On further ordering bicyclic graphs with respect to the Laplacian spectra radius

Guangqing Jin
College of Mathematical Science,
Tianjin Normal University,
Tianjin, 300387
China

Liancui Zuo
College of Mathematical Science,
Tianjin Normal University,
Tianjin, 300387
China
Corresponding author: lczuo@163.com

Abstract: Among all the n-vertex bicyclic graphs, the first eight largest Laplacian spectra radii had been obtained in the past, all of which are no smaller than $n-1$. In this paper, it is first obtained that all the bicyclic graphs on n vertices with Laplacian spectra radii at least $n-2$ must contain two adjacent vertices which cover all the vertices except possibly two and one of the two adjacent vertices must have the degree at least $n-3$. Then the total forty-two such graphs are further ordered, and the ninth to the forty-first largest Laplacian spectra radii among all the n-vertex bicyclic graphs are finally determined in this way.

Key–Words: Bicyclic graph; the Laplacian spectra radius; the Laplacian characteristic polynomial

1 Introduction

In this paper, all the graphs we discussed are simple graphs. Let G be a graph with vertex set $V_G = \{v_1, v_2, \dots, v_n\}$ and edge set E_G. Denote by $N_G(v_i)$ or simply $N(v_i)$ the set of all neighbours of a vertex v_i of G and by d_i the degree of v_i. Let $D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix $L(G)$ of G is defined by $L(G) = D(G) - A(G)$, where $A(G)$ is the $(0,1)$-adjacent matrix of G. The Laplacian characteristic polynomial $\det(xI - L(G))$ is denoted by $\Phi(G,x)$. It is well known that $L(G)$ is positive semi-definite, symmetric and singular. We denote the ith largest eigenvalue of $L(G)$ simply by $\mu_i(G)$ and order them in non-increasing order, i.e., $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$, and $\mu_1(G)$ is called the Laplacian spectra radius of G, denoted by $\mu(G)$ in this paper. For $v \in V_G$, let d_v denote the degree of v in G.

Dragoš Cvetković once introduced in his book [1] that the motivation for founding the theory of graph spectra has come from applications in Chemistry and Physics. One of the main applications of graph spectra to Chemistry is the application in a theory of unsaturated conjugated hydrocarbons known as the Hückel molecular orbital theory. In chemistry, there is a closed relation between the molecular orbital energy levels of π-electrons in conjugated hydrocarbons and the eigenvalues of the corresponding molecular graph. In [2], Ivan Gutman further exploited and discovered the chemical applications of the Laplacian spectrum of molecular graphs when studying the Wiener number.

Additionally, the Laplacian spectral radius of a graph has numerous applications in theoretical chemistry, combinatorial optimization, communication networks, etc. For related reference, one may see [3]. In [4], we can see that the Laplacian spectral radius is also related to the algebraic connectivity, i.e., to the second smallest eigenvalue (here $\overline{G}$ denote the complement of G) from the equation $\mu_{n-1}(\overline{G}) = n - \mu_1(G)$. More importantly, Cvetković in [5] pointed out twelve directions in graph spectra theory which can offer us further research. It's also mentioned that one important direction is to classify and order graphs with respect to graph spectra. So far, much research had been made on ordering certain class of graphs with respect to graph spectra, especially with respect to the spectra radius and the Laplacian spectra radius, and many relevant results in this direction have been obtained. In [6-8], the ordering of unicyclic graphs with respect to the spectra radius has been extended to the eleventh. In [9-11], the ordering of bicyclic graphs with respect to the spectra radius has been extended to the tenth. While with respect to the Laplacian spectra radius, the ordering of unicyclic graphs has been extended to the thirteenth in [12-14] and the ordering of bicyclic graphs to the eighth in [15] and [16].

Recall that a bicyclic graph is a connected graph in which the number of edges equals to the number of vertices plus one. With respect to the Laplacian spectra radius, He et al. [15] determined the ordering of the n-vertex bicyclic graphs from the first one to the fourth together with the corresponding graphs (see the graphs G_1, $G_{1'}$, G_2, G_3, G_4, $G_{4'}$ of Fig.1). It turns out that the Laplacian spectra radii of all these graphs are in the interval $(n-1, n]$. In [16], Li et al. first identified the remaining n-vertex bicyclic graphs with Laplacian spectra radii in $[n-1, n]$ by characterizing such graphs and then obtained the further ordering of the fifth to the eighth largest Laplacian spectra radii together with the corresponding graphs (see the graphs G_5, G_6, G_7, G_8, $G_{8'}$ of Fig. 1). In detail, Li et al. obtained the partial ordering of some graphs easily by applying algebraic method. For the remaining graphs, which is difficult to order by algebraic method, they applied some Mathematica method to completely solve this problem.

As one of the directions in graph spectra pointed out by Cvetković in [5], ordering the bicyclic graphs with respect to their Laplacian spectra is of interest and significance. In this paper, designed to extend the above ordering of n-vertex bicyclic graphs, we first characterize the n-vertex bicyclic graphs with Laplacian spectra radii in $[n-2, n]$ based on [16] and then identify the remaining bicyclic graphs with $\mu(G) \in [n-2, n]$ (forty-two graphs in total as you can see in Fig. 2). Finally, we focus on ordering them by algebraic method and consequently obtain the ninth to the forty-first largest Laplacian spectra radii. Even for the readers who has little knowledge of the Mathematica, it is easy to understand. As we can see in this paper, the bicyclic graphs with Laplacian spectra radii in $[n-2, n]$ is forty-two more than the bicyclic graphs with Laplacian spectra radii in $[n-1, n]$ though the interval which we considered is just expanded by value one than that considered by Li et al. Therefore, much more difficult work is needed on the further ordering. That is what we intend to work on next.

We first introduce the following lemmas that are useful for our results.

Lemma 1 *([17]) Let G be a connected graph on $n \geq 2$ vertices, and maximum degree $\Delta(G)$. Then $\mu(G) \geq \Delta(G)+1$, with equality attained if and only if $\Delta(G) = n-1$.*

Lemma 2 *([16]) Let G be a bicyclic graph on n vertices, and let v_i and v_j be any two vertices of G. If v_i and v_j are adjacent then $d_i + d_j \leq n+2$; otherwise $d_i + d_j \leq n+1$.*

Lemma 3 *([18]) Let G be a graph on $n > 1$ vertices. Then*

$$\mu(G) \leq \max\{d_i + m_i\},$$

where $m_i = \frac{\sum_{v_i v_j \in E_G} d_j}{d_i}$.

Lemma 4 *([19]) Let G be a graph on $n \geq 2$ vertices. Then*

$$\mu(G) \leq \max\{d_i + d_j - |N(v_i) \cap N(v_j)| \mid v_i v_j \in E_G\}.$$

Lemma 5 *([20]) Let G be a graph on n vertices. Then $\mu(G) \leq n$, with the equality holds if and only if $\overline{G}$ is disconnected, where $\overline{G}$ is the complement of G.*

Lemma 6 *([15]) Let G be a connected rooted graph on n vertices and root r, which consists of a subgraph H (with at least two vertices) and $n-|H|$ pendent edges (neither in H) attached at vertex v of H (note that $|H|$ denotes the order of H). Then*

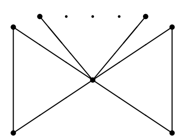
$$\Phi(G, x) = (x-1)^{n-|H|}\Phi(H, x)$$
$$-(n-|H|)x(x-1)^{n-|H|-1}\Phi(L_v(H), x),$$

where $L_v(H)$ denotes the principle submatrix of $L(H)$ obtained from $L(H)$ by deleting the row and the column corresponding to the vertex v.

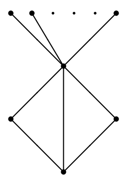
2 The main results

2.1 Bicyclic graphs with $\mu(G)$ at least $n-2$.

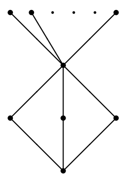
In this section, we study the structure of n-vertex bicyclic graphs whose Laplacian spectra radii are in the interval $[n-2, n]$. As the only eleven graphs (shown in Fig. 1) whose Laplacian spectra radii are in the interval $[n-1, n]$ have already been obtained, here we have determined the remaining forty-two graphs with $\mu(G) \in [n-2, n-1)$ that are shown in Fig. 2.



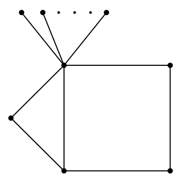
G_1



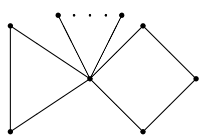
$G_{1'}$



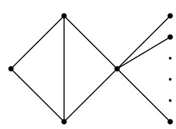
G_2



G_3



G_4



$G_{4'}$

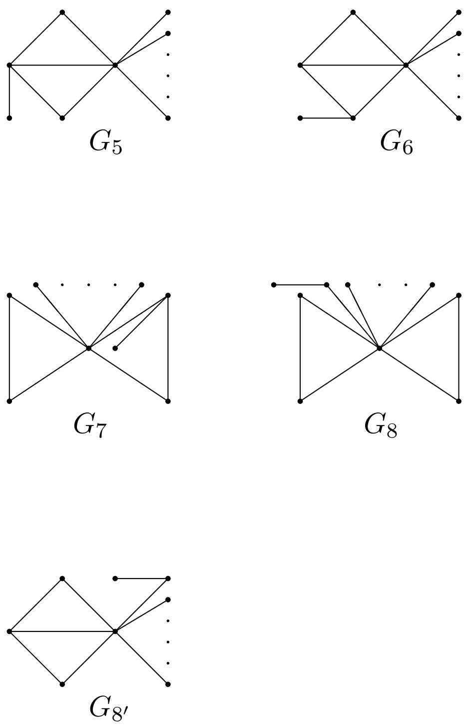


Fig. 1. n-vertex bicyclic graphs with $\mu \in [n-1, n]$

Lemma 7 *Let G be a graph on $n \geq 2$ vertices with $\mu(G) \in [n-2, n]$. Then G contains two adjacent vertices that cover all its vertices, except possibly two.*

Proof: If $d_u + d_v - |N(u) \cap N(v)| < n - 2$ for any $uv \in E_G$, then by Lemma 4, we have $\mu(G) < n - 2$, a contradiction. Therefore, there must exist two adjacent vertices u and v such that

$$d_u + d_v - |N(u) \cap N(v)| \geq n - 2.$$

That is, $|N(u) \cup N(v)| \geq n - 2$. Then it follows that there exists at most two vertices that are not covered by u and v. This completes the proof. □

Lemma 8 *Let G be an n-vertex bicyclic graph with $\mu(G) \in [n-2, n]$ and let a and b denote the two adjacent vertices in G as described in the above lemma7. For each $v \in V_G$, if $v \notin \{a, b\}$, then we have $d_v \leq 5$.*

Proof: We give a proof by contradiction. Assume that there exists a vertex v such that $v \notin \{a, b\}$ but $d_v \geq 6$. Let p and q denote the two vertices possibly uncovered by a and b.

Case 1. If $v \in N(a) \cup N(b)$, for $d_v \geq 6$, then v is adjacent to at least four neighbors of a or b, which contradicts the bicyclic structure of G.
Case 2. If v is exactly one of p and q, say $v = p$. It is easy to see that v can be only adjacent to the neighbors of a or b (not counting a and b) and v has at least five such neighbors (for $d_v \geq 6$), which contradicts the bicyclic structure of G, too.

For the sake of simplicity, we still define a, b, p and q as above. Moreover, we assume that $\Delta(G) < n - 1$, for otherwise, by Lemma 1 we have that trivial case $\mu(G) = n$. □

Lemma 9 *Let G be a n-vertex bicyclic graph with $\mu(G) \in [n-2, n]$. If $n \geq 14$ then $d_a \geq n - 3$ or $d_b \geq n - 3$.*

Proof: By Lemma 3, there exists at least one vertex v in G such that $d_v + m_v \geq n - 2$.

Case 1. $d_v = 1$.

Let u be the unique neighbor of v, then $d_u \geq n - 2 - d_v = n - 3$. If $u = a$ or $u = b$, then we are done. Otherwise, $d_u \leq 5$. Then we have $d_u + d_v \leq 6$ which contradicts $d_u + d_v \geq n - 2$ for $n \geq 14$.

Let $N(v) = \{u_1, u_2, \cdots, u_d\}$, where $d = d_v$. Then $d_v + m_v = d + \frac{1}{d}\sum_{i=1}^{d} d_{u_i}$.

Case 2. $d_v = 2$.

We have $N(v) = \{u_1, u_2\}$. By Lemma 2, we have $d_{u_1} + d_{u_2} \leq n + 2$. So $d_v + m_v \leq 2 + \frac{n+2}{2}$, then $d_v + m_v < n - 2$ for $n > 10$, thus $d_v \neq 2$.

Case 3. $d_v = 3$.

We have $N(v) = \{u_1, u_2, u_3\}$ and there exists at least one vertex which is different from a and b, say u_3. Clearly, $d_{u_3} \leq 5$ by Lemma 2.2. Similarly, we have $d_{u_1} + d_{u_2} \leq n + 2$. So $d_v + m_v \leq 3 + \frac{n+2+5}{3}$, then $d_v + m_v < n - 2$ for $n > 11$, hence $d_v \neq 3$.

Case 4. $d_v = 4$.

We have $N(v) = \{u_1, u_2, u_3, u_4\}$ and there exist at least two vertices both of which are different from a and b, say u_3 and u_4. Clearly, $d_{u_3} \leq 5$ and $d_{u_4} \leq 5$. Then

$$d_v + m_v \leq 4 + \frac{n+2+5+5}{4},$$

so $d_v + m_v < n - 2$ for $n > 12$, therefore $d_v \neq 4$.

Case 5. $d_v = 5$.

Similarly, we have

$$d_v + m_v \leq 5 + \frac{n+2+5+5+5}{5}.$$

But then $d_v + m_v < n - 2$ for $n > 13$, and therefore $d_v \neq 5$.

Case 6. $d_v > 5$.

By Lemma 8 we have $v = a$ or b. Without loss of generality, we assume that $v = a$. Then here $d = d_a$. It is not difficult to see that among all the neighbors of a and b there are at most two pairs of such neighbors which can be joined by edges. Therefore, at most two edges are counted twice. Since G has $n + 1$ edges, we have $\sum_{i=1}^{d} d_{u_i} \leq (n + 1) + 2$. Now we have $d_a + m_a \leq d + \frac{n+3}{d}$. If this inequality is less than $n - 2$, then we are done. Otherwise, we have that $d + \frac{n+3}{d} \geq n - 2$ holds. Solving a corresponding quadratic equation in d we obtain that

$$d \geq \frac{n - 2 + \sqrt{(n-2)^2 - 4(n+3)}}{2}.$$

For $n > 11$, we have

$$\sqrt{(n-2)^2 - 4(n+3)} > n - 6,$$

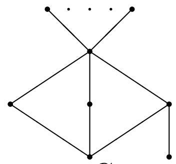
then we obtain $d > n - 4$. That is $d = d_a \geq n - 3$. Note that the other region for d can be rejected as then $d < 2$ which contradicts the assumption $d > 5$. □

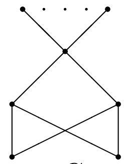
Theorem 10 *If G is an n-vertex bicyclic graph with $n \geq 14$ whose Laplacian spectra radius is in the interval $[n-2, n-1)$, then G is one of the forty-two graphs from Fig. 2.*

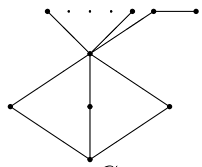
Proof: By Lemma 1, we know that $\Delta(G) \leq n - 3$ for $\mu(G) < n - 1$, but, by Lemma 9, we have $\Delta(G) \geq n - 3$. Therefore $\Delta(G) = n - 3$. It is exactly the case when the two uncovered vertices p and q arise. Clearly, $|N(p) \cap N(a)| \leq 3$ and $|N(q) \cap N(a)| \leq 3$, for otherwise, it contradicts the bicyclic structure. Note that the positions of p and q in the graph is symmetric, so we don't differentiate p from q.

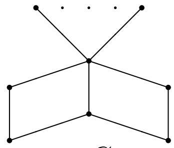
Observing vertices p and q we have that G_9 and G_{10} are the all graphs with $|N(p) \cap N(a)| = 3$ and $|N(q) \cap N(a)| = 1$.

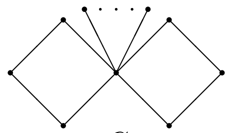
Similarly, G_{13} is the unique graph with $|N(p) \cap N(a)| = 3$ and $|N(q) \cap N(a)| = 0$, $G_{9'}$, G_{11} and G_{12} are the all graphs with $|N(p) \cap N(a)| = 2$ and $|N(q) \cap N(a)| = 2$, G_{14}, G_{15}, G_{16}, $G_{16'}$, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22} and $G_{22'}$ are the all graphs with $|N(p) \cap N(a)| = 2$ and $|N(q) \cap N(a)| = 1$, G_{26}, $G_{26'}$ and G_{27} are the graphs with $|N(p) \cap N(a)| = 2$ and $|N(q) \cap N(a)| = 0$, G_{23}, G_{24}, G_{25}, $G_{25'}$, G_{28}, $G_{28'}$, G_{29}, G_{30}, G_{31}, G_{32}, G_{33}, G_{34}, G_{35}, G_{36}, $G_{36'}$, $G_{36''}$ and G_{41} are the graphs with $|N(p) \cap N(a)| = 1$ and $|N(q) \cap N(a)| = 1$, and G_{37}, G_{38}, G_{39}, G_{40} and $G_{40'}$ are the graphs with $|N(p) \cap N(a)| = 1$ and $|N(q) \cap N(a)| = 0$.

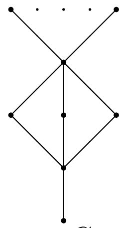

G_9

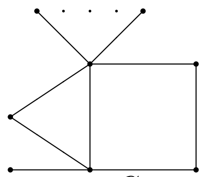

$G_{9'}$

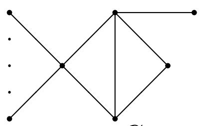

G_{10}

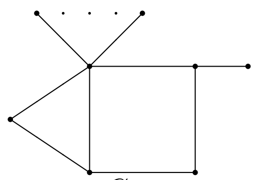

G_{11}

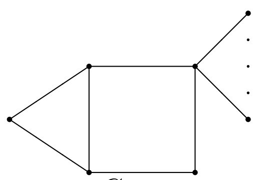

G_{12}

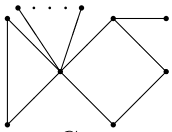

G_{13}


G_{14}

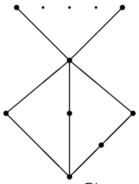

G_{15}


G_{16}

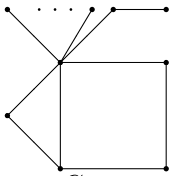

$G_{16'}$



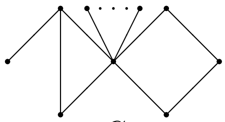
G_{17}



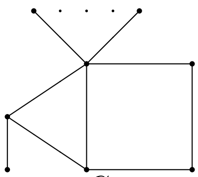
G_{18}



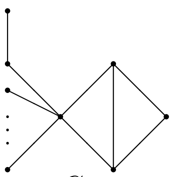
G_{19}



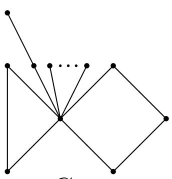
G_{20}



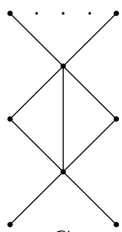
G_{21}



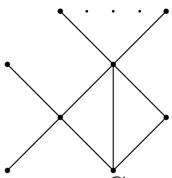
G_{22}



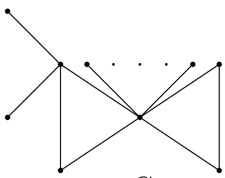
$G_{22'}$



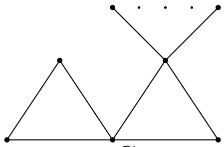
G_{23}



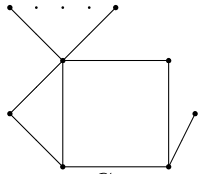
G_{24}



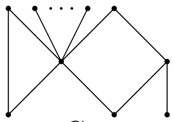
G_{25}



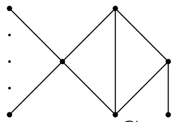
$G_{25'}$



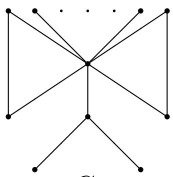
G_{26}



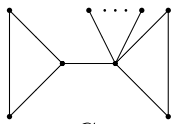
G_{27}



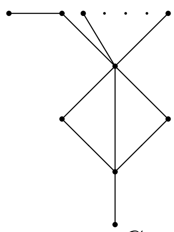
$G_{27'}$



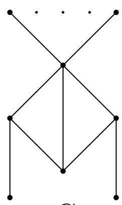
G_{28}



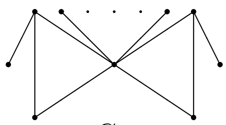
$G_{28'}$



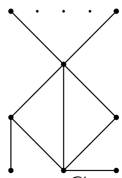
G_{29}



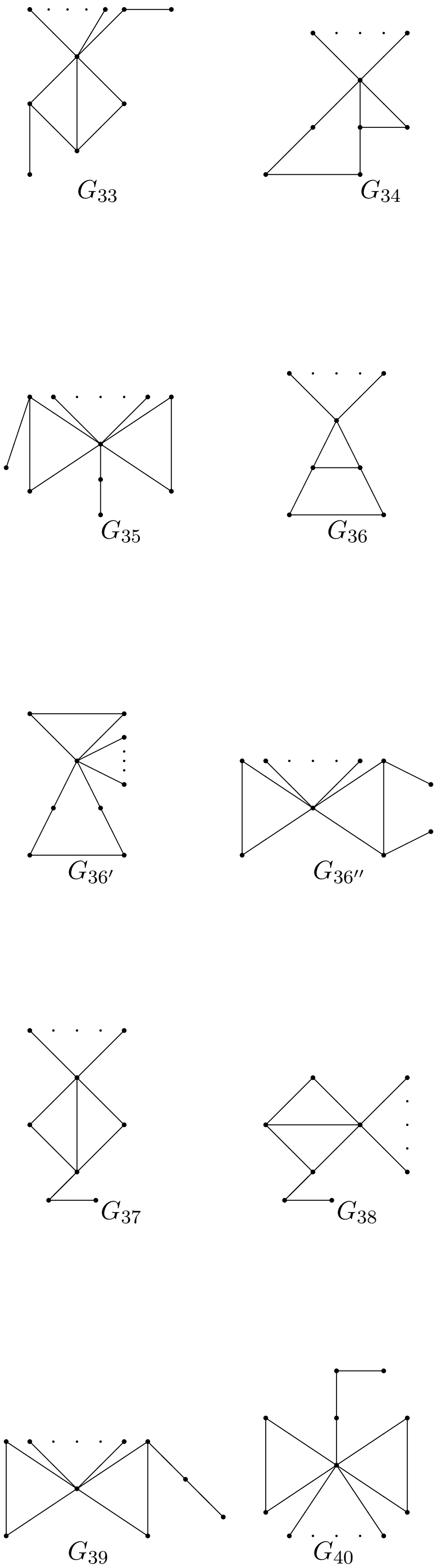
G_{30}



G_{31}



G_{32}



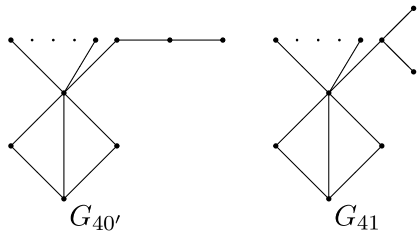


Fig. 2. Bicyclic graphs with $\mu \in [n-2, n-1)$

2.2 Ordering the graphs in Fig. 2

In this section, we will give the ordering of the graphs in Fig. 2 with respect to their Laplacian spectra radii. Let

$$
\begin{aligned}
g_9(x) &= x^3-(n+3)x^2+(5n-8)x-2n;\\
g_{9'}(x) &= x^3-(n+3)x^2+(5n-8)x-2n;\\
g_{10}(x) &= x^5-(n+6)x^4+(8n+3)x^3\\
&\quad -(19n-20)x^2+(14n-11)x-3n;\\
g_{11}(x) &= x^4-(n+5)x^3+(7n-1)x^2\\
&\quad -(13n-17)x+5n;\\
g_{12}(x) &= x^3-(n+2)x^2+(4n-6)x-2n;\\
g_{13}(x) &= x^4-(n+5)x^3+(7n-4)x^2\\
&\quad -(10n-14)x+3n;\\
g_{14}(x) &= x^5-(n+8)x^4+(10n+12)x^3\\
&\quad -(32n-24)x^2+(37n-46)x-11n;\\
g_{15}(x) &= x^5-(n+8)x^4+(10n+11)x^3\\
&\quad -(31n-28)x^2+(31n-37)x-8n;\\
g_{16}(x) &= x^5-(n+8)x^4+(10n+13)x^3\\
&\quad -(33n-24)x^2-(39n-50)x-11n;\\
g_{16'}(x) &= x^5-(n+8)x^4+(10n+13)x^3\\
&\quad -(33n-24)x^2-(39n-50)x-11n;\\
g_{17}(x) &= x^4-(n+5)x^3\\
&\quad +(7n-1)x^2-(13n-19)x+4n;\\
g_{18}(x) &= x^5-(n+7)x^4+(9n+8)x^3\\
&\quad -(26n-22)x^2+(27n-30)x-8n;\\
g_{19}(x) &= x^7-(n+10)x^6+(12n+31)x^5\\
&\quad -(55n+14)x^4+(121n-86)x^3\\
&\quad -(131n-128)x^2+(64n-43)x-11n;\\
g_{20}(x) &= x^6-(n+8)x^5+(10n+15)x^4\\
&\quad -(35n-16)x^3+(51n-53)x^2\\
&\quad -(30n-22)x+6n;\\
g_{21}(x) &= x^6-(n+9)x^5+(11n+21)x^4\\
&\quad -(43n-11)x^3+(72n-73)x^2
\end{aligned}
$$

$$\begin{aligned}
& -(49n-43)x+11n; \\
g_{22}(x) &= x^5-(n+5)x^4+(7n+1)x^3 \\
& -(15n-17)x^2+(10n-8)x-2n; \\
g_{22'}(x) &= x^5-(n+5)x^4+(7n+1)x^3 \\
& -(15n-17)x^2+(10n-8)x-2n; \\
g_{23}(x) &= x^4-(n+6)x^3 \\
& +(8n-3)x^2-(13n-20)x+4n; \\
g_{24}(x) &= x^5-(n+8)x^4+(10n+11)x^3 \\
& -(31n-28)x^2+(32n-44)x-8n; \\
g_{25}(x) &= x^4-(n+5)x^3 \\
& +(7n-3)x^2-(11n-17)x+3n; \\
g_{25'}(x) &= x^4-(n+5)x^3 \\
& +(7n-3)x^2-(11n-17)x+3n; \\
g_{26}(x) &= x^6-(n+9)x^5+(11n+21)x^4 \\
& -(43n-11)x^3+(73n-80)x^2 \\
& -(52n-62)x+11n; \\
g_{27}(x) &= x^4-(n+4)x^3 \\
& +(6n-4)x^2-(8n-12)x+2n; \\
g_{27'}(x) &= x^4-(n+4)x^3 \\
& +(6n-4)x^2-(8n-12)x+2n; \\
g_{28}(x) &= x^3-(n+2)x^2+(4n-7)x-n; \\
g_{28'}(x) &= x^3-(n+2)x^2+(4n-7)x-n; \\
g_{29}(x) &= x^6-(n+8)x^5+(10n+13)x^4 \\
& -(33n-20)x^3+(44n-51)x^2 \\
& -(23n-18)x+4n; \\
g_{30}(x) &= x^4-(n+5)x^3 \\
& +(7n-2)x^2-(12n-18)x+4n; \\
g_{31}(x) &= x^4-(n+4)x^3 \\
& +(6n-3)x^2-(9n-13)x+3n; \\
g_{32}(x) &= x^6-(n+9)x^5+(11n+19)x^4 \\
& -(41n-17)x^3+(63n-71)x^2 \\
& -(39n-37)x+8n; \\
g_{33}(x) &= x^7-(n+10)x^6+(12n+30)x^5 \\
& -(54n+8)x^4+(114n-95)x^3 \\
& -(115n-126)x^2+(51n-37)x-8n; \\
g_{34}(x) &= x^6-(n+9)x^5+(11n+22)x^4 \\
& -(44n-8)x^3+(78n-81)x^2 \\
& -(59n-65)x+14n; \\
g_{35}(x) &= x^6-(n+7)x^5+(9n+10)x^4 \\
& -(28n-18)x^3+(36n-42)x^2 \\
& -(18n-14)x+3n; \\
g_{36}(x) &= x^3-(n+1)x^2+(3n-5)x-n; \\
g_{36'}(x) &= x^3-(n+1)x^2+(3n-5)x-n; \\
g_{36''}(x) &= x^3-(n+1)x^2+(3n-5)x-n; \\
g_{37}(x) &= x^5-(n+7)x^4+(9n+6)x^3 \\
& -(24n-26)x^2+(21n-32)x-4n; \\
g_{38}(x) &= x^5-(n+7)x^4+(9n+7)x^3 \\
& -(25n-27)x^2+(22n-34)x-4n; \\
g_{39}(x) &= x^5-(n+6)x^4+(8n+4)x^3 \\
& -(20n-22)x^2+(17n-26)x-3n; \\
g_{40}(x) &= x^4-(n+3)x^3+(5n-4)x^2 \\
& -(6n-10)x+n; \\
g_{40'}(x) &= x^4-(n+3)x^3+(5n-4)x^2 \\
& -(6n-10)x+n;
\end{aligned}$$

and

$$g_{41}(x)=x^3-(n+2)x^2+(4n-6)x-n-3.$$

In the following, we will give the complete ordering of the forty-two graphs and determine the ninth to the forty-first largest values of the Laplacian spectra radii among all the n-vertex bicyclic graphs.

Theorem 11 *Let $G_i(i = 9, 9', 10, \cdots, 41)$ be the graphs as shown in Fig. 2. When $n \geq 14$, we have:*

$$\begin{aligned}
&\mu(G_9)=\mu(G_{9'})>\mu(G_{10})>\mu(G_{11})>\mu(G_{12}) \\
&>\mu(G_{13})>\mu(G_{14})>\mu(G_{15})>\mu(G_{16}) \\
&=\mu(G_{16'})>\mu(G_{17})>\mu(G_{18})>\mu(G_{19})>\mu(G_{20}) \\
&>\mu(G_{21})>\mu(G_{22})=\mu(G_{22'})>\mu(G_{23})>\mu(G_{24}) \\
&>\mu(G_{25})=\mu(G_{25'})>\mu(G_{26})>\mu(G_{27})=\mu(G_{27'}) \\
&>\mu(G_{28})=\mu(G_{28'})>\mu(G_{29})>\mu(G_{30})>\mu(G_{31}) \\
&>\mu(G_{32})>\mu(G_{33})>\mu(G_{34})>\mu(G_{35})>\mu(G_{36}) \\
&=\mu(G_{36'})=\mu(G_{36''})>\mu(G_{37})>\mu(G_{38})>\mu(G_{39}) \\
&>\mu(G_{40})=\mu(G_{40'})>\mu(G_{41}).
\end{aligned}$$

Proof: Applying Lemma 6, we can obtain all the Laplacian characteristic polynomials by routine calculations. Also we can see that $\mu(G_i)$ is the largest root of $g_i(x)$ for $i = 9, 9', 10, \cdots, 40', 41$.

In order to prove our result, we will apply three kinds of methods to compare every two Laplacian spectra radii step by step. All the three methods are related to the monotonicity of the function corresponding to the Laplacian characteristic polynomial. We will make a full introduction in the following text. Note that $\Delta(G_i) = n-3$ for all i, then by Lemma 1.1 we have that $\mu(G_i) \in (n-2, n-1)$ for $i = 9, 9', 10, \cdots, 40', 41$.

For $\Phi(G_9, x) = \Phi(G_{9'}, x)$, $\Phi(G_{16}, x) = \Phi(G_{16'}, x)$, $g_{22}(x) = g_{22'}(x)$, $\Phi(G_{25}, x) = \Phi(G_{25'}, x)$, $g_{27}(x) = g_{27'}(x)$, $\Phi(G_{28}, x) = \Phi(G_{28'}, x)$, $g_{36}(x) = g_{36'}(x) = g_{36''}(x)$ and $g_{40}(x) = g_{40'}(x)$, we have $\mu(G_9) = \mu(G_{9'})$, $\mu(G_{16}) = \mu(G_{16'})$, $\mu(G_{22}) = \mu(G_{22'})$, $\mu(G_{25}) = \mu(G_{25'})$, $\mu(G_{27}) = \mu(G_{27'})$, $\mu(G_{28}) = \mu(G_{28'})$,

$\mu(G_{36}) = \mu(G_{36'}) = \mu(G_{36''})$ and $\mu(G_{40}) = \mu(G_{40'})$.

Method 1. For some pair of graphs, by comparing their Laplacian characteristic polynomials we can see their difference is either constantly greater than 0 or constantly less than 0 in the interval $(n-2, n-1)$. Combining with the corresponding function image, then we can compare their roots in the interval $(n-2, n-1)$ easily. The following specific examples will further illustrate this method. That is to say, for such pair of graphs we can compare their Laplacian spectra radii by Method 1. By applying Method 1, we will obtain the partial ordering as follows: $\mu(G_9) > \mu(G_{10}) > \mu(G_{11})$, $\mu(G_{13}) > \mu(G_{14})$, $\mu(G_{17}) > \mu(G_{18})$, $\mu(G_{19}) > \mu(G_{20})$, $\mu(G_{23}) > \mu(G_{24})$, $\mu(G_{37}) > \mu(G_{38})$, and $\mu(G_{39}) > \mu(G_{40}) > \mu(G_{41})$.

Clearly,

$$\begin{aligned}&\Phi(G_{10}, x) - \Phi(G_9, x) = x^2(x-2)(x-1)^{n-8}\\&\cdot[x^3 - (n+2)x^2 + (4n-4)x - 2n - 2].\end{aligned}$$

Let

$$f(x) = x^3 - (n+2)x^2 + (4n-4)x - 2n - 2.$$

Then we have $f''(x) = 6x - 2(n+2) > 0$ for $x \in (n-2, n-1)$ and $n \geq 14$. Therefore $f'(x) > f'(n-2) = (n-4)^2 > 0$. This is to say that $f(x)$ is strictly monotone increasing when $x \in (n-2, n-1)$. Then $f(x) > f(n-2) = 2(n-5) > 0$ and

$$\begin{aligned}&\Phi(G_{10}, x) - \Phi(G_9, x)\\&= x^2(x-2)(x-1)^{n-8}f(x) > 0, \forall x \in (n-2, n-1).\end{aligned}$$

Since $\Phi(G_{10}, x) > \Phi(G_9, x)$, we have $\mu(G_9) > \mu(G_{10})$.

Similarly,

$$\begin{aligned}&\Phi(G_{11}, x) - \Phi(G_{10}, x) =\\&x(x-1)^{n-8}[2x^3 - (2n+2)x^2 + (6n-7)x - 3n]\\&> 0, \quad \forall x \in (n-2, n-1),\end{aligned}$$

thus we have $\mu(G_{10}) > \mu(G_{11})$.

It is easy to verify that

$$\begin{aligned}&\Phi(G_{14}, x) - \Phi(G_{13}, x) =\\&x(x-1)^{n-7}[2x^3 - (2n-2)x^2 + (4n-10)x - n]\\&> 0, \quad \forall x \in (n-2, n-1),\end{aligned}$$

so $\mu(G_{13}) > \mu(G_{14})$.

It is no difficult to prove that

$$\begin{aligned}&\Phi(G_{18}, x) - \Phi(G_{17}, x) =\\&-x(x-1)^{n-7}[2x^3 - (2n+5)x^2 + (7n-3)x - 4n]\\&> 0, \quad \forall x \in (n-2, n-1),\end{aligned}$$

hence $\mu(G_{17}) > \mu(G_{18})$.

It is clear that

$$\begin{aligned}&(x-2)g_{20}(x) - g_{19}(x)\\&= (x-1)[x^2 - (n-1)x + n] > 0\end{aligned}$$

for $x \in (n-2, n-1)$, then $\mu(G_{19}) > \mu(G_{20})$.

Since

$$\begin{aligned}&\Phi(G_{24}, x) - \Phi(G_{23}, x)\\&= 2x^2(x-1)^{n-5}[x - (n-2)]\\&> 0, \quad \forall x \in (n-2, n-1)\end{aligned}$$

we have $\mu(G_{23}) > \mu(G_{24})$.

Because

$$\begin{aligned}&\Phi(G_{38}, x) - \Phi(G_{37}, x)\\&= x^2(x-2)(x-1)^{n-6}[x - (n-2)] > 0\end{aligned}$$

for $x \in (n-2, n-1)$, we obtain that $\mu(G_{37}) > \mu(G_{38})$.

It is not difficult to verify that

$$\begin{aligned}&\Phi(G_{40}, x) - \Phi(G_{39}, x) =\\&x^2(x-2)(x-3)(x-1)^{n-7}[x - (n-2)] > 0\end{aligned}$$

for $x \in (n-2, n-1)$, hence $\mu(G_{39}) > \mu(G_{40})$.

It is obvious that

$$\begin{aligned}&\Phi(G_{40}, x) - \Phi(G_{41}, x)\\&= x(x-1)^{n-7}[x^4 - (2n-4)x^3\\&+(11n-49)x^2 - (14n-84)x + n - 24] < 0\end{aligned}$$

for $x \in (n-2, n-1)$, so $\mu(G_{40}) > \mu(G_{41})$.

Method 2. For some pair of graphs, by applying Euclidean algorithm to their Laplacian characteristic polynomials we can see the remainder is either constantly greater than 0 or constantly less than 0 in the interval $(n-2, n-1)$. If so, we can compare the Laplacian spectra radii of this pair of graphs by Method 2. The following specific examples will further illustrate this method. By using Method 2, we can obtain the partial ordering as follows: $\mu(G_{25}) > \mu(G_{26}) > \mu(G_{27})$, $\mu(G_{28}) > \mu(G_{29})$ and $\mu(G_{35}) > \mu(G_{36}) = \mu(G_{36''}) > \mu(G_{37})$.

With the help of mathematic procedure, we can obtain that

$$\begin{aligned}g_{26}(x) &= (x^2 - 4x + 4)g_{25}(x)\\&+2x^3 - 2nx^2 + (4n-6)x - n.\end{aligned}$$

It is not difficult to show that

$$2x^3 - 2nx^2 + (4n-6)x - n > 0$$

for $x \in (n-2, n-1)$. Then we have $g_{26}(\mu(G_{25})) > 0 = g_{26}(\mu(G_{26}))$ for $\mu(G_{25}) \in (n-2, n-1)$.

Since $g_{26}(x)$ is monotone increasing in the interval $(n-2, n-1)$, we have $\mu(G_{25}) > \mu(G_{26})$.

Similarly, we can get

$$\begin{aligned} g_{26}(x) &= (x^2-5x+5)g_{27}(x) \\ &\quad -x^3+nx^2+(2-2n)x+n. \end{aligned}$$

It is easy to verify that

$$-x^3+nx^2+(2-2n)x+n<0$$

for $x \in (n-2, n-1)$. Then we have $g_{26}(\mu(G_{27})) < 0 = g_{26}(\mu(G_{26}))$ for $\mu(G_{27}) \in (n-2, n-1)$. Since $g_{26}(x)$ is monotone increasing in the interval $(n-2, n-1)$, we have $\mu(G_{26}) > \mu(G_{27})$.

Since

$$\begin{aligned} g_{29}(x) &= (x^3-6x^2+8x-6)g_{28}(x) \\ &\quad -7x^2+(9n-24)x-2n, \end{aligned}$$

and $-7x^2+(9n-24)x-2n > 0$ for $x \in (n-2, n-1)$, we have $g_{29}(\mu(G_{28})) > 0 = g_{29}(\mu(G_{29}))$ for $\mu(G_{28}) \in (n-2, n-1)$. Because $g_{29}(x)$ is monotone increasing in the interval $(n-2, n-1)$, we obtain that $\mu(G_{28}) > \mu(G_{29})$.

Because $g_{35}(x) = (x^3-6x^2+9x-3)g_{36}(x)-x$, we have $g_{35}(\mu(G_{36})) < 0 = g_{35}(\mu(G_{35}))$. Since $g_{35}(x)$ is monotone increasing in the interval $(n-2, n-1)$, we obtain that $\mu(G_{35}) > \mu(G_{36})$.

It is obvious that

$$g_{37}(x) = (x^2-6x+5)g_{36}(x)+x^2-7x+n,$$

and $x^2-7x+n > 0$ for $x \in (n-2, n-1)$. Then we have $g_{37}(\mu(G_{36})) > 0 = g_{37}(\mu(G_{37}))$. Since $g_{37}(x)$ is monotone increasing in the interval $(n-2, n-1)$, we have $\mu(G_{36}) > \mu(G_{37})$.

Method 3. For some pair of graphs, by comparing their Laplacian characteristic polynomials we can see their difference is neither constantly greater than 0 nor constantly less than 0 in the interval $(n-2, n-1)$. However, in certain subinterval of $(n-2, n-1)$ their difference is either constantly greater than 0 or constantly less than 0. The following specific examples will illustrate how to determine this subinterval. In this way, we can compare their Laplacian spectra radii by Method 1 in this subinterval. In this case, by applying Method 3, we will obtain the partial ordering as follows: $\mu(G_{11}) > \mu(G_{12}) > \mu(G_{13})$, $\mu(G_{14}) > \mu(G_{15}) > \mu(G_{16}) > \mu(G_{17})$, $\mu(G_{18}) > \mu(G_{19})$, $\mu(G_{20}) > \mu(G_{21}) > \mu(G_{22}) = \mu(G_{22'}) > \mu(G_{23})$, $\mu(G_{24}) > \mu(G_{25})$, $\mu(G_{27}) > \mu(G_{28})$, $\mu(G_{29}) > \mu(G_{30}) > \mu(G_{31}) > \mu(G_{32}) > \mu(G_{33}) > \mu(G_{34}) > \mu(G_{35})$ and $\mu(G_{38}) > \mu(G_{39})$.

Let

$$\begin{aligned} &h_1(x) = xg_{12}(x)-g_{11}(x) = \\ &3x^3-(3n+5)x^2+(11n-17)x-5n. \end{aligned}$$

Clearly, $\mu(G_{12})$ is the largest root of $xg_{12}(x)$ and $\mu(G_{11})$ is the largest root of $g_{11}(x)$. Furthermore, we have $h_1(n-2) = -10 < 0$ and $h_1(n-1) = 3n^2-17n+9 > 0$. Let α_1 be the largest root of $h_1(x)$, then $\alpha_1 \in (n-2, n-1)$. We can get $g_{11}(x) = m_1(x)h_1(x)+r_1(x)$ and $xg_{12}(x) = m_2(x)h_1(x)+r_1(x)$, where

$$r_1(x) = \frac{-8}{9}x^2+(\frac{8n}{9}-\frac{17}{9})x-\frac{5n}{9}.$$

It can be checked that $r_1(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{11}(\alpha_1) = r_1(\alpha_1) < 0 = g_{11}(\mu(G_{11}))$, therefore $\mu(G_{11}) > \alpha_1$. Similarly, we have $\mu(G_{12}) > \alpha_1$. For $h_1(x) > 0$ when $x \in (\alpha_1, n-1)$ (here $(\alpha_1, n-1)$ is just the subinterval we need to determine as mentioned above), we have $xg_{12}(x) > g_{11}(x)$. So $\mu(G_{11}) > \mu(G_{12})$.

Let

$$\begin{aligned} h_2(x) &= (x-2)g_{12}(x)-g_{13}(x) \\ &= x^3-(n-2)x^2-2x+n. \end{aligned}$$

It is obvious that $\mu(G_{12})$ is the largest root of $(x-2)g_{12}(x)$ and $\mu(G_{13})$ is the largest root of $g_{13}(x)$. Let α_2 be the largest root of $h_2(x)$, then we have $\alpha_2 \in (n-2, n-1)$ for $h_2(n-2) = -n+4 < 0$ and $h_2(n-1) = n^2-3n+3 > 0$. Moreover, We can get $(x-2)g_{12}(x) = m_3(x)h_2(x)+r_2(x)$ and $g_{13}(x) = m_4(x)h_2(x)+r_2(x)$, where

$$r_2(x) = 12x^2-11nx+10n.$$

Finally, it can be checked that $r_2(x) > 0$ for $x \in (n-2, n-1)$. Then we have $g_{13}(\alpha_2) = r_2(\alpha_2) > 0 = g_{13}(\mu(G_{13}))$. It can be seen that $\mu(G_{13}) \in (n-2, \alpha_2)$ as well as $\mu(G_{12})$. For $h_2(x) < 0$ when $x \in (n-2, \alpha_2)$, then we have $(x-2)g_{12}(x) < g_{13}(x)$. So $\mu(G_{12}) > \mu(G_{13})$.

Let

$$\begin{aligned} h_3(x) &= g_{14}(x)-g_{15}(x) \\ &= x^3-(n+4)x^2+(6n-9)x-3n. \end{aligned}$$

Clearly, $\mu(G_{14})$ is the largest root of $g_{14}(x)$ and $\mu(G_{15})$ is the largest root of $g_{15}(x)$. Let α_3 be the largest root of $h_3(x)$, then we have $\alpha_3 \in (n-2, n-1)$ for $h_3(n-2) = -6 < 0$ and $h_3(n-1) = n^2-8n+4 > 0$ for $n \geq 14$. We can get $g_{14}(x) = m_5(x)h_3(x)+r_3(x)$ and $g_{15}(x) = m_6(x)h_3(x)+r_3(x)$, where

$$r_3(x) = 8x^2-(5n+1)x+4n.$$

Moreover, it is easy to verify that $r_3(x) > 0$ for $x \in (n-2, n-1)$. Then we have $g_{14}(\alpha_3) = r_3(\alpha_3) > 0 = g_{14}(\mu(G_{14}))$. It can be seen that $\mu(G_{14}) \in (n-$

$2, \alpha_3)$ and so as $\mu(G_{15})$. Since $h_3(x) < 0$ for $x \in (n-2, \alpha_3)$, we have $g_{14}(x) < g_{15}(x)$. So $\mu(G_{14}) > \mu(G_{15})$.

Let

$$\begin{aligned} h_4(x) &= g_{16}(x) - g_{15}(x) \\ &= 2x^3 - (2n+4)x^2 + (8n-13)x - 3n, \end{aligned}$$

and α_4 be the largest root of $h_4(x)$, then we have $\alpha_4 \in (n-2, n-1)$ for $h_4(n-2) = -6 < 0$ and $h_4(n-1) = 2n^2 - 12n + 7 > 0$ $(n \geq 14)$. We can get $g_{15}(x) = m_7(x)h_4(x) + r_4(x)$ and $g_{16}(x) = m_8(x)h_4(x) + r_4(x)$, where

$$r_4(x) = \frac{-5x}{4} + \frac{n}{4}.$$

It is easy to check that $r_4(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{15}(\alpha_4) = r_4(\alpha_4) < 0 = g_{15}(\mu(G_{15}))$. It can be seen that $\mu(G_{15}) \in (\alpha_4, n-1)$ as well as $\mu(G_{16})$. For $h_4(x) > 0$ when $x \in (\alpha_4, n-1)$, then $g_{15}(x) < g_{16}(x)$. So $\mu(G_{15}) > \mu(G_{16})$.

Let

$$\begin{aligned} h_5(x) &= (x-3)g_{17}(x) - g_{16}(x) \\ &= x^3 - (n+2)x^2 + (4n-7)x - n. \end{aligned}$$

It is clear that $\mu(G_{17})$ is the largest root of $(x-3)g_{17}(x)$ and $\mu(G_{16})$ is the largest root of $g_{16}(x)$. Let α_5 be the largest root of $h_5(x)$, then we have $\alpha_5 \in (n-2, n-1)$ for $h_5(n-2) = -2 < 0$ and $h_5(n-1) = n^2 - 6n + 4 > 0$ $(n \geq 14)$. We can get $g_{16}(x) = m_9(x)h_5(x) + r_5(x)$ and $(x-3)g_{17}(x) = m_{10}(x)h_5(x) + r_5(x)$, where

$$r_5(x) = -2x^2 + (n+6)x - 3n.$$

It is not difficult to verify that $r_5(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{16}(\alpha_5) = r_5(\alpha_5) < 0 = g_{16}(\mu(G_{16}))$. It can be seen that $\mu(G_{16}) \in (\alpha_5, n-1)$ and so as $\mu(G_{17})$. For $h_5(x) > 0$ when $x \in (\alpha_5, n-1)$, we obtain that $g_{16}(x) < (x-3)g_{17}(x)$. So $\mu(G_{16}) > \mu(G_{17})$.

Let

$$\begin{aligned} h_6(x) &= g_{19}(x) - (x-1)(x-2)g_{18}(x) \\ &= 2x^4 - (2n+6)x^3 + (10n-6)x^2 \\ &\quad -(14n-17)x + 5n. \end{aligned}$$

It is obvious that $\mu(G_{18})$ is the largest root of $(x-1)(x-2)g_{18}(x)$ and $\mu(G_{19})$ is the largest root of $g_{19}(x)$. Let α_6 be the largest root of $h_6(x)$, then we have $\alpha_6 \in (n-2, n-1)$ because $h_6(n-2) = -6n + 22 < 0$ and $h_6(n-1) = 2n^3 - 16n^2 + 34n - 15 > 0$ $(n \geq 14)$. We can get $g_{19}(x) = m_{11}(x)h_6(x) + r_6(x)$ and $(x-1)(x-2)g_{18}(x) = m_{12}(x)h_6(x) + r_6(x)$, where

$$r_6(x) = -x^3 + 4x^2 - \frac{19}{4}x + \frac{n}{4}.$$

It is not difficult to show that $r_6(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{19}(\alpha_6) = r_6(\alpha_6) < 0 = g_{19}(\mu(G_{19}))$. It can be seen that $\mu(G_{19}) \in (\alpha_6, n-1)$ as well as $\mu(G_{18})$. For $h_6(x) > 0$ when $x \in (\alpha_6, n-1)$, then $(x-1)(x-2)g_{18}(x) < g_{19}(x)$. Thus $\mu(G_{18}) > \mu(G_{19})$.

Let

$$\begin{aligned} h_7(x) &= (x-1)g_{20}(x) - xg_{21}(x) \\ &= 2x^5 - (2n+10)x^4 + (14n+4)x^3 \\ &\quad -(32n-32)x^2 + (25n-22)x - 6n. \end{aligned}$$

Clearly, $\mu(G_{20})$ is the largest root of $(x-1)g_{20}(x)$ and $\mu(G_{21})$ is the largest root of $xg_{21}(x)$. Let α_7 be the largest root of $h_7(x)$, then we have $\alpha_7 \in (n-2, n-1)$ since $h_7(n-2) = -7n^2 + 50n - 84 < 0$ and $h_7(n-1) = 2n^4 - 22n^3 + 79n^2 - 103n + 38 > 0$ $(n \geq 14)$. We can get $(x-1)g_{20}(x) = m_{13}(x)h_7(x) + r_7(x)$ and $xg_{21}(x) = m_{14}(x)h_7(x) + r_7(x)$, where

$$\begin{aligned} r_7(x) = \; & -2x^4 + (4 + \tfrac{5n}{2})x^3 + (15-12n)x^2 \\ & +(\tfrac{23n}{2} - 11)x - 3n. \end{aligned}$$

It is not difficult to prove that $r_7(x) > 0$ for $x \in (n-2, n-1)$. Then we have $(\alpha_7 - 1)g_{20}(\alpha_7) = r_7(\alpha_7) > 0 = (\mu(G_{20}) - 1)g_{20}(\mu(G_{20}))$. Since $(x-1)g_{20}(x)$ is monotone increasing in the interval $(n-2, n-1)$, we have $\mu(G_{20}) \in (n-2, \alpha_7)$ and so as $\mu(G_{21})$. Since $h_7(x) < 0$ for $x \in (n-2, \alpha_7)$, we have $(x-1)g_{20}(x) < xg_{21}(x)$. So $\mu(G_{20}) > \mu(G_{21})$.

Let

$$\begin{aligned} h_8(x) &= (x-4)g_{22}(x) - g_{21}(x) \\ &= 2x^3 - (2n+3)x^2 + (7n-11)x - 3n. \end{aligned}$$

It is obvious that $\mu(G_{22})$ is the largest root of $(x-4)g_{22}(x)$ and $\mu(G_{21})$ is the largest root of $g_{21}(x)$. Let α_8 be the largest root of $h_8(x)$, then we have $\alpha_8 \in (n-2, n-1)$ for $h_8(n-2) = -6 < 0$ and $h_8(n-1) = 2n^2 - 11n + 6 > 0$ $(n \geq 14)$. We can get $(x-4)g_{22}(x) = m_{15}(x)h_8(x) + r_8(x)$ and $g_{21}(x) = m_{16}(x)h_8(x) + r_8(x)$, where

$$r_8(x) = -\frac{3}{16}x^2 + (\frac{39}{16} - \frac{5n}{16})x - \frac{n}{16}.$$

It is easy to show that $r_8(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{21}(\alpha_8) = r_8(\alpha_8) < 0 = g_{21}(\mu(G_{21}))$. It can be seen that $\mu(G_{21}) \in (\alpha_8, n-1)$ as well as $\mu(G_{22})$. For $h_8(x) > 0$ when $x \in (\alpha_8, n-1)$, we obtain that $(x-4)g_{22}(x) > g_{21}(x)$. Hence $\mu(G_{21}) > \mu(G_{22})$.

Let

$$h_9(x) = g_{22}(x) - xg_{23}(x)$$
$$= x^4 - (n-4)x^3 - (2n+3)x^2 + (6n-8)x - 2n.$$

Clearly, $\mu(G_{22})$ is the largest root of $g_{22}(x)$ and $\mu(G_{23})$ is the largest root of $xg_{23}(x)$. Let α_9 be the largest root of $h_9(x)$, then we have $\alpha_9 \in (n-2, n-1)$ since $h_9(n-2) = -n^2 + 6n - 12 < 0$ and $h_9(n-1) = n^3 - 2n^2 - 3n + 2 > 0$ $(n \geq 14)$. We can get $g_{22}(x) = m_{17}(x)h_9(x) + r_9(x)$ and $xg_{23}(x) = m_{18}(x)h_9(x) + r_9(x)$, where

$$r_9(x) = 40x^3 - (39n+2)x^2 + (66n-80)x - 20n.$$

It is not difficult to verify that $r_9(x) > 0$ for $x \in (n-2, n-1)$. Then we have $g_{22}(\alpha_9) = r_9(\alpha_9) > 0 = g_{22}(\mu(G_{22}))$. It can be seen that $\mu(G_{22}) \in (n-2, \alpha_9)$ and so as $\mu(G_{23})$. Because $h_9(x) < 0$ for $x \in (n-2, \alpha_9)$, we have $g_{22}(x) < xg_{23}(x)$. Thus $\mu(G_{22}) > \mu(G_{23})$.

Let

$$h_{10}(x) = (x-3)g_{25}(x) - g_{24}(x)$$
$$= x^3 - (n+2)x^2 + (4n-7)x - n.$$

It is clear that $\mu(G_{25})$ is the largest root of $(x-3)g_{25}(x)$ and $\mu(G_{24})$ is the largest root of $g_{24}(x)$. Let α_{10} be the largest root of $h_{10}(x)$, then we have $\alpha_{10} \in (n-2, n-1)$ for $h_{10}(n-2) = -2 < 0$ and $h_{10}(n-1) = n^2 - 6n + 4 > 0$ $(n \geq 14)$. We can obtain that $g_{24}(x) = m_{19}(x)h_{10}(x) + r_{10}(x)$ and $(x-3)g_{25}(x) = m_{20}(x)h_{10}(x) + r_{10}(x)$, where

$$r_{10}(x) = -2x^2 + (2n-2)x - 2n.$$

It is obvious that $r_{10}(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{24}(\alpha_{10}) = r_{10}(\alpha_{10}) < 0 = g_{24}(\mu(G_{24}))$. It can be seen that $\mu(G_{24}) \in (\alpha_{10}, n-1)$ and so as $\mu(G_{25})$. Since $h_{10}(x) > 0$ for $x \in (\alpha_{10}, n-1)$, we obtain that $(x-3)g_{25}(x) > g_{24}(x)$. Therefore $\mu(G_{24}) > \mu(G_{25})$.

Let

$$h_{11}(x) = xg_{28}(x) - g_{27}(x)$$
$$= 2x^3 - (2n+3)x^2 + (7n-12)x - 2n.$$

Clearly, $\mu(G_{28})$ is the largest root of $xg_{28}(x)$ and $\mu(G_{27})$ is the largest root of $g_{27}(x)$. Let α_{11} be the largest root of $h_{11}(x)$, then we have $\alpha_{11} \in (n-2, n-1)$ for $h_{11}(n-2) = -4 < 0$ and $h_{11}(n-1) = 2n^2 - 11n + 7 > 0$ $(n \geq 14)$. We can get $g_{27}(x) = m_{21}(x)h_{11}(x) + r_{11}(x)$ and $xg_{28}(x) = m_{22}(x)h_{11}(x) + r_{11}(x)$, where

$$r_{11}(x) = -\frac{7}{4}x^2 + (\frac{7n}{4} - 3)x - \frac{n}{2}.$$

It is easy to verify that $r_{11}(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{27}(\alpha_{11}) = r_{11}(\alpha_{11}) < 0 = g_{27}(\mu(G_{27}))$. It can be seen that $\mu(G_{27}) \in (\alpha_{11}, n-1)$ as well as $\mu(G_{28})$. For $h_{11}(x) > 0$ when $x \in (\alpha_{11}, n-1)$, then $xg_{28}(x) > g_{27}(x)$. So $\mu(G_{27}) > \mu(G_{28})$.

Let

$$h_{12}(x) = (x-1)^3 g_{30}(x) - xg_{29}(x)$$
$$= 3x^5 - (3n+12)x^4 + (18n-4)x^3$$
$$-(32n-38)x^2 + (20n-18)x - 4n.$$

It is obvious that $\mu(G_{30})$ is the largest root of $(x-1)^3 g_{30}(x)$ and $\mu(G_{29})$ is the largest root of $xg_{29}(x)$. Let α_{12} be the largest root of $h_{12}(x)$, then we have $\alpha_{12} \in (n-2, n-1)$ since $h_{12}(n-2) = -6n^2 + 42n - 68 < 0$ and $h_{12}(n-1) = 3n^4 - 30n^3 + 98n^2 - 120n + 45 > 0$ $(n \geq 14)$. We can get $xg_{29}(x) = m_{23}(x)h_{12}(x) + r_{12}(x)$ and $(x-1)^3 g_{30}(x) = m_{24}(x)h_{12}(x) + r_{12}(x)$, where

$$r_{12}(x) = -\frac{14}{3}x^4 + (\frac{14n}{3} + \frac{31}{9})x^3$$
$$+(-\frac{115n}{9} + \frac{136}{9})x^2 + (\frac{88n}{9} - 10)x - \frac{20n}{9}.$$

It is not difficult to prove that $r_{12}(x) < 0$ for $x \in (n-2, n-1)$. Then we have $\alpha_{12}g_{29}(\alpha_{12}) = r_{12}(\alpha_{12}) < 0 = \mu(G_{29})g_{29}(\mu(G_{29}))$. It can be seen that $\mu(G_{29}) \in (\alpha_{12}, n-1)$ and so as $\mu(G_{30})$. For $h_{12}(x) > 0$ when $x \in (\alpha_{12}, n-1)$, then $(x-1)^3 g_{30}(x) > xg_{29}(x)$. Thus $\mu(G_{29}) > \mu(G_{30})$.

Let

$$h_{13}(x) = g_{31}(x) - g_{30}(x)$$
$$= x^3 - (n+1)x^2 + (3n-5)x - n.$$

Clearly, $\mu(G_{30})$ and $\mu(G_{31})$ are the largest roots of $g_{30}(x)$ and $g_{31}(x)$, respectively. Let α_{13} be the largest root of $h_{13}(x)$, then we have $\alpha_{13} \in (n-2, n-1)$ since $h_{13}(n-2) = -2 < 0$ and $h_{13}(n-1) = n^2 - 5n + 3 > 0$ $(n \geq 14)$. We can obtain that $g_{30}(x) = m_{23}(x)h_{13}(x) + r_{13}(x)$ and $g_{31}(x) = m_{24}(x)h_{13}(x) + r_{13}(x)$, where

$$r_{13}(x) = -x[x - (n-2)].$$

It is easy to verify that $r_{13}(x) < 0$ for $x \in (n-2, n-1)$. Then we have $g_{30}(\alpha_{13}) = r_{13}(\alpha_{13}) < 0 = g_{30}(\mu(G_{30}))$. It can be seen that $\mu(G_{30}) \in (\alpha_{13}, n-1)$ as well as $\mu(G_{31})$. For $h_{13}(x) > 0$ when $x \in (\alpha_{13}, n-1)$, we have $g_{31}(x) > g_{30}(x)$. Hence $\mu(G_{30}) > \mu(G_{31})$.

Let

$$h_{14}(x) = (x^3 - 6x^2 + 9x - 3)g_{31}(x) - (x-1)g_{32}(x)$$
$$= 2x^5 - (2n+6)x^4 + (10n-5)x^3$$
$$-(15n-18)x^2 + (7n-2)x - n.$$

Obviously, $\mu(G_{31})$ and $\mu(G_{32})$ are the largest roots of $(x^3-6x^2+9x-3)g_{31}(x)$ and $(x-1)g_{32}(x)$, respectively. Let α_{14} be the largest root of $h_{14}(x)$, then we have $\alpha_{14}\in(n-2,n-1)$ because $h_{14}(n-2)=-5n^2+31n-44<0$ and $h_{14}(n-1)=2n^4-18n^3+52n^2-54n+17>0$ $(n\geq 14)$. We can get $(x^3-6x^2+9x-3)g_{31}(x)=m_{25}(x)h_{14}(x)+r_{14}(x)$ and $(x-1)g_{32}(x)=m_{26}(x)h_{14}(x)+r_{14}(x)$, where

$$r_{14}(x)=(\frac{n}{2}-\frac{1}{4})x^3+(-\frac{23n}{4}+\frac{31}{2})x^2$$
$$+(\frac{41n}{4}-\frac{55}{2})x-\frac{13n}{4}.$$

It is not difficult to verify that $r_{14}(x)>0$ for $x\in(n-2,n-1)$. Then we have $(\alpha_{14}-1)g_{32}(\alpha_{14})=r_{14}(\alpha_{14})>0=(\mu(G_{32})-1)g_{32}(\mu(G_{32}))$. It can be seen that $\mu(G_{32})\in(n-2,\alpha_{14})$ and so as $\mu(G_{31})$. For $h_{14}(x)<0$ when $x\in(n-2,\alpha_{14})$, then $(x-1)g_{32}(x)>(x^3-6x^2+9x-3)g_{31}(x)$. So $\mu(G_{31})>\mu(G_{32})$.

Clearly,

$$\begin{aligned}&(x-1)g_{32}(x)-g_{33}(x)=\\&x[-2x^4+(2n+6)x^3-(10n-7)x^2\\&+(13n-18)x-4n].\end{aligned}$$

Let

$$\begin{aligned}h_{15}(x)=\ &-2x^4+(2n+6)x^3\\&-(10n-7)x^2+(13n-18)x-4n.\end{aligned}$$

It is obvious that $\mu(G_{32})$ and $\mu(G_{33})$ are the largest roots of $(x-1)g_{32}(x)$ and $g_{33}(x)$, respectively. Let α_{15} be the largest root of $h_{15}(x)$, then we have $\alpha_{15}\in(n-2,n-1)$ for $h_{15}(n-2)=4(n-4)>0$ and $h_{15}(n-1)=-2n^3+16n^2-35n+17<0$ $(n\geq 14)$. We can get $(x-1)g_{32}(x)=m_{27}(x)h_{15}(x)+r_{15}(x)$ and $g_{33}(x)=m_{28}(x)h_{15}(x)+r_{15}(x)$, where

$$r_{15}(x)=(-\frac{1}{4})x^3+(\frac{n}{4}-\frac{1}{2})x^2-x.$$

It is not difficult to show that $r_{15}(x)<0$ for $x\in(n-2,n-1)$. Then we have $g_{33}(\alpha_{15})=r_{15}(\alpha_{15})<0=g_{33}(\mu(G_{33}))$. It can be seen that $\mu(G_{33})\in(\alpha_{15},n-1)$ as well as $\mu(G_{32})$. For $h_{15}(x)<0$ when $x\in(\alpha_{15},n-1)$, then $(x-1)g_{32}(x)<g_{33}(x)$. Thus $\mu(G_{32})>\mu(G_{33})$.

Let

$$\begin{aligned}&h_{16}(x)=(x-1)g_{34}(x)-g_{33}(x)\\&=x^5-(n+6)x^4+(8n+6)x^3-(22n-20)x^2\\&+(22n-28)x-6n.\end{aligned}$$

It is clear that $\mu(G_{33})$ and $\mu(G_{34})$ are the largest roots of $g_{33}(x)$ and $(x-1)g_{34}(x)$, respectively. Let α_{16} be the largest root of $h_{16}(x)$, then we have $\alpha_{16}\in(n-2,n-1)$ for $h_{16}(n-2)=-2(n^2-9n+20)<0$ and $h_{16}(n-1)=n^4-12n^3+50n^2-80n+35>0$ $(n\geq 14)$. We can get $(x-1)g_{34}(x)=m_{29}(x)h_{16}(x)+r_{16}(x)$ and $g_{33}(x)=m_{30}(x)h_{16}(x)+r_{16}(x)$, where

$$r_{16}(x)=-4x^4+(4n+13)x^3$$
$$+(14-21n)x^2+(27n-37)x-8n.$$

It can be verified that $r_{16}(x)<0$ for $x\in(n-2,n-1)$. Then we have $g_{33}(\alpha_{16})=r_{16}(\alpha_{16})<0=g_{33}(\mu(G_{33}))$. It can be seen that $\mu(G_{33})\in(\alpha_{16},n-1)$ and so as $\mu(G_{34})$. Since $h_{16}(x)>0$ for $x\in(\alpha_{16},n-1)$, we have $(x-1)g_{34}(x)>g_{33}(x)$. Hence $\mu(G_{33})>\mu(G_{34})$.

Let

$$\begin{aligned}&h_{17}(x)=(x-3)g_{35}(x)-(x-1)g_{34}(x)\\&=2x^4-(2n+7)x^3+(11n-6)x^2\\&-(16n-23)x+5n.\end{aligned}$$

Clearly, $\mu(G_{34})$ and $\mu(G_{35})$ are the largest roots of $(x-1)g_{34}(x)$ and $(x-3)g_{35}(x)$, respectively. Let α_{17} be the largest root of $h_{17}(x)$, then we have $\alpha_{17}\in(n-2,n-1)$ for $h_{17}(n-2)=-4n+18<0$ and $h_{17}(n-1)=2n^3-17n^2+40n-20>0$ $(n\geq 14)$. We can obtain that $(x-1)g_{34}(x)=m_{31}(x)h_{17}(x)+r_{17}(x)$ and $(x-3)g_{35}(x)=m_{32}(x)h_{17}(x)+r_{17}(x)$, where

$$r_{17}(x)=(-\frac{3}{16})x^3+(\frac{3n}{16}-\frac{1}{4})x^2-(\frac{n}{8}+\frac{5}{16})x+\frac{n}{16}.$$

It is easy to verify that $r_{17}(x)<0$ for $x\in(n-2,n-1)$. Then we have $(\alpha_{17}-1)g_{34}(\alpha_{17})=r_{17}(\alpha_{17})<0=(\mu(G_{34})-1)g_{34}(\mu(G_{34}))$. It can be seen that $\mu(G_{34})\in(\alpha_{17},n-1)$ as well as $\mu(G_{35})$. Since $h_{17}(x)>0$ for $x\in(\alpha_{17},n-1)$, we have $(x-3)g_{35}(x)>(x-1)g_{34}(x)$. So $\mu(G_{34})>\mu(G_{35})$.

Let

$$\begin{aligned}&h_{18}(x)=g_{39}(x)-g_{38}(x)\\&=x^4-(n+3)x^3+(5n-5)x^2-(5n-8)x+n.\end{aligned}$$

It is obvious that $\mu(G_{38})$ and $\mu(G_{39})$ are the largest roots of $g_{38}(x)$ and $g_{39}(x)$, respectively. Let α_{18} be the largest root of $h_{18}(x)$, then we have $\alpha_{18}\in(n-2,n-1)$ for $h_{18}(n-2)=4-n<0$ and $h_{18}(n-1)=n^3-8n^2+17n-9>0$ $(n\geq 14)$. We can get $g_{38}(x)=m_{33}(x)h_{18}(x)+r_{18}(x)$ and $g_{39}(x)=m_{34}(x)h_{18}(x)+r_{18}(x)$, where

$$r_{18}(x)=-x[x-(n-2)].$$

It is not difficult to show that $r_{18}(x)<0$ for $x\in(n-2,n-1)$. Then we have $g_{38}(\alpha_{18})=r_{18}(\alpha_{18})<$

$0 = g_{38}(\mu(G_{38}))$. It can be seen that $\mu(G_{38}) \in (\alpha_{18}, n-1)$ and so as $\mu(G_{39})$. For $h_{18}(x) > 0$ when $x \in (\alpha_{18}, n-1)$, then $g_{39}(x) > g_{38}(x)$. Thus $\mu(G_{38}) > \mu(G_{39})$.

It can be seen that for every pair of graphs we can definitely compare their Laplacian spectra radii by one of the above three methods. In this way, we finally obtained the complete ordering and the ninth to the forty-first largest Laplacian spectra radii among all the n-vertex bicyclic graphs.

Acknowledgements: The research was supported by the National Science Natural Foundation in China (NSFC-61103073).

References:

[1] Dragoš Cvetković, Michael Doob, Horst Sachs, *Spectra of Graphs* (3rd edition), John Wiley and Sons, 1999.

[2] Ivan Gutman, Shyi-Long Lee, Chih-Hung Chu, Yeung-Long Luo, Chemical applications of the Laplacian spectrum of molecular graphs: studies of the Wiener number, *Indian Journal of Chemistry*, 33, 1994, (A): pp.603-608.

[3] B. Mohar, Some applications of Laplacian eigenvalues of graphs, Notes taken by Martin Juvan, in: G. Hahn, et al. (Eds.), *Graph Symmetry: Algebraic Methods and Applications*, in: NATO ASI Ser., Ser. C, Math. Phys. Sci., vol. 497, Kluwer Academic Publishers, Dordrecht, 1997, 225-275.

[4] N. M. M. Abreu, Old and new results on algebraic connectivity of graphs, *Linear Algebra and its Appl.*, 423, 2007, (1): pp.53-73.

[5] D. M. Cvetković, *Some possible directions in further investigation of graph spetra, Alegebra Methods in Graphs Theory*, Vol 1, North-Holland, Amsterdam, 1981, 47-67.

[6] Hong Yuan, Bounds of the eigenvalues of unicyclic graphs, *Journal of East China Normal University*, 1986, (1): 31-34 (in Chinese).

[7] Chen Ailian, The order of the unicyclic graphs by their largest eigenvalues, *Journal of Mathematical Study*, 36, 2003, (1): pp.87-94 (in Chinese).

[8] Wu Jianhai, Lin Wenshui, The further order of the unicyclic graphs by their largest eigenvalues, *Journal of Mathematical Study*, 38, 2005, (3): pp.302-308.

[9] C. X. He, Y. Liu, J. Y. Shao, On the spectra radii of bicyclic graphs, *Journal of Mathematical Research and Exposition*, 27, 2007, (3): pp.445-454.

[10] J. Qi, On the spectra radius of bicyclic graph of order n, *Journal of Hainan Normal University*, 19, 2006, (4):pp.289-300 (in Chinese).

[11] Wang Xingke, Tan Shangwang, The ordering of bicyclic graphs by their spectral radii, *Acta Mathematica Sinica, Chinese Series*, 53, 2010, (3): pp.469-476.

[12] Guo Shuguang, The largest eigenvalues of Laplacian matrices of unicyclic graphs, *Appl. Math. J. Chinese Univ. Ser A*, 16, 2001, (2): pp.131-135.

[13] Y. Liu, J. Y. Shao, X. Y. Yuan, some results on the ordering of Laplacian spectra radii of unicyclic graphs, *Discrete Appl. Math.*, 156, 2008, (14):pp.2679-2697.

[14] Liu Ying, Liu Yue, Ordering of unicyclic graphs with Laplacian spectral radii, *Journal of Tongji Univ(Nature Science)*, 36, 2008, (6): pp.841-843.

[15] C. X. He, J. Y. Shao, On the Laplacian spectra radii of bicyclic graphs, *Discrete Math.*, 308, 2008, pp.5981-5995.

[16] Shuchao Li, Slobodan K. Simić et al., On ordering bicyclic graphs with respect to the Laplacian spectra radius, *Applied. Math. Letters*, 24, 2011, pp.2186-2192.

[17] R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra and its Appl.*, 197, 1994, pp. 143-176.

[18] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Appl.*, 285, 1998,pp. 33-35.

[19] K. C. Das, An improved upper bound for Laplacian graph eigenvalues, *Linear Algebra and its Appl.*, 368, 2003, pp. 269-278.

[20] W. N. Anderson, T. D. Morley, Eigenvalues of Laplacian of a graph, *Linear and Multilinear Algebra*, 18, 1985, pp. 141-145.