraphs, and so we will first introduce Catlin's reduction method.

2.1 Catlin's Reduction Method

Let H be a connected subgraph of G. The *contraction* G/H is the graph obtained from G by contracting all edges of H and deleting any resulting loops, i.e., replacing H by a new vertex v_H such that the number of edges in G/H joining any $v \in G - V(H)$ to v_H in G/H equals the number of edges joining v in G to H. A graph G is *contractible* to a graph G' if G contains pairwise vertex-disjoint connected subgraphs $H_1, \cdots, H_c$ with $\bigcup_{i=1}^{c} V(H_i) = V(G)$ such that G' is obtained from G by successively contracting each H_i $(1 \leq i \leq c)$. Each subgraph $H \in \{H_1, \cdots, H_c\}$ is called the *preimage* of the vertex v_H of G'. A vertex v_H in G' is nontrivial if v_H is the contraction image of a nontrivial connected subgraph H of G.

A graph G is *collapsible* ([5]) if for every subset $R \subseteq V(G)$ with $|R|$ even, G has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $G - E(\Gamma_R)$ is connected. We use $\mathcal{CL}$ and $\mathcal{SL}$ to denote the families of collapsible graphs and supereulerian graphs, respectively.

As shown in the proposition below, collapsible graphs can be characterized in different form, which is often used in applications.

Proposition 1 *A graph G is collapsible if and only if for every subset $R \subseteq V(G)$ with $|R|$ even, G has a spanning connected subgraph L_R such that $O(L_R) = R$.*

Proof. To prove this proposition, we define the symmetric difference of sets X and Y as $X \oplus Y = (X \cup Y) - (X \cap Y)$. For any subset $R \subseteq V(G)$ with $|R|$ even, let $R' = R \oplus O(G)$.

If G is collapsible, then G has a subgraph $\Gamma_{R'}$ such that $O(\Gamma_{R'}) = R'$ and $G - E(\Gamma_{R'})$ is connected. Let $L_R = G - E(\Gamma_{R'})$. Then L_R is spanning and connected with $O(L_R) = R$.

Conversely, suppose that G has a spanning connected subgraph $L_{R'}$. Let $\Gamma_R = G - E(L_{R'})$. Then $G - E(\Gamma_R)$ is connected and $O(\Gamma_R) = R$. Thus G is collapsible. □

It follows immediately that the trivial graph K_1, and cycles of length at most 3 are both supereulerian and collapsible, but $C_4 \in \mathcal{SL} - \mathcal{CL}$. Note that being collapsible is stronger than being supereulerian. To show this, for a collapsible graph G with $u, v \in V(G)$, we let $R = \{u, v\}$ if $u \neq v$, and $R = \emptyset$ if $u = v$. Then $|R|$ is even. By the definition of collapsible graphs, G has a spanning subgraph H_R such that $O(H_R) = R$. Thus H_R is a spanning (u, v)-trail in G. When $u = v$, H_R is a spanning Eulerian subgraph of G, and so $\mathcal{CL} \subset \mathcal{SL}$.

In [5], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, \cdots, H_c$ such that $\bigcup_{i=1}^{c} V(H_i) = V(G)$. The *reduction* of G, denoted by G', is the graph obtained from G by contracting each maximal collapsible subgraph H_i, $(1 \leq i \leq c)$, into a single vertex v_i. A graph G is *reduced* if $G = G'$.

A subgraph H is a *dominating subgraph* of G if $E(G - V(H)) = \emptyset$. What makes the reduction method and the collapsible graphs so useful is the following theorem.

Theorem 2 *(Catlin, Theorems 3 and 8 of [5]) Let G be a connected graph and G' the reduction of G. Then each of the following holds.*
(a) $G \in \mathcal{CL}$ if and only if $G' = K_1$.
(b) $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.
(c) G has a dominating Eulerian subgraph if and only if G' has a dominating Eulerian subgraph containing all nontrivial vertices of G'.

To determine whether a graph G is supereulerian, Theorem 2 suggests that we firstly contract all nontrivial collapsible subgraphs of G to end up with a reduced graph G'. If G' has a smaller order than G, then it often makes the problem easier to solve.

Theorem 2 also suggests that collapsible graphs are the contractible configurations of the supereulerian problem, in the sense of Theorem 2(b). Are collapsible graphs the only contractible configurations for the supereulerian problem? Catlin investigated this problem, and he proved the following result.

Let H be a graph. For any pairing (a family of mutually disjoint 2-subsets)

$$A = \{\{v_1, v_1''\}, \{v_2, v_2''\}, \cdots, \{v_k, v_k''\}\}$$

of k disjoint pairs of vertices in $V(H)$, let $H(A)$ denote the supergraph of H obtained by adding to H the k paths $P_1, P_2, \cdots, P_k$ such that P_j is the path $v_j v_j' v_j'' (1 \leq j \leq k)$ and $v_j' \notin V(H) (1 \leq j \leq k)$.

Theorem 3 *(Catlin, Theorem 2 of [8]) Let G be a graph, and let H be a connected proper subgraph of G. Let $G_1, G_2, \cdots, G_p$ denote the components of $G - E(H)$ having at least one vertex not in $V(H)$. If $H(A) \in \mathcal{SL}$ for every pairing A in $V(H)$ satisfying*

$$\{v_i, v_i''\} \in A, v_i \in V(G_s), v_i'' \in V(G_t) \Longrightarrow s = t,$$

then $G \in \mathcal{SL} \Longleftrightarrow G/H \in \mathcal{SL}$.

The following theorem characterizes a property of even graphs.

Theorem 4 *(Jaeger, [37]) Let G be a graph and let $X \subseteq E(G)$. There is an even subgraph H of G with $X \subseteq E(H)$ if and only if X contains no edge-cut of G with odd cardinality.*

Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Hence a graph

G has two edge-disjoint spanning trees if and only if $F(G) = 0$. Jaeger [37] and Catlin [5] found that graphs with small values of $F(G)$ are mostly supereulerian graphs.

Suppose that G is a graph with 2 edge-disjoint spanning trees. Let X be the edge set of one of the two edge-disjoint spanning trees of G. Then X contains no edge-cut of G, and so Theorem 4 implies the following.

Theorem 5 *(Jaeger, [37]) If $F(G) = 0$, then G is supereulerian.*

As $\mathcal{CL} \subset \mathcal{SL}$, Catlin and others improved Jaeger's result by allowing $F(G) > 0$ but not greater than 2.

Theorem 6 *Let G be a connected graph.*
(i) (Catlin, Theorem 2 of [5]) If $F(G) = 0$, then $G \in \mathcal{CL}$.
(ii) (Catlin, Theorem 7 of [5]) If $F(G) \leq 1$, then $G \in \mathcal{CL}$ if and only if $\kappa'(G) = 1$.
(iii) (Catlin et al, Theorem 1.3 of [10]) If $F(G) \leq 2$, then $G \in \mathcal{CL}$ if and only if the reduction of G is not in $\{K_2\} \cup \{K_{2,s} : s \geq 1\}$.

An immediate corollary from Theorem 6 (iii) is the following.

Corollary 7 *(Catlin et al, Theorem 1.5 of [10]) Let G be a connected graph. If $F(G) \leq 2$, then exactly one of the following holds:*
(a) G is supereulerian,
(b) G has a cut edge,
(c) the reduction of G is $K_{2,s}$, for some odd integer $s \geq 3$.

It is natural to consider similar characterizations of graphs G with larger values of $F(G)$. As shown in Theorems 12 and 14 below, there will be many non-collapsible reduced graphs with $F(G) = 3$, and so an enumerative characterization would not be feasible. We close this section with the following conjecture.

Conjecture 8 *Let G be a 3-edge-connected graph. If $F(G) \leq 3$, then G is collapsible if and only if the reduction of G is not the Petersen graph.*

2.2 Collapsible Graphs with Small Orders

Catlin's reduction method using collapsible graphs is often used in inductive arguments. The induction basis would normally require the examination of the collapsibility of graphs with small orders. Many have been investigating the collapsibility of graphs with small orders. Theorems below describe some collapsible graphs with at most 9 vertices.

Let L_1, L_3, L_4, L_5, L_6 and L_7 be graphs depicted in Figure 1. Define L_2 to be any graph obtained from L_1 by contracting an edge incident with a vertex of degree 2 in L_1.

Theorem 9 *(Chen and Lai, Lemma 2.3 of [20]) The graphs $L_1, L_2, L_3, L_4, L_5, L_6$ and L_7 are all collapsible.*

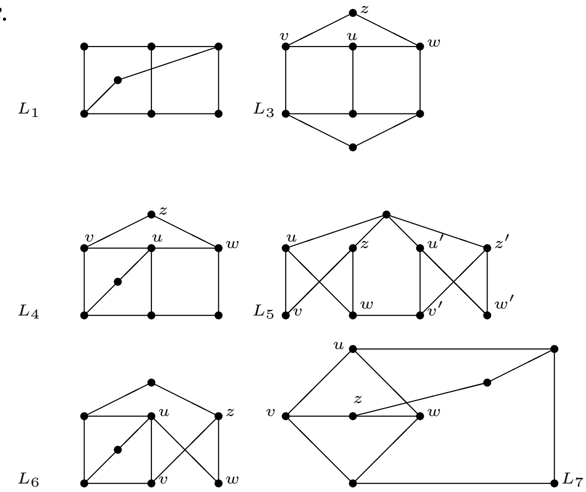

Figure 1. The graphs $L_1, L_3, L_4, L_5, L_6, L_7$

Theorem 10 *(Li et al, Lemma 2.1 of [53]) Let G be a connected simple graph with $n \leq 8$ vertices and with $D_1(G) = \emptyset$, $|D_2(G)| \leq 2$. Then either G is one of the three graphs in Figure 2, or the reduction of G is K_1 or K_2.*

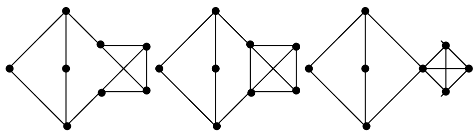

Figure 2. Graphs in Theorem 10

Next we define a class of graphs and summarize the results of reduced graphs with small orders.

Definition 11 *The Petersen graph is denoted by P. Let s_1, s_2, s_3, m, l, t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $K \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from K by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3$ (mod 3). Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for* *a $K'_{2,t}(u,u',u'')$ and a $K''_{2,t}(u,u',u'')$, respectively. Let $S(m,l)$ be the graph obtained from a $K_{2,m}(u,u')$ and a $K'_{2,l}(w,w',w'')$ by identifying u with w, and w'' with u'; let $J(m,l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w,w',w'')$ by identifying w,w'' with the two ends of an edge in $K_{2,m+1}$, respectively; let $J'(m,l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w,w',w'')$ by identifying w,w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. Define $\mathcal{F} = \{K_1,\ K_2,\ K_{2,t},\ K'_{2,t},\ K''_{2,t},\ K_{1,3}(s,s',s''),\ S(m,l),\ J(m,l),\ J'(m,l),\ P\}$, where t,s,s',s'',m,l are nonnegative integers.*

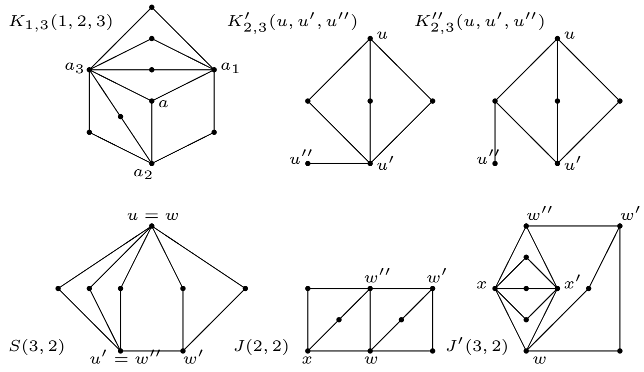

Figure 3. Some graphs in $\mathcal{F}$ with small parameters

The following theorems studied the reduced graphs with at most 11 vertices, which are useful while applying reduction techniques to study supereulerian graphs.

Theorem 12 *(Chen and Lai, Theorem 2.4 of [20]) If G is a connected reduced graph with $|V(G)| \le 11$ and $F(G) \le 3$, then $G \in \mathcal{F}$.*

Theorem 13 *(Chen, [13]) Let G be a reduced graph of order at most 11 with $\kappa'(G) \ge 3$, then $G \in \{K_1, P\}$.*

Theorem 14 *(Li et al, Lemma 1 of [58]) Let G be a 2-edge-connected reduced graph with $|V(G)| \le 11$ and $F(G) \le 3$, then*
(i) If $|D_2(G)| = 4$, $|D_3(G)| = 2$, $|D_i(G)| = 0$, $i \ge 5$, then $G \in \mathcal{S}$ or $G \in \{S(1,2),\ J(2,2)\}$.
(ii) If $|D_2(G)| = 5$, $|D_3(G)| = 1$, $|D_5(G)| = 1$, $|D_i(G)| = 0$, $i \ge 6$, then $G \in \mathcal{S}$ or $G = S(3,2)$.
(iii) If $|D_2(G)| = 6$, $|D_5(G)| = 2$, $|D_3(G)| = 0$, $|D_i(G)| = 0$, $i \ge 6$, then $G \in \mathcal{S}$ or $G = S(4,1)$.

Define $\mathcal{F}' = \{S(1,2),\ S(2,3),\ S(1,4),\ J(2,2),\ K_{2,3},\ K_{2,5}\}$, where the graphs in $\mathcal{F}'$ are defined as in Definition 11.

Theorem 15 *(Lai and Liang, Theorem 3.1 of [46]) If G is a 2-edge-connected reduced graph which satisfies $|D_2(G)|+|D_3(G)| \le 6$ and $|D_3(G)|+|D_5(G)| \le 2$, then G is supereulerian if and only if $G \notin \mathcal{F}'$.*

Catlin's reduction method is often used with an inductive argument, which is deployed by taking the reduction of the graph under consideration. Therefore, the study of reduced graphs with small orders is of particular importance in such applications. Currently, the investigations of reduced graphs with smaller orders are focused on how they are used. There has not been a systematic study on the structures and on the generic and characteristic properties of reduced graphs with smaller orders.

3 Degree Conditions

The decision problems for Hamiltonian and supereulerian graphs are both NP-complete (see [30], [71]). Degree conditions have been useful tools to seek sufficient conditions for Hamiltonian graphs, so it is also natural to use degree conditions to study supereulerian graphs. In this section, we survey the degree conditions for supereulerian graphs that have appeared since 1992.

Chen and Lai studied the best possible lower bounds for degree sums of two end-vertices of each edge in a simple graph G that guarantees the existence of a spanning Eulerian subgraph, and characterized the structures of the extremal graphs.

Theorem 16 *(Chen and Lai, Theorem 7 of [18]) If G is a 3-edge-connected graph and if $d(u) + d(v) \ge \frac{n}{5} - 2$ for every edge $uv \in E(G)$, then either $G \in \mathcal{SL}$ or equality holds and G is contractible to the Petersen graph.*

Theorem 17 *(Chen, [16], [18]). Let G be a k-edge-connected simple graph of order n, where $k \in \{2,3\}$. Let G' be the reduction of G and $l = |D_2(G)|$. If for any $uv \in E(G)$, $d(u) + d(v) \ge \frac{2n}{(k-1)5} - 2$, then exactly one of the following holds:*
(a) $G \in \mathcal{CL}$;
(b) $k = 2$ and $l < n/5 - 19$, and either
(b1) $G' = K_{2,c-2}$, where $c \le \max\{5, 3+l\}$, or
(b2) $n = 5s$ ($s > 19$), $G' = C_5$, and the preimage of each vertex of $C_5 \in \{K_s, K_s - e\}$.
(c) $k = 3$, $n = 10s$ ($s > 24$), $G' = P$, and the preimage of each vertex of $P \in \{K_s, K_s - e\}$.

Theorem 18 *(Chen and Lai, Theorem 1.3 of [20]) Let G be a 3-edge-connected simple graph with order n. If $n > 306$ and if for every edge $uv \in E(G)$, $d(u) + d(v) \ge n/6 - 2$, then G is supereulerian if and only if G cannot be contracted to the Petersen graph.*

A similar result was recently conducted by Li and Yang, with a weaker assumption on edge connectivity.

Theorem 19 *(Li and Yang, Theorem 2 of [56]) Let G be a connected graph of order $n \geq 4$ and $d(x) + d(y) \geq n$ for any $xy \in E(G)$. Then exactly one of the following holds:*
(i) G is collapsible.
(ii) The reduction of G is $K_{1,t-1}$ for $t \geq 3$ such that all of the vertices of degree 1 are trivial and they have the same neighbor in G, $t \leq n/2$. Moreover, if $t = 2$, then $G - v$ is collapsible for a vertex v in K_2.
(iii) G is $K_{2,n-2}$.

What is the best possible lower bound of degree sums for which all the exceptional cases have the Petersen graph structure? Towards this end, Chen and Lai conjectured the following.

Conjecture 20 *(Chen and Lai, [20]) Let G be a 3-edge-connected simple graph with order n. If n is sufficiently large and if for every edge $uv \in E(G)$, $d(u)+d(v) > n/9-2$, then either G is supereulerian or G can be contracted to the Petersen graph.*

It is shown in [20] that an infinite family of graphs have been constructed to show that the lower bound of this conjecture, if valid, will be best possible.

Let $\sigma_k(G)$ denote the minimum degree sum of k independent vertices in G. Chen studied the sufficient conditions on $\sigma_2(G)$ with a control on the girth of the graph G.

Theorem 21 *(Chen, Theorem 4 of [14]) Let G be a 2-edge-connected simple graph of order n with girth g. Let G' be the reduction of G. If for some integer $p \geq 2$, $n \geq 4(g-2)p^2$ and if $d(u) + d(v) \geq \frac{2}{g-2}\left(\frac{n}{p} - 4 + g\right)$, whenever $uv \notin E(G)$, then exactly one of the following holds:*
(a) $G \in \mathcal{CL}$;
(b) G_1 is nontrivial with order $c \leq p$.
(b1) If $c = p = 4$, then $p = 4$ and G_1 is the 4-cycle.
(b2) If $c = p \geq 5$, then for some integer $s > 0$, $n = (g-2)ps$ and $\delta(G) = \frac{1}{g-2}\left(\frac{n}{p} - 4 + g\right)$ such that either
(i) $g = 3$, the preimage of each vertex v_i in G_1 is at most $\frac{d_{G_1}(v_i)}{2}$ edges short of being a K_s.
(ii) $g = 4$, the preimage of each vertex v_i in G_1 is at most $\frac{d_{G_1}(v_i)}{2}$ edges short of being a $K_{s,s}$.

A set of k independent vertices is referred as a k-independent set. As two nonadjacent vertices form a 2-independent set, it is natural to generalize the degree sum condition on one 2-independent set to a collection of 2-independent sets. This was done by Chen.

Theorem 22 *(Chen, Theorem 4 of [17]) Let G be a 2-edge-connected simple graph of order n with girth at least g, where $g \in \{3, 4\}$. If n is sufficiently large and if for any m vertex-disjoint 2-independent sets $\{S_2^1, S_2^2, \cdots, S_2^m\}$, the degree sum $\sum_{i=1}^{m}(S_2^i) \geq \frac{2m}{g-2}(\frac{n}{5} - 4 + g)$, where $m \in \{1, 2, 3\}$, then exactly one of the following holds:*
(a) G is collapsible (hence supereulerian)
(b) the reduction of G is in $\{C_4, C_5, K_{2,3}\}$.

Chen also studied the degree sum conditions on 3-independent sets, with a control on the girth and obtained the following results.

Theorem 23 *(Chen, Theorem 5 of [17]) Let G be a 2-edge-connected simple graph of order n with girth g, where $g \in \{3, 4\}$. Let G' be the reduction of G. If for every 3-independent set $\{u, v, w\}$ of $V(G)$, $d(u) + d(v) + d(w) \geq \frac{3}{g-2}(\frac{n}{5} - 4 + g)$, then exactly one of the following holds:*
(a) G is collapsible (hence supereulerian)
(b) $G' \in \{C_4, C_5\}$, and so G is supereulerian but not collapsible
(c) $G' \in \{K_{2,3}, K'_{2,3}\}$, and so G is non-supereulerian.

Theorem 24 *(Chen, Theorem 7 of [17]) Let G be a 2-edge-connected simple graph of order n with girth $g \in \{3, 4\}$. Let G' be the reduction of G. If for any 3-independent set $\{u, v, w\}$, $d(u) + d(v) + d(w) \geq \frac{n}{2(g-2)} + 2(g-2)$, then exactly one of the following holds:*
(a) G is supereulerian
(b) $G' \in \{K_{2,3}, K'_{2,3}\}$.

Theorem 25 *(Chen, Theorem 8 of [17]) Let G be a 3-edge-connected simple graph of order n with girth $g \in \{3, 4\}$. If n is sufficiently large and if for any 3-independent set $\{u, v, w\}$, $d(u) + d(v) + d(w) > \frac{3}{g-2}(\frac{n}{14} - 4 + g)$, then exactly one of the following holds:*
(a) G is supereulerian
(b) the reduction of G is the Petersen graph.

As mentioned in the introduction, having a connected $[2, 2]$-factor implies Hamiltonian property and having a 2-edge-connected even $[2, \lceil\Delta(G)\rceil]$-factor implies supereulerian property. The following result is a generalization of Ore's theorem ([70]) and other theorems.

Theorem 26 *(Kouider and Mahéo, [38]) Let $b \geq 2$ be an integer. If G is a 2-edge-connected graph with $\sigma_2(G) \geq 4n/(2+b)$, then G has a 2-edge-connected even $[2, 2\lceil b/2\rceil]$-factor (hence G is supereulerian).*

A *2-trail* is a trail that uses every vertex at most twice. So if G has a closed spanning 2-trail, then G is supereulerian. Ellingham et al proved the following.

Theorem 27 *(Ellingham et al, Theorems 3.1 and 4.1 of [26]) Both of the following statements are true.*
(1) If $\sigma_3(G) \geq n$, then G has either a Hamilton path or a closed spanning 2-trail.
(2) If G is 2-edge-connected and $\sigma_3(G) \geq n$, then G has a closed spanning 2-trail, unless $G \cong K_{2,3}$ or $K^*_{2,3}$ (the 6-vertex graph obtained from $K_{2,3}$ by subdividing one edge).

Degree conditions are very customary in seeking the existence of spanning Eulerian subgraphs. The results of Chen indicate that other graph invariants such as the girth of a graph, can also be involved in the investigation. It remains to be studied whether the limitation of the girth g can be relaxed in some of the theorems above.

4 Edge Connectivity and Supereulerian Graphs

Any supereulerian graph must be 2-edge-connected. Unlike Hamiltonian problems, high edge-connectivity will also guarantee the existence of a spanning Eulerian subgraph. By the well-known theorem of Nash-Williams ([66]) and Tutte ([75]) on spanning tree packing, graphs with edge-connectivity at least $2k$ must have at least k edge-disjoint spanning trees. This has been applied by Jaeger [37] and Catlin [5] who independently proved that every 4-edge-connected graph is supereulerian. Since the Petersen graph is 3-edge-connected and non-supereulerian, it becomes a natural question to study which 3-edge-connected graphs are supereulerian. Limiting the number of edge cuts in a 3-edge-connected graph, the following results of Catlin and Lai show the progress in this direction.

Theorem 28 *(Catlin et al, Theorem 1.6 of [10]) If $\kappa'(G) \geq 3$ and G has at most 9 edge cuts of size 3, then G is collapsible (hence supereulerian).*

Theorem 29 *(Catlin and Lai, Theorem 3.12 of [11]) If $\kappa'(G) \geq 3$ and G has at most 10 edge cuts of size 3, then G is either supereulerian or contractible to the Petersen graph.*

Theorem 30 *(Catlin and Lai, Theorem 3.14 of [11]) If $\kappa'(G) \geq 3$ and G has at most 11 edge cuts of size 3, then exactly one of these holds:*
(a) G is supereulerian;
(b) The reduction of G is the Petersen graph;
(c) The reduction of G is non-supereulerian graph of order between 17 and 19, with girth at least 5, with exactly 11 vertices of degree 3, and with the remaining vertices independent and of degree 4.

Theorem 30 raised an open problem whether graphs stated in Theorem 30 (iii) exist or not. In a recent paper [61], it is shown that no such graphs exist.

Theorem 31 *(Li ea al, [61]) Let G be a 3-edge-connected graph. If G has at most 11 edge-cuts of size 3, then the following are equivalent:*
(i) G is supereulerian;
(ii) The reduction of G is not the Petersen graph.

How many 3-edge-cuts can we have in this direction? We need the concept of snarks. A snark is a 2-edge-connected 3-regular graph with edge chromatic number equal to 4. It is known [24, 35]) that the Petersen graph is the smallest snark and no snarks of order between 11 and 17. There are two snarks of order 18, called the Blanuša snarks (see [24, 35]). By the definition of snarks, any cyclically 4-edge-connected snark is essentially 4-edge-connected and non-supereulerian. Motivated by the structure of Blanuša snarks, Catlin made the following conjecture.

Conjecture 32 *(Catlin, Conjecture 3 of [6]) If $\kappa'(G) \geq 3$ and G has at most 17 edge cuts of size 3, then G is either supereulerian or contractible to the Petersen graph.*

As subdividing edges in a graph G would decrease the density of the graph, what would be the smallest edge-connectivity to guarantee that a subdivided graph remains to be collapsible? Luo et al considered this problem. They obtained the following generalization of Theorem 5.

Theorem 33 *(Luo et al, Theorem 3.1 of [63]) Let $r \geq 4$ be an integer and let $k = \lfloor r/2 \rfloor$. Let G be an r-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leq r + k - 2$ and let G_X be the graph obtained from G by subdividing every edge in X. Then exactly one of the following holds:*
(i) G_X is collapsible;
(ii) X is an edge cut of G and $|X| \geq r$.

An edge cut X in a connected graph G is *essential* if at least two components of $G - X$ are nontrivial.

A graph is *essentially k-edge-connected* if it does not have an essential edge cut with fewer than k edges. Clearly, a k-edge-connected graph is also essentially k-edge-connected, but not vice versa. In fact, a large essential edge connectivity does not guarantee a large edge connectivity. To see that, we simply attach one pendent edge to a large complete graph to get a graph with large essential edge connectivity but with a 1-edge-cut. Hence, relaxing the edge connectivity condition to the essential edge connectivity is another research direction.

The proof arguments used by Zhan in [81] actually proved the following.

Theorem 34 *(Zhan, [81]) Every 3-edge-connected, essentially 7-edge-connected graph is collapsible, hence it is also supereulerian.*

Therefore, one direction to study 3-edge-connected supereulerian graphs has been to determine the smallest essential edge-connectivity k such that every 3-edge-connected, essentially k-edge-connected graph is supereulerian.

Conjecture 35 *(Chen and Lai, Conjectures 8.6 and 8.7 of [19], see also [65])*
(i) Every 3-edge-connected, essentially 6-edge-connected graph is collapsible.
(ii) Every 3-edge-connected, essentially 5-edge-connected graph is supereulerian.

From the definition of essential edge connectivity, we know that if a graph G is essentially k-edge-connected, then for any edge $e = uv \in E(G)$ with $|E(G-e)| \geq 1$, $d(u) + d(v) - 2 \geq k$, but not vice versa. By adding a degree condition, the following is a partial result towards Conjecture 35.

Theorem 36 *(Yang et al, [80]) For $e = uv \in E(G)$, define $d(e) = d(u) + d(v) - 2$ the edge-degree of e. The edge-degree of G is $\min\{d(e) : e \in E(G)\}$. Both of the following hold.*
(1) (Theorem 3.7 of [80]) Every 3-edge-connected, essentially 6-edge-connected graph with edge-degree at least 7 is collapsible (hence supereulerian).
(2) (Theorem 3.2 of [80]) Every 3-edge-connected, essentially 5-edge-connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible (hence supereulerian).

5 Extremal and Structural Conditions

There have been efforts trying to investigate the Eulerian subgraph problems using extremal and structural approaches. This section will be surveying such efforts. Chen considered the minimum size of a simple graph to warrant the property of being supereulerian, and characterized all extremal graphs.

Theorem 37 *(Chen, [15]) Let n, m and p be natural numbers, $m, p \geq 2$. Let G be a 2-edge-connected simple graph on $n > p+6$ vertices containing no K_{m+1}. If $|E(G)| \geq \binom{n-p+1-k}{2} + (m-1)\binom{k+1}{2} + 2p - 4$ (*), where $k = \left\lfloor \frac{n-p+1}{m} \right\rfloor$, then either G is supereulerian, or G can be contracted to a non-supereulerian graph of order less than p, or equality holds in (*) and G can be contracted to $K_{2,p-2}$ (p is odd) by contracting a complete m-partite graph $T_{m,n-p+1}$ of order $n-p+1$ in G.*

A Ramsey type of result on Eulerian subgraph problems was obtained by Lai.

Theorem 38 *(Lai, Theorem 1 of [40]) Let G be a simple graph with at least 61 vertices, and let G^c denote the complement of G. One of the following holds.*
(a) G is supereulerian.
(b) G^c is supereulerian.
(c) Both G and G^c have a vertex of degree 1.
(d) One of G or G^c is contractible to a $K_{2,t}$ for some odd integer $t \geq 3$, and the other has either one or two vertices of degree 1.
(e) One of G and G^c is contractible to $K_{1,p}$ for some integer $p \geq 1$, and the other has exactly one isolated vertex.

Graphs with the property that every 2-edge-induced subgraph is supereulerian are also studied, with a forbidden induced minor characterization. To describe this characterization, we introduce the subdivided wheels. A *wheel* W_n is the graph obtained from the n-cycle $C_n = v_1v_2\cdots v_nv_1$, where $n \geq 2$, by adding an extra vertex v and new edges $\{vv_i : 1 \leq i \leq n\}$. Define the *subdivided wheel* W_n^* to be the graph obtained from W_n by replacing each edge v_iv_{i+1}, $(1 \leq i \leq n, (\text{mod } n))$ by a path of length 2, $v_iv_i'v_{i+1}$ (say), where $\{v_1', \cdots, v_n'\} \cap V(W_n) = \emptyset$. Note that $W_2^* \cong K_{2,3}$.

A graph H is a *minor* of G if H is isomorphic to the contraction image of a subgraph of G. We call H an *induced minor* of G if H is isomorphic to the contraction image of an induced subgraph of G.

Theorem 39 *Let G be a 2-edge-connected graph.*
(a) (Lai, Theorem 1.1 of [41]) Every 2-edge-connected subgraph of G is supereulerian if and only if G does not have an induced minor isomorphic to a subdivided wheel.
(b) (Lai, Corollary 1.2 of [41]) If G has no induced

minor isomorphic to a member in $\{W_n^* : n \geq 2\}$, *then G is supereulerian.*

For an integer $l \geq 0$, define $\mathcal{SE}(l)$ to be the family of graphs such that $G \in \mathcal{SE}(l)$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq l$, G has a spanning Eulerian subgraph H with $X \subseteq E(H)$. The graphs in $\mathcal{SE}(0)$ are known as supereulerian graphs. Let $f(l)$ be the minimum value of k such that every k-edge-connected graph is in $\mathcal{SE}(l)$. Theorem 5 implies that $f(0) = 4$. In this sense, the following result improves Theorem 5.

Theorem 40 *(Lai, Theorem 3.3 of [42]) Let* l *be an integer. Then* $f(l) = \begin{cases} 4, & 0 \leq l \leq 2 \\ l+1, & l \geq 3 \text{ and } l \text{ is odd} \\ l, & l \geq 4 \text{ and } l \text{ is even} \end{cases}$

The study in this direction for 3-edge-connected graphs is focused on Eulerian subgraphs that contain given edges. Chen et al proved the following theorem. This result later plays an important role in proving a conjecture of Kuipers and Veldman [39, 29] on 3-edge-connected Hamiltonian claw-free graphs by Lai, Shao and Zhan [44], where claw-free means there is no induced subgraph isomorphic to $K_{1,3}$.

Theorem 41 *(Chen et al, Theorem 1.1 of [21]) Let G be a 3-edge-connected graph and let* $S \subseteq V(G)$ *be a vertex subset such that* $|S| \leq 12$. *Then either G has an Eulerian subgraph H such that* $S \subseteq V(H)$, *or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.*

An open problem with an extremal nature has been proposed. For a graph G, let $SE(G)$ denote the set of all spanning Eulerian subgraphs of G. Note that it is possible that $SE(G) = \emptyset$.

Problem 42 *(Chen and Lai, Problem 8.8 of [19]) Determine*

$$\min_{G \in \mathcal{SL}} \max \left\{ \frac{|E(H)|}{|E(G)|} : H \in SE(G) \right\}.$$

A progress has been made towards this problem by Li et al.

Theorem 43 *(Li et al, Theorem and Corollary of [54]) If* $G \in \mathcal{SL}$ *and* $G \neq K_1$, *then there are infinite families of graphs G such that* $L = \frac{|E(H)|}{|E(G)|} \leq \frac{2}{3}$, *where H is a spanning Eulerian subgraph of G. Moreover, if G is an r-regular graph, and if* $r \neq 5$, *then* $L \geq \frac{2}{3}$; *if* $r = 5$, *then* $L > \frac{3}{5}$.

We conclude this section with a conjecture.

Conjecture 44 *(Li et al, Conjecture of [54]) Let G be a simple graph. If* $G \in \mathcal{SL}$, *and* $G \neq K_1$, *then G has a spanning Eulerian subgraph H with* $|E(H)| \geq \frac{3}{5}|E(G)|$.

6 Planar Supereulerian Graphs

Planar graphs are one of the families of graphs that have been intensively studied. There are also quite a few researches in supereulerian planar graphs. As the Petersen graph is not planar, the next theorem provides evidence to support Conjecture 8.

Theorem 45 *(Lai et al, Theorem 1.3 of [43]) Every 3-edge-connected planar graph with* $F(G) \leq 3$ *is collapsible.*

Let s_1, s_2, s_3 be positive integers. Denote $K_4(s_1, s_2, s_3)$, $T(s_1, s_2)$, $C_3(s_1, s_2)$, $S(s_1, s_2)$ and $K_{2,3}(s_1, s_2)$ to be the graphs depicted in Figure 4, where the $s_i\,(i = 1, 2, 3)$ vertices and the two vertices connected by the two lines shown in each of the graphs form a K_{2,s_i} graph. Define $\mathcal{F}_1 = \{\, K_4(s_1, s_2, s_3),\ T(s_1, s_2),\ C_3(s_1, s_2),\ S(s_1, s_2),\ K_{2,3}(s_1, s_2)\}$, where s_1, s_2 and s_3 are positive integers. We further define $\mathcal{F}_2 = \mathcal{F}_1 \cup \{\, K_{2,t} \mid t \geq 2 \,\}$. It is routine to verify that each graph G in $\mathcal{F}_2$ is reduced and $F(G) \leq 3$. The following result is a planar version of Conjecture 8. It also partially explains why we need 3-edge-connectedness when studying reduced graphs G with $F(G) = 3$.

Theorem 46 *(Lai et al, Theorem 1.3 of [43]) Let G be a 2-edge-connected planar graph. If* $F(G) \leq 3$, *then either G is collapsible or the reduction of G is a graph in* $\mathcal{F}_2$.

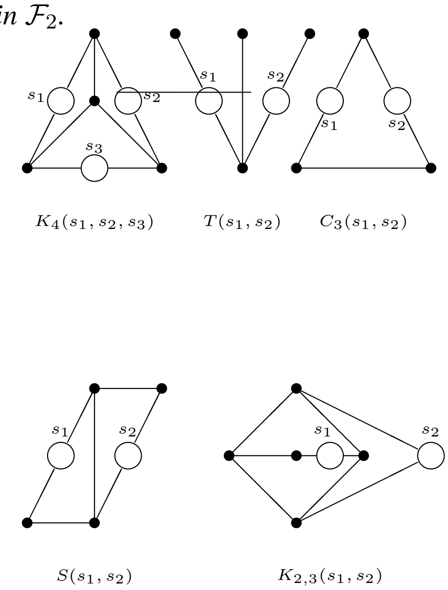

Figure 4

Compared with Theorem 41, one can do much better within planar graphs, as shown in the following theorem.

Theorem 47 *(Chen et al, Theorem 1.2 of [21]) Let G be a 3-edge-connected planar graph, and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leq 23$. Then there is an Eulerian subgraph in G containing S.*

Since 3-cycles are collapsible, we often assume that graphs under considerations have girth at least 4 when investigating collapsibility. Compared with Theorem 37, the extremal size bound for planar graphs can be reduced from quadratic to linear, as shown in the following results.

Theorem 48 *(Lai et al, Theorem 1.9 of [43]) Let G be a simple planar graph of order $n \geq 4$ with $\kappa'(G) \geq 3$ and girth $g(G) \geq 4$. If $|E(G)| \geq 2n - 5$, then G is collapsible, and so it is supereulerian.*

Theorem 49 *(Lai et al, Theorem 1.10 of [43]) Let G be a planar graph with $\kappa'(G) \geq 3$ such that every edge of G is in a face of degree at most 6. If either G has at most two faces of degree 5 and no faces of degree bigger than 5, or G has exactly one face of degree 6 and no other faces of degree bigger than 4, then G is collapsible (hence supereulerian).*

Theorem 50 *(Lai et al, Theorem 1.11 of [43]) If G is a 2-edge-connected simple planar graph with order $n \geq 6$ and $|E(G)| \geq 3n - 8$, then $F(G) = 0$, and so G is collapsible.*

Theorem 51 *(Lai et al, Theorem 1.12 of [43]) If G is a 2-edge-connected simple planar graph with $n \geq 9$ vertices and with $|E(G)| \geq 3n - 12$ edges, then exactly one of the following holds.*
(i) G is supereulerian.
(ii) G has a maximal collapsible subgraph H with order $n - 4$ such that G/H is a $K_{2,3}$.

7 Characterizations within Graph Family $C_h(l, k)$

Since determining whether a graph is supereulerian is NP-complete, researchers have been looking for families of graphs in which they can completely characterize supereulerian graphs in terms of some structural descriptions. Catlin and Li are the pioneers in this maneuver.

For integers h, k and l with $3 \geq h \geq 2$, $k \geq 0$ and $l > 0$, let $C_h(l, k)$ denote the family of h-edge-connected graphs G such that for every edge cut X with size 2 or 3, each component of $G - X$ has at least $(|V(G)| - k)/l$ vertices. A sequence of papers have been published, pushing the characterizations deeper and deeper.

Theorem 52 *(Catlin and Li, Theorem 6 of [12]) If $G \in C_2(5, 0)$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$ or $K_{2,5}$.*

Theorem 53 *(Broersma and Xiong, Theorem 4 of [4]) If $G \in C_2(5, 2)$ and $n \geq 13$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$.*

Theorem 54 *(Li et al, Theorem 1.3 and Corollary 1.1 of [53]) If $G \in C_2(6, 0)$, then G is supereulerian if and only if G cannot be contracted to $K_{2,3}$, $K_{2,5}$ or $S(1, 2)$.*

Theorem 55 *(Li et al, Theorem 14 of [58]) Let $G \in C_2(6, 5)$ be a graph with $n > 35$. Then G is supereulerian if and only if G cannot be contracted to a member in $\{S(1, 2), S(2, 3), S(1, 4), J(2, 2), K_{2,3}, K_{2,5}\}$.*

Theorem 56 *(Lai and Liang, Theorem 1.6 of [46]) Let $k > 0$ be an integer. Then there exists an integer $N(k) \leq 7k$ such that, for any graph $G \in C_2(6, k)$ with $|V(G)| > N(k)$, G is supereulerian if and only if G cannot be contracted to a member in $\{S(1, 2), S(2, 3), S(1, 4), J(2, 2), K_{2,3}, K_{2,5}\}$.*

Similar studies are also conducted for 3-edge-connected graphs.

Theorem 57 *(Li et al, Theorem 2.4 of [62]) If $G \in C_2(7, 0)$, then G is not supereulerian if and only if G can be contracted to one of the nine specified graphs.*

Theorem 58 *(Lai et al, Theorem 1.6 of [43]) Let G be a simple planar graph of order n. If $G \in C_3(16, 0)$, then G is supereulerian.*

Theorem 59 *(Li and Li, Theorem 2 of [60]) Let G be a simple graph. If $G \in C_3(10, 3)$, then G is supereulerian if and only if G cannot be contracted to the Petersen graph.*

Theorem 60 *(Niu and Xiong, Theorem 11 of [69]) Let G be a simple graph. If $G \in C_3(10, k)$ with $\kappa'(G) \geq 3$, then when $|V(G)| > 11k$, G is supereulerian if and only if G cannot be contracted to the Petersen graph.*

Theorem 61 *(Li et al, Theorem 7 of [59]) Let G be a simple graph. If $G \in C_3(12, 1)$ with $\kappa'(G) \geq 3$, then G is supereulerian if and only if G cannot be contracted to the Petersen graph.*

Further researches in this direction within 2-edge-connected graphs will be facing the difficulty of characterizing a divergent list of exceptional graphs as the parameters l and k become bigger. New structural characterizations would be expected for such parameters. For 3-edge-connected graphs, we present a conjecture as the concluding remark for this section.

Conjecture 62 *Let G be a simple graph. If $G \in C_3(17, 0)$ with $\kappa'(G) \geq 3$, then G is supereulerian if and only if G cannot be contracted to the Petersen graph.*

A Blanuša snark (see [24, 35]) has 18 edge-cuts of size 3 and it is not supereulerian and not contractible to the Petersen graph. An infinite family of graphs can be constructed from either of the two Blanuša snarks to show the sharpness of Conjecture 62.

8 Supereulerian Index

Let G be a graph. The *line graph* of G, denoted $L(G)$, has vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. For a connected graph G, the *n-th iterated line graph* $L^n(G)$ is defined recursively by $L^0(G) = G$, $L^n(G) = L(L^{n-1}(G))$.

Let H be a graph. A graph G is H-free if G does not have an induced subgraph isomorphic to H. A $K_{1,3}$-free graph is also called a claw-free graph. Beineke [1] and Robertson [72] (see also [34]) indicated that every line graph must be claw-free.

The supereulerian index $s(G)$ of a graph G is the smallest integer k such that the k-th iterated line graph of G is supereulerian. In 1997, Ryjáček [73] introduced the concept of the closure $cl(G)$ of graph G.

Let G be a claw-free graph. The neighborhood $N_G(v)$ of v is the set of vertices that are adjacent to v. We use $G[N_G(v)]$ to denote the induced subgraph of G by $N_G(v)$. Then for any $v \in V(G)$, $G[N_G(v)]$ is either connected (in this case, v is referred as a *locally connected vertex of G*) or is a disjoint union of two cliques. If $G[N_G(v)]$ is connected and not a clique, then the *local completion* of G at v is a graph obtained from G by adding edges to join nonadjacent vertices in $N_G(v)$. The closure of G, denoted by $cl(G)$, is the graph obtained from G by repeating applications of this local completion, until every locally connected vertex has its neighborhood being a clique. This construction was introduced by Ryjáček [73], and he proved the following very useful result.

Theorem 63 *(Ryjáček, [73]) Let G be a claw-free graph. Then*
(i) $cl(G)$ is uniquely determined.
(ii) $cl(G)$ is the line graph of some triangle-free simple graph.
(iii) G is Hamiltonian if and only if $cl(G)$ is Hamiltonian.

Xiong and Li proved that the supereulerian index of a graph G remains unchanged under taking the reduction of G, and also under taking the closure of G when G is claw-free.

Theorem 64 *(Xiong and Li, Corollary 3.4 of [78]) Let G be a graph and H be a collapsible subgraph of G. Then $s(G) = s(G/H)$.*

Theorem 65 *(Xiong and Li, Theorem 4.2 of [78]) Let G be a connected claw-free graph with at least three edges other than a path. Then $s(G) = s(cl(G))$.*

Clark and Wormald [25] defined indices of graphs for many Hamiltonian properties. For these properties, it is shown [45] that if a graph G has such properties, then so does $L(G)$. Compared with other indices of graphs, the study of supereulerian index is just at the beginning.

9 Other Conditions

A *biclaw* is defined as the graph obtained from two vertex-disjoint claws by adding an edge between the two vertices of degree 3 in each of the claws. In [55], Li proposed the following conjecture.

Conjecture 66 *(Li, Conjecture 2b.32 of [28], see also [55]) There exists a constant c such that every connected bipartite biclaw-free graph G with $\delta(G) \geq c$ is Hamiltonian.*

Motivated by this conjecture, it is shown that higher minimum degree of connected biclaw-free bipartite graphs are supereulerian.

Theorem 67 *(Lai and Yao, Theorem 2.1 of [50]) Every connected bipartite biclaw-free graph G with $\delta(G) \geq 5$ is supereulerian.*

The well-known Thomassen conjecture [74] and Matthews and Sumner [64] conjecture state that every 4-connected claw-free graph is Hamiltonian. As a Hamilton cycle is a connected [2,2]-factor, the following is a weakened version of this conjecture.

Theorem 68 *(Chen et al, Theorem 1 of [22]) Every connected, essential 4-edge-connected claw-free simple graph with $\delta(G) \geq 3$ is collapsible and has a connected even $[2, 4]$-factor.*

The result was later improved by Li et al.

Theorem 69 *(Li et al, Theorem 4 of [57]) Every supereulerian $K_{1,k}$-free ($k \geq 2$) graph contains a connected even $[2, 2k-2]$-factor.*

The circumference of a graph G is the length of a longest cycle of G. Lai et al used the circumference as a control to study supereulerian graphs, and obtained the following result.

Theorem 70 *(Lai et al, Theorem 4 of [48]) Every 3-edge-connected graph G with circumference $c(G) \leq 8$ is supereulerian.*

Han et al used the Chvátal-Erdös condition for Hamiltonian graphs to study supereulerian graphs and obtained the following.

Theorem 71 *(Han et al, Theorem 3 of [33]) Let G be a 2-connected simple graph and let P denote the Petersen graph. Let $\alpha(G)$ be the maximum number of independent vertices in the graph G. If $\kappa(G) \geq \alpha(G) - 1$, then exactly one of the following holds.*
(a) G is supereulerian.
(b) $G \in \{P, K_{2,3}, K_{2,3}(1), K_{2,3}(2), K'_{2,3}\}$.
(c) G is one of the two 2-connected graphs obtained from $K_{2,3}$ and $K_{2,3}(1)$ by replacing a vertex whose neighbors have degree three in $K_{2,3}$ and $K_{2,3}(1)$ with a complete graph of order at least three.

A supereulerian graph with small matching number was recently characterized by the next theorem.

Theorem 72 *(Lai and Yan, Theorem 2 of [49]) Let $\alpha'(G)$ be the maximum number of independent edges in the graph G. If G is a 2-edge-connected simple graph and $\alpha'(G) \leq 2$, then G is supereulerian if and only if G is not $K_{2,t}$ for some odd t.*

Characterizations of some potential degree sequences for supereulerian graphs have also been studied. A sequence $d = (d_1, d_2, \cdots, d_n)$ is *graphic* if there exists a simple graph G having d as its degree sequence, and is *multi-graphic* if there exists a multigraph G having d as its degree sequence. In either case, the graph G is a *d-realization*. The next theorem characterizes potential degree sequences of supereulerian graphs.

Theorem 73 *(Fan et al, Theorem 1.2 of [27]) Let $d = (d_1, d_2, \cdots, d_n)$ be a non-increasing graphic sequence. Then d has a supereulerian realization if and only if either $n = 1$ and d_1=0, or $n \geq 3$ and $d_n \geq 2$.*

By Theorem 5, a graph G having two edge-disjoint spanning trees is collapsible, and whence it is supereulerian. A characterization of potential degree sequence to have k edge-disjoint spanning trees is given below.

Theorem 74 *(Lai et al, Theorem 1.1 of [47]) A non-increasing graphic sequence $d = (d_1, d_2, \cdots, d_n)$ has a realization G with k edge-disjoint spanning trees if and only if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and both of the following hold.*
(i) $d_n \geq k$.
(ii) $\sum_{i=1}^{n} d_i \geq 2k(n-1)$.

The multigraph version of these theorems was proved by Gu et al in [32].

10 Generalizations

As a supereulerian graph is one that has a spanning even subgraph with one component, it is natural to consider graphs with a spanning even subgraph with more than one components. If a graph G has a spanning even subgraph with at most k components such that each of its components is nontrivial, then the line graph of G will have a 2-factor with at most k components. This motivates a direction of generalization of supereulerian graphs and their applications to line graphs.

Jackson and Yoshimoto showed that the number of components of a line graph $L(G)$ can be upper bounded by a linear function of the order of G.

Theorem 75 *(Jackson and Yoshimoto, Theorem 5 of [36]) Let G be a simple graph of order n with $\delta(G) \geq 3$. Then $L(G)$ has a 2-factor with at most $\max\{1, \lfloor \frac{3n-4}{10} \rfloor\}$ components.*

Xiong et al used spanning even subgraphs to improve the result of Jackson and Yoshimoto.

Theorem 76 *(Xiong et al, Theorem 7 of [79]) Let G be a simple graph of order n with $\delta(G) \geq 3$. Then $L(G)$ has a 2-factor with at most $\max\{1, \lfloor \frac{2n-2}{7} \rfloor\}$ components.*

Niu and Xiong also used spanning even subgraphs to prove the following result on bounded number of components of an even factor in line graphs.

Theorem 77 *(Niu and Xiong, [67]) Let G be a connected simple graph of order n and k a positive integer such that $\delta(G) \geq \lfloor \frac{n}{k} \rfloor - 1$. If n is sufficiently large relative to k, then G has an even factor with at most k components.*

Xiong [77] defined a closure operation $cl^{se}(G)$ on claw-free graph G and proved the following theorem.

Theorem 78 *(Xiong, Theorem 4 of [77]) Let G be a claw-free graph. Then G has an even factor with at most k components if and only if its closure $cl^{se}(G)$ has an even factor with at most k components.*

A few more generalizations of supereulerian graphs will be introduced below. A graph is *k-supereulerian* if it has a spanning even subgraph with at most k components. Note that a graph G is 1-supereulerian if and only if G is supereulerian. Niu et al. [68] proved the following results.

Theorem 79 *(Niu et al, Theorem 2 of [68]) Let G be a connected graph and G' be the reduction of G. Then G is k-supereulerian if and only if G' is k-supereulerian.*

Theorem 80 *(Niu et al, Theorem 5 of [68]) Let k be a positive integer and G be a connected graph. If $F(G) \leq k$, then exactly one of the following holds:*
(a) G is k-supereulerian;
(b) G can be contracted to a tree of order $k+1$.

Theorem 81 *(Niu et al, Theorem 8 of [68]) Let G be a 2-edge-connected graph on n vertices. If $\delta(G) \geq n/(3+k) - 1$ and $n > 4(3+k)$, then G is k-supereulerian.*

As a strengthening of the supereulerian property, Chen considered graphs with a spanning Eulerian subgraph that contains some given edges and avoids some other given edges.

Theorem 82 *(Chen et al, Theorem 4.2 of [23]) Let $r \geq 4$. For a graph G, let $X \subseteq E(G)$ and $Y \subseteq E(G)$ such that $X \cap Y = \emptyset$, $|Y| \leq \lfloor (r+1)/2 \rfloor$, $|X \cup Y| = |X|+|Y| \leq r$, and $r-|Y|$ is even and $\kappa'(G-Y) \geq 3$. Then G has a spanning Eulerian subgraph H such that $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$ for any such X and Y if and only if G is r-edge-connected.*

A further generalization of supereulerian graphs can be defined along this line. Given two nonnegative integers s and t, a graph G is *(s,t)-supereulerian* if for any disjoint sets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, there is a spanning Eulerian subgraph H of G that contains X and avoids Y. Note that a graph G is $(0,0)$-supereulerian if and only if G is supereulerian. Lei et al first proved that graphs have such properties when edge-connectivity is high enough.

Theorem 83 *(Lei et al, [51]) Let s and t be non-negative integers with $s \leq 2$. Suppose that G is a $(t+2)$-edge-connected locally connected graph on n vertices. For any disjoint sets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, exactly one of the following holds:*
(i) G has a spanning Eulerian subgraph H such that $X \subset E(G)$ and $Y \bigcap E(H) = \emptyset$.
(ii) The reduction of $(G-Y)_X$ is a member of $\{K_1, K_2, K_{2,t}(t \geq 1)\}$.

In another paper, Lei et al investigated (s,t)-supereulerian graphs with high local connectivity. They obtained the following results.

Theorem 84 *(Lei et al, Theorem 1.3 of [52]) Let $k \geq 1$ be an integer. If G is a connected, locally k-edge-connected graph, then G is (s,t)-supereulerian for all pairs of nonnegative integers s and t with $s+t \leq k-1$.*

Theorem 85 *(Lei et al, Theorem 1.4 of [52]) For $k \geq 1$, let s and t be nonnegative integers such that $s+t \leq k$. Let G be a connected, locally k-edge-connected graph, then for any disjoint sets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, there is a spanning eulerian subgraph H that contains X and avoids Y if and only if $G-Y$ is not contractible to K_2 or $K_{2,l}$ with l odd.*

Corollary 86 *(Lei et al, Corollary 1.5 of [52]) Let G be a connected, locally k-edge-connected graph. Let s and t be nonnegative integers such that $s+t \leq k$. Then*
(i) If $t < k$ and $k \geq 3$, then G is (s,t)-supereulerian.
(ii) If $\kappa'(G) \geq k+2$ and $k \geq 3$, then G is (s,t)-supereulerian.

The studies on the generalizations of supereulerian graphs and their applications are just at the beginning. They are motivated by the study of supereulerian graphs as well as their applications in line graphs and other applications. Further research tools for studying such generalizations are expected to be developed.

11 Concluding Remarks

The supereulerian graph problems, and the more general Eulerian subgraph problems which seek the existence of Eulerian subgraphs with given properties in given graphs, have also been studied in digraphs and matroids. This survey has not reviewed efforts in these directions. Moreover, this survey does not review the applications of supereulerian graphs to the study of Hamiltonicity of claw-free graphs and to other areas. Interested readers can find related surveys by Faudree,

E. Flandrin, and Z. Ryjáček in [28] and by Gould in [31], in addition to the former surveys of Catlin [7] and its supplement [19].

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