Exponential stabilization of 1-d wave equation with input delay

Han Wang
Tianjin University,
Department of Mathematics
Tianjin 300072
China
hotty6@126.com

Gen Qi Xu
Tianjin University,
Department of Mathematics
Tianjin 300072
China
gqxu@tju.edu.cn

Abstract: In this paper, we consider the stabilization problem of 1-d wave equation with input delay. Suppose that the wave system is fixed at one end whereas a control force is applied at other end. Here we consider the control force of the form $\alpha u(t) + \beta u(t - \tau)$ where τ is the time delay. In this paper we find a feedback control law that stabilizes exponentially the system for any $|\alpha| \neq |\beta|$ and $\tau > 0$.

Key–Words: wave system; input delay; feedback control; exponential stabilization

1 Introduction

In the present paper we consider the stabilization problem of 1-d wave equation with input delay of the following form

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), t > 0 \\ w(0,t) = 0, \\ w_x(1,t) = -\alpha u(t) - \beta u(t-\tau) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x),\ x \in (0,1) \\ u(s) = \varphi(s),\ s \in (-\tau, 0). \end{cases} \tag{1}$$

Observe that the control force term $\alpha u(t) + \beta u(t-\tau)$ is a special case of the general form $\int_{-\tau}^{0} u(t+s) d\eta(s)$ that denotes the input delay or memory. In 2006, Xu et al. in [1] had discussed the stabilization problem of the system (1). They proved that the system (1) can be stabilized exponentially under the feedback control law $u(t) = w_t(1,t)$ provided that $\alpha > \beta > 0$, but destabilized for $0 < \alpha < \beta$. After then, this result was extended to the multi-dimensional wave equation with a delay term in the boundary or internal feedbacks (see, [2] [3]). A similar result was obtained in [4] for networks of 1-d wave equations with delay in the nodal feedbacks. In 2009, Benhassi et al. in [5] proved that a class of abstract second order evolution equations with delay can be stabilized exponentially by the velocity feedback when $\alpha > \beta > 0$. For similar results, we also refer to [6] and the references therein. All results mentioned above show that $\alpha > \beta > 0$ is a stability criterion for the system with delay damping. Recently, this result was again verified in [7] for Euler-Bernoulli beam with delay in boundary control and in [8] for Timoshenko system with internal delay.

Since the coefficients α and β are determined by the controller, they are inherent property of the controller, usually they are unknown. The stability criterion about $\alpha > \beta > 0$ only is a special case of the controller. Indeed, we cannot determine whether or not $\alpha > \beta > 0$ hold in practice. Therefore, to find an anti-delay feedback control law for any $\alpha, \beta \in \mathbb{R}$ with $|\alpha| + |\beta| \neq 0$ becomes an important task.

It is well known that when $\beta = 0$ and $\alpha > 0$ that means controller has no delay, the system (1) can be stabilized exponentially by the velocity feedback control law $u(t) = w_t(1,t)$. When $\beta > 0$ and $\alpha = 0$ that means the controller is of full delay, however, the system is destabilized by the same control law (see [9]). Obviously, the velocity feedback is not a suitable candidate. In order to find a feedback control law to stabilize the system (1) for any $\alpha, \beta \in \mathbb{R}$ with $|\alpha| + |\beta| \neq 0$, our idea is to modify the velocity feedback into the form

$$u(t) = \beta w_t(1,t) + \alpha f(w(.,t), w_t(.,t))$$

where f is a linear functional. To determine the expression of this functional, we firstly translate system (1) into a control system without delay, and then by the duality principle obtain the observation for the system without delay and hence by collocated feedback to determine the form of function f. Finally we prove that the feedback control law can stabilize exponentially the system (1). Clearly the key step is to translate system (1) into a system without delay, the method of the classical Smith predictor shows that this step is realizable.

In what follows, we describe the designing procedure of the feedback control law. Suppose that the full state of the system is measurable. Let

$(w(x,t), w_t(x,t))$ be the state of system (1). Analogous to the Smith Predictor we introduce an auxiliary system as follows

$$\begin{cases} \widehat{w}_{ss}(x,s,t) = \widehat{w}_{xx}(x,s,t),\ 0 \le s \le \tau, t \ge 0, \\ \widehat{w}(0,s,t) = 0, \quad s \in (0,\tau), \\ \widehat{w}_x(1,s,t) = -\beta u(t-\tau+s), \quad s \in (0,\tau), \\ \widehat{w}(x,0,t) = w(x,t),\ \widehat{w}_s(x,0,t) = w_t(x,t),\ t \ge 0. \end{cases} \tag{2}$$

This auxiliary system is not a Smith Predictor because the control of the auxiliary system is only a partial control of the original system.

We take the state of (2) at $s = \tau$, denoted by

$$p(x,t) = \widehat{w}(x,\tau,t) \quad \textit{and} \quad q(x,t) = \widehat{w}_s(x,\tau,t).$$

Using (1.1) we derive the following system without delay

$$\begin{cases} p_t(x,t) = q(x,t) - \alpha a(x)u(t), \quad t > 0, \\ q_t(x,t) = p_{xx}(x,t) - \alpha b(x)u(t), \\ p(0,t) = 0, \quad q(0,t) = 0 \\ p_x(1,t) = -\beta u(t), \\ p(x,0) = E_0(w_0,w_1)(x) + \beta\int_{-\tau}^0 a_0(x,s)\varphi(s)ds, \\ q(x,0) = E_1(w_0,w_1)(x) - \beta\int_{-\tau}^0 a_1(x,s)\varphi(s)ds. \end{cases} \tag{3}$$

where $a_0(x,s), a_1(x,s)$, $a(x)$ and $b(x)$ are real functions and E_0 and E_1 are bounded linear operators on $L^2[0,1]$, they will be determined later.

In order to obtain the feedback control law, we consider the dual system of (3). The duality theory shows that observation system corresponding to (3) is

$$\begin{cases} w_t(x,t) = v(x,t), \quad 0 < x < 1, t > 0 \\ v_t(x,t) = w_{xx}(x,t) \\ w(0,t) = w_x(1,t) = 0 \\ w(x,0) = w_0(x), \quad v(x,0) = v_0(x) \\ y(t) = U(w,v) \\ = \beta v(1,t) + \alpha\int_0^1 w_x(x,t)a'(x)dx \\ \quad +\alpha\int_0^1 v(x,t)b(x)dx \end{cases} \tag{4}$$

This observation gives an expression of the functional f. So we can consider the system with control and observation

$$\begin{cases} p_t(x,t) = q(x,t) - \alpha a(x)u(t),\ t > 0, \\ q_t(x,t) = p_{xx}(x,t) - \alpha b(x)u(t), \\ p(0,t) = 0, \quad q(0,t) = 0 \\ p_x(1,t) = -\beta u(t), \\ p(x,0) = p_0(x), \ \ q(x,0) = q_0(x) \\ y(t) = U(p,q) \\ = \beta q(1,t) + \alpha\int_0^1 p_x(x,t)a'(x)dx \\ \quad +\alpha\int_0^1 q(x,t)b(x)dx. \end{cases} \tag{5}$$

We take the feedback control $u(t)$ as

$$\begin{aligned} u(t) &= U(p,q) \\ &= \beta q(1,t) + \alpha\int_0^1 p_x(x,t)a'(x)dx \\ &\quad +\alpha\int_0^1 q(x,t)b(x)dx. \end{aligned} \tag{6}$$

Now acting this control signal on both systems (3) and (1) respectively, we get the closed loop systems

$$\begin{cases} p_t(x,t) = q(x,t) - \alpha a(x)U(p,q),\ t > 0, \\ q_t(x,t) = p_{xx}(x,t) - \alpha b(x)U(p,q), \\ p(0,t) = 0, \quad q(0,t) = 0, \\ p_x(1,t) = -\beta U(p,q), \\ p(x,0) = E_0(w_0,w_1)(x) + \beta\int_{-\tau}^0 a_0(x,s)\varphi(s)ds, \\ q(x,0) = E_1(w_0,w_1)(x) - \beta\int_{-\tau}^0 a_1(x,s)\varphi(s)ds. \end{cases} \tag{7}$$

and

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), t > 0 \\ w(0,t) = 0, \\ w_x(1,t) = -\alpha U(p,q)(t) - \beta U(p,q)(t-\tau) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x),\ x \in (0,1) \\ u(s) = \varphi(s),\ s \in (-\tau,0). \end{cases} \tag{8}$$

With control (6), the energy function of (7) defined by

$$E(p,q)(t) = \tfrac{1}{2}\int_0^1 [|p_x(x,t)|^2 + |q(x,t)|^2]dx$$

has property that $\frac{dE(p,q)(t)}{dt} = -U^2(p,q)(t)$.

In order to establish a relationship between (7) and (8), we consider the error of both systems

$$\begin{aligned} e(t,\tau) \ &= \ \tfrac{1}{2}\int_0^1 |w_x(x,t+\tau) - p_x(x,t)|^2dx \\ &\quad +\tfrac{1}{2}\int_0^1 |w_t(t+\tau) - q(x,t)|^2dx. \end{aligned}$$

We shall prove the following result.

Lemma 1 *There exists a positive constant* $M(\alpha,\tau)$ *such that*

$$\begin{aligned} &\tfrac{1}{2}\int_0^1 |w_x(x,t+\tau) - p_x(x,t)|^2 + |w_t(t+\tau) - q(x,t)|^2dx \\ &\le M(\alpha,\tau)\int_0^\tau |U(p,q)(t+s)|^2ds. \end{aligned}$$

Based on Lemma 1, we can prove the following result about the system (1).

Theorem 2 *Suppose that control is determined by (6), and let the energy function of the system (1) be defined as*

$$E(w,w_t) = \frac{1}{2}\int_0^1 [|w_x(x,t)|^2 + |w_t(x,t)|^2]dx.$$

Then the following statements are true:

1) If the energy of system (7) decays exponentially, then the energy of system (8) decays exponentially.

2) If the energy of system (7) decays asymptotically, then the energy of system (8) also decays asymptotically.

Theorem 2 shows that the stability of system (7) implies the stability of (8). So we only need to pay our attention to stability analysis of (7) in next step. The following is main result of this paper.

Theorem 3 *Let $\mu_n = (n - \frac{1}{2})\pi, n \in \mathbb{N}$. Then the following statements are true:*

1) If $|\alpha| \neq |\beta|$, then the energy of the system (7) decays exponentially for any $\tau > 0$.

2) When $|\alpha| = |\beta|$, the stability of the system (7) depends upon the time delay τ. More precisely, there are three cases:

(a)if $\min\{\inf_n |1 + e^{-i\mu_n\tau}|, \inf_n |1 - e^{-i\mu_n\tau}|\} > 0$, then the system (7) is exponentially stable;

(b) if $\min\{\inf_n |1+e^{-i\mu_n\tau}|, \inf_n |1-e^{-i\mu_n\tau}|\} = 0$, but $\tau \notin \{\tau > 0 \mid e^{-i\mu_n\tau} = \pm 1, \forall n \in \mathbb{N}\}$, then the system (7) is asymptotically stable;

(c) if $\tau \in \{\tau > 0 \mid e^{-i\mu_n\tau} = \pm 1, \forall n \in \mathbb{N}\}$, the system (7) is unstable.

Theorem 3 shows that the feedback control law (6) is desired, that stabilize exponentially system (1) provided that $|\alpha| \neq |\beta|$.

The rest is organized as follows. In section 2, we prove Lemma 1 and Theorem 2. In section 3, we shall prove theorem 3. Since it has a long and complex verification of some facts, we shall complete the proof by several subsections. In subsection 3.1, we derive the system (3) from (1) and (2) and give the explicit expression of the function $a(x), b(x), a_0(x,\theta)$ and $a_1(x,\theta)$, but the computations leave in Appendix. In subsection 3.2, we prove that the system (5) is L^2 well-posed system, this fact will be used in the discussion of the exponential stabilization. In subsection 3.3, we prove the exact observability of the system (3). In subsection 3.4, we prove the stabilization result in Theorem 3, especially the exponential stability. In section 4, we give a conclusion remark.

2 Proofs of Lemma 1 and Theorem 2

In this section, we shall prove the results of Lemma 1 and Theorem 2. Let $\mathcal{H} = H_E^1(0,1) \times L^2(0,1)$ where $H_E^k(0,1) = \{f \in H^k(0,1) \mid f(0) = 0\}$ be equipped with the inner product

$$((f,g),(w,v))_{\mathcal{H}} = \int_0^1 \left[f'(x)\overline{w'(x)} + g(x)\overline{v(x)}\right] dx.$$

Clearly, $\mathcal{H}$ is a Hilbert space. In the sequel we always work in the this space.

We begin with proving Lemma 1.

Proof of Lemma 1 Here we mainly estimate the error

$$\begin{array}{rcl} e(t,\tau) & = & \frac{1}{2}\int_0^1 |p_x(x,t) - w_x(x,t+\tau)|^2 dx \\ & & + \frac{1}{2}\int_0^1 |q(x,t) - w_t(x,t+\tau)|^2 dx \end{array}$$

Let $w(x,t)$ and $\widehat{w}(x,s,t)$ be the solutions of the system (1) and the auxiliary system (2) respectively, and let

$$e(x,s,t) = w(x,t+s) - \widehat{w}(x,s,t).$$

Clearly, $e(x,\tau,t) = w(x,t+\tau) - p(x,t)$, $e_s(x,\tau,t) = w_t(x,t+\tau) - q(x,t)$, and $e(x,s,t)$ satisfies the following equation

$$\begin{cases} e_{ss}(x,s,t) = e_{xx}(x,s,t), s \in (0,\tau),\ t > 0, \\ e(0,s,t) = 0, \\ e_x(1,s,t) = -\alpha u(t+s) \\ e(x,0,t) = e_s(x,0,t) = 0. \end{cases} \tag{9}$$

Putting

$$\begin{array}{rcl} e(t,s) & = & \frac{1}{2}\int_0^1 |w_x(x,t+s) - \widehat{w}_x(x,s,t)|^2 dx \\ & & + \frac{1}{2}\int_0^1 |w_s(x,t+s) - \widehat{w}_s(x,s,t)|^2 dx \end{array}$$

and

$$\rho(t,s) = \int_0^1 x e_s(x,s,t) e_x(x,s,t) dx,$$

we have $\frac{\partial e(t,s)}{\partial s} = e_s(1,s,t)e_x(1,s,t)$ and

$$\frac{\partial \rho(t,s)}{\partial s} = \frac{1}{2}|e_s(1,s,t)|^2 + \frac{1}{2}|e_x(1,s,t)|^2 - e(t,s).$$

Hence $e(t,s) = \int_0^s e_s(1,s,t)e_x(1,s,t)ds$ and

$$\begin{array}{rcl} \rho(t,s) & = & \frac{1}{2}\int_0^s |e_s(1,s,t)|^2 ds + \frac{1}{2}\int_0^s |e_x(1,s,t)|^2 ds \\ & & - \int_0^s e(t,s) ds \end{array}$$

where we have used $e(t,0) = \rho(t,0) = 0$. Therefore, for $\gamma > 1$ we have

$$\begin{array}{rl} & e(t,s) = \displaystyle\int_0^s e_s(1,s,t)e_x(1,s,t)ds \\ \leq & \displaystyle\frac{1}{\gamma}\int_0^s |e_s(1,s,t)|^2 + \gamma\int_0^s |e_x(1,s,t)|^2 ds \\ = & \displaystyle\frac{1}{\gamma}\left[\rho(t,s) + \int_0^s e(t,s)ds - \int_0^s |e_x(1,s,t)|^2 ds\right] \\ & \displaystyle+\gamma\int_0^s |e_x(1,s,t)|^2 ds \\ \leq & \displaystyle\frac{1}{\gamma}\left[e(t,s) + \int_0^s e(t,s)ds\right] \\ & \displaystyle+(\gamma - \frac{1}{\gamma})\int_0^s |e_x(1,s,t)|^2 ds \end{array}$$

where we used inequality

$$\rho(t,s) \le \frac{1}{2}\int_0^1 |e_x(s,t)|^2 + |e_s(x,s,t)|^2 dx = e(t,s).$$

Thus the Gronwall inequality gives

$$e(t,\tau) \le \left[\frac{(\gamma+1)}{(\gamma-1)}\int_0^\tau e^{\frac{\tau-\mu}{\gamma-1}} d\mu + (\gamma+1)\right]\int_0^\tau |e_x(1,r,t)|^2 dr$$

Taking $\gamma = 2$ and $e_x(1,s,t) = -\alpha U(p,q)$ yields

$$e(t,\tau) \le [3e^\tau + 3]\,|\alpha^2| \int_0^\tau |u(t+s)|^2 ds.$$

Therefore, we have

$$\frac{1}{2}\int_0^1 |w_x(x,t+\tau) - p_x(x,t)|^2 + |w_t(x,t+\tau) - q(x,t)|^2 dx \le M(\alpha,\tau)\int_0^\tau |U(p,q)(t+s)|^2 ds$$

where $M(\alpha,\tau) = 3\alpha^2(1+e^\tau)$. The proof of Lemma 1 is then complete. ☐

Remark 4 *Note that* $p(x,t) = \widehat{w}(x,\tau,t)$ *and* $q(x,t) = \widehat{w}_s(x,\tau,t)$. *The control signal actually is*

$$u(t) = \begin{cases} \beta\widehat{w}_s(1,\tau,t) + \alpha\int_0^1 \widehat{w}_s(x,\tau,t)b(x)dx \\ +\alpha\int_0^1 \widehat{w}_x(x,\tau,t)a'(x)dx, \quad t \in (0,+\infty) \\ \varphi(\theta), \quad \theta \in [-\tau, 0]. \end{cases}$$

Proof of Theorem 2 We define the energy function of the system (7) as

$$E(p,q)(t) = \frac{1}{2}\int_0^1 [|p_x(x,t)|^2 + |q(x,t)|^2]dx.$$

then we have

$$\begin{aligned} &\frac{dE(p,q)}{dt} \\ =& \int_0^1 [p_x(x,t)p_{xt}(x,t) + q(x,t)q_t(x,t)]dx \\ =& \int_0^1 p_x(x,t)(q_x(x,t) - \alpha a'(x)U(p,q))dx \\ & + q(x,t)(p_{xx}(x,t) - \alpha b(x)U(p,q))dx \\ =& \int_0^1 [p_x(x,t)q_x(x,t) + q(x,t)p_{xx}(x,t)]dx \\ & -\alpha U(p,q)\int_0^1 [p_x(x,t)a'(x) + q(x,t)b(x)]dx \\ =& -U^2(p,q) \end{aligned}$$

where we have used the boundary conditions $p_x(1,t) = -\beta U(p,q)$ and $q(0,t) = 0$. Thus it holds that

$$\int_0^\tau U^2(p,q)(t+s)ds = E(p,q)(t) - E(p,q)(t+\tau).$$

Further, we have

$$\begin{aligned} &\frac{1}{2}\int_0^1 [|w_x(x,t+\tau)|^2 + |w_t(x,t+\tau)|^2]dx \\ \le\ & 2\frac{1}{2}\int_0^1 [|p_x(x,t)|^2 + |q(x,t)|^2]dx \\ & + \frac{1}{2}\int_0^1 |w_x(,t+\tau) - p_x(x,t)|^2 dx \\ & + \frac{1}{2}\int_0^1 |w_t(x,t+\tau) - q(x,t)|^2 dx \\ \le\ & 2E(p,q)(t) + 2M(\alpha,\tau)\int_0^\tau U^2(p,q)(t+s)ds \\ =\ & 2E(p,q)(t) + 2M(\alpha,\tau)[E(p,q)(t) - E(p,q)(t+\tau)]. \end{aligned}$$

The assertions of Theorem 2 are followed from the above inequality. ☐

3 Proof of Theorem 3

In this section, we shall prove the results of Theorem 3. Since the proof contains a complex procedure, we shall divide it into several subsections and then use a series propositions to complete it.

3.1 Coefficients in equation (3)

From introduction we see that the key point of this approach is to determine the property of functions $a(x), b(x), a_0(x,\theta)$ and $a_1(x,\theta)$ appearing in (3) when we translate the system (1) into (3). In this subsection, we introduce the property of the coefficients in Eqs. (3), but the complex computations will be postponed in appendix.

Let $\mu_n = n\pi - \frac{\pi}{2}, n \in \mathbb{N}$. We observe that $\pm i\mu_n, n \in \mathbb{N}$ are the eigenvalues of the system (1) without control and $\varphi_n(x) = \sqrt{2}\sin\mu_n x$ are the corresponding eigenfunctions. Further, the following properties are true.

Lemma 5 *Let* $\mu_n = (n - \frac{1}{2})\pi, n \in \mathbb{N}$ *and* $\varphi_n(x) = \sqrt{2}\sin\mu_n x$. *Then it holds that*

$$\int_0^1 \varphi_n(x)\varphi_m(x)dx = \delta_{nm}$$

and

$$\int_0^1 |\varphi_n'(x)|^2 dx = \mu_n^2, \quad |\varphi_n(1)| = \sqrt{2}.$$

The family $\{\varphi_n; n \in \mathbb{N}\}$ *is a normalized orthogonal basis for* $L^2[0,1]$.

Lemma 6 *Let $a(x)$, $b(x)$, $a_0(x,\theta)$, $a_0(x,\theta)$ be the functions appearing in (3), and $E(w_0,w_1)(x)$ and $E_1(w_0,w_1)(x)$ be the operators in (3). Then we have*

$$\begin{cases} a(x) = \sum\limits_{n=1}^{\infty} [\varphi_n(1)\sin\mu_n\tau]\frac{\varphi_n(x)}{\mu_n}, \\ b(x) = \sum\limits_{n=1}^{\infty} [\varphi_n(1)\cos\mu_n\tau]\varphi_n(x), \\ a_0(x,\theta) = \sum\limits_{n=1}^{\infty} [\varphi_n(1)\sin\mu_n\theta]\frac{\varphi_n(x)}{\mu_n}, \\ a_1(x,\theta) = \sum\limits_{n=1}^{\infty} [\varphi_n(1)\cos\mu_n\theta]\varphi_n(x), \end{cases} \tag{10}$$

and

$$\begin{cases} E_0(w_0,w_1)(x) \\ = \sum\limits_{n=0}^{\infty} \left[a_n^{(0)}\cos\mu_n\tau + a_n^{(1)}\frac{\sin\mu_n\tau}{\mu_n}\right]\varphi_n(x), \\ E_1(w_0,w_1)(x) \\ = \sum\limits_{n=0}^{\infty} \left[-a_n^{(0)}\mu_n\sin\mu_n\tau + a_n^{(1)}\cos\mu_n\tau\right]\varphi_n(x) \end{cases} \tag{11}$$

where

$$a_n^{(0)} = \int_0^1 w_0(x)\varphi_n(x)dx, \quad a_n^{(1)} = \int_0^1 w_1(x)\varphi_n(x)dx.$$

The proof of Lemma 6 has a complex calculation, we leave it in appendix.

3.2 L^2 well-posed of the system (5)

In this subsection we shall prove the well-posed-ness of the system (5). Firstly let us recall the well-posed system.

Definition 7 *Let H, U and Y be Hilbert spaces and $A : D(A) \subset H \to H$ be the generator of C_0 semigroup $T(t)$ on H. Denote H_{-1} by the completion space of H in norm $||R(\lambda,A)x||_H$. Let $B : U \to H_{-1}$ and $C : D(A) \to Y$ be the linear operators for $T(t)$. Consider the linear system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \in H \\ y(t) = Cx(t) + Du(t) \end{cases}$$

If for any $t > 0$, there exists a positive constants M_t such that for all $u \in L^2_{loc}(\mathbb{R}_+, U)$, it holds that

$$||x(t)||^2 + \int_0^t |y(s)|^2 ds \le M_t(||x_0||^2 + \int_0^t ||u(s)||^2 ds),$$

then the linear system is said to be well-posed one.

The following proposition we need the admissibility of the operators B and C for $T(t)$, for their definition we refer to the papers [10],[11], also see [12] or [13].

Proposition 8 *If B and C are admissible for $T(t)$, then the system is well-posed if and only if for any $T > 0$, there exists $M_T > 0$ such that for all $u \in L^2_{loc}(\mathbb{R}_+, U)$,*

$$\int_0^T \left| C\int_0^t T(t-s)Bu(s)ds \right|^2 dt \le M_T \int_0^T ||u(s)||^2 ds.$$

The following proposition give an equivalent condition.

Proposition 9 *[14] Suppose that B and C are admissible for $T(t)$. Let $h(\lambda)$ be the transform function from u to y. Then the linear system is well-posed if and only if there exists $\eta > 0$ such that*

$$\sup_{\Re\lambda \ge \eta} ||h(\lambda)|| < \infty.$$

In what follows, we shall prove the well-posed-ness of the system (5) based on proposition 9.

Theorem 10 *The (5) is a well-posed linear system.*

Proof. Taking the Laplace transform for equation (5) with $w_0 = w_1 = 0$, we get

$$\begin{array}{l} \lambda\widehat{p}(x,\lambda) = \widehat{q}(x,\lambda) - \alpha a(x)\widehat{u}(\lambda) \\ \lambda\widehat{q}(x,\lambda) = \widehat{p}_{xx}(x,\lambda) - \alpha b(x)\widehat{u}(\lambda) \\ \widehat{p}(0,\lambda) = 0, \quad \widehat{p}_x(1,\lambda) = -\beta\widehat{u}(\lambda) \\ \widehat{y}(\lambda) = \beta\widehat{q}(1,\lambda) \\ \qquad +\alpha\int_0^1 a'(x)\widehat{p}_x(x,\lambda) + b(x)\widehat{q}(x,\lambda)dx \end{array} \tag{12}$$

Thanks to $\{\varphi_n, n \in \mathbb{N}\}$ being an orthogonal basis for $L^2[0,1]$, we can set

$$\widehat{p}(x,\lambda) = \sum_{n=1}^{\infty} (\widehat{p},\varphi_n)\varphi_n, \quad \widehat{q}(x,\lambda) = \sum_{n=1}^{\infty} (\widehat{q},\varphi_n)\varphi_n.$$

From (12) we get

$$\begin{array}{l} \lambda(\widehat{p},\varphi_n) = (\widehat{q},\varphi_n) - \alpha\varphi_n(1)\frac{\sin\mu_n\tau}{\mu_n}\widehat{u}(\lambda) \\ \lambda(\widehat{q},\varphi_n) = -\mu_n^2(\widehat{p},\varphi_n) - [\alpha\cos\mu_n\tau + \beta]\widehat{u}(\lambda)\varphi_n(1) \end{array} \tag{13}$$

Thus we have

$$(\widehat{p},\varphi_n) = \frac{-\varphi_n(1)}{\lambda^2 + \mu_n^2}[\alpha\lambda\frac{\sin\mu_n\tau}{\mu_n} + (\alpha\cos\mu_n\tau + \beta)]\widehat{u}(\lambda)$$

and

$$\begin{array}{l} (\widehat{q},\varphi_n) = \frac{\mu_n\varphi_n(1)}{\lambda^2+\mu_n^2}[\alpha\sin\mu_n\tau]\varphi_n(1)\widehat{u}(\lambda) \\ + \frac{-\lambda\varphi_n(1)}{\lambda^2+\mu_n^2}[\alpha\cos\mu_n\tau + \beta]\widehat{u}(\lambda). \end{array}$$

Now we calculate $\widehat{y}(\lambda)$:

$$\begin{aligned}\widehat{y}(\lambda) &= \sum_{n=1}^{\infty}\beta(\widehat{q},\varphi_n)\varphi_n(1)\\ &\quad+\alpha(\widehat{p},\varphi_n)\int_0^1 a'(x)\varphi_n'(x)dx\\ &\quad+\alpha(\widehat{q},\varphi_n)\int_0^1 b(x)\varphi_n(x)dx\\ &= \sum_{n=1}^{\infty}[\alpha\cos\mu_n\tau+\beta](\widehat{q},\varphi_n)\varphi_n(1)\\ &\quad+\sum_{n=1}^{\infty}\alpha(\widehat{p},\varphi_n)\mu_n\sin\mu_n\tau\varphi_n(1)\\ &= -2\lambda\sum_{n=1}^{\infty}\frac{[\alpha\cos\mu_n\tau+\beta]^2+(\alpha\sin\mu_n\tau)^2}{\lambda^2+\mu_n^2}\widehat{u}(\lambda)\end{aligned}$$

So the transform function is

$$h(\lambda)=-2\lambda\sum_{n=1}^{\infty}\frac{[\alpha\cos\mu_n\tau+\beta]^2+(\alpha\sin\mu_n\tau)^2}{\lambda^2+\mu_n^2}.$$

A straightforward calculation shows that $\sup_{\Re\geq 1}|h(\lambda)| < \infty$. Therefore, the assertion follows from proposition 9. □

3.3 The exact observability of the system (3)

The duality theory shows that observation system corresponding to (3) is (4), i.e.,

$$\begin{cases} w_t(x,t)=v(x,t), \quad 0<x<1, t>0\\ v_t(x,t)=w_{xx}(x,t)\\ w(0,t)=w_x(1,t)=0\\ w(x,0)=w_0(x), v(x,0)=v_0(x)\\ y(t)=U(w,v)\\ =\beta v(1,t)+\alpha\int_0^1 w_x(x,t)a'(x)dx\\ \quad+\alpha\int_0^1 v(x,t)b(x)dx\end{cases}\tag{14}$$

Theorem 11 *Let $\mathcal{H}$ be defined as before. If $|\alpha|\neq|\beta|$ or $|\alpha|=|\beta|$ and $\tau>0$ satisfies condition*

$$\min\{\inf_n|1-e^{-i\mu_n\tau}|,\inf_n|1+e^{-i\mu_n\tau}|\}=\delta>0,$$

then the system (14) is exactly observable for any $T>2$.

Proof. Since $\{\varphi_n; n\in\mathbb{N}\}$ is a normalized orthogonal basis for $L^2[0,1]$, we can expand $w_0(x)$ and $w_1(x)$ with $(w_0,w_1)\in\mathcal{H}$ into the fourier series

$$w_0(x)=\sum_{n=1}^{\infty}a_n^{(0)}\varphi_n, \quad w_1(x)=\sum_{n=1}^{\infty}a_n^{(1)}\varphi_n.$$

So we have

$$w(x,t)=\sum_{n=1}^{\infty}[a_n^{(0)}\cos\mu_n t+a_n^{(1)}\frac{\sin\mu_n t}{\mu_n}]\varphi_n(x)$$

and

$$\begin{aligned}v(x,t)&=w_t(x,t)\\ &=\sum_{n=1}^{\infty}[-a_n^{(0)}\mu_n\sin\mu_n t+a_n^{(1)}\cos\mu_n t]\varphi_n(x)\end{aligned}$$

A simple calculation gives

$$\begin{aligned}y(t)&=\beta v(1,t)+\alpha\int_0^1[a'(x)w_x(x,t)+b(x)v(x,t)]dx\\ &=\sum_{n=1}^{\infty}\varphi_n(1)[-(\alpha\cos\mu_n\tau+\beta)\mu_n a_n^{(0)}\\ &\qquad+\alpha a_n^{(1)}\sin\mu_n\tau]\sin\mu_n t\\ &+\sum_{n=1}^{\infty}\varphi_n(1)[(\alpha\cos\mu_n\tau+\beta)a_n^{(1)}\\ &\qquad+\alpha a_n^{(0)}\mu_n\sin\mu_n\tau]\cos\mu_n t\end{aligned}$$

Note that the function family $\{\sin\mu_n t,\cos\mu_n t; n\in\mathbb{N}\}$ is an orthogonal basis for $L^2[0,2]$, so we have

$$\int_0^2|y(t)|^2dt=2\sum_{n=1}^{\infty}|\beta+\alpha e^{-\mu_n\tau}|^2[|a_n^{(1)}|^2+|\mu_n a_n^{(0)}|^2].$$

Clearly, when $|\alpha|\neq|\beta$, we have $|\beta+\alpha e^{-\mu_n\tau}|^2\geq(|\beta|-|\alpha|)^2=\delta^2>0$ and hence

$$\begin{aligned}\int_0^2|y(t)|^2dt &> 2\delta^2\sum_{n=1}^{\infty}[|a_n^{(1)}|^2+|\mu_n a_n^{(0)}|^2]\\ &= 2\delta^2||(w_0,w_1)||^2_{\mathcal{H}}.\end{aligned}$$

If $|\alpha|=|\beta|$ and τ satisfy the condition $\min\{\inf_n|1+e^{-i\mu_n\tau}|,\inf_n|1-e^{-i\mu_n\tau}|\}=\delta>0$, then we also have

$$\begin{aligned}\int_0^2|y(t)|^2dt &> 2\delta^2\sum_{n=1}^{\infty}[|a_n^{(1)}|^2+|\mu_n a_n^{(0)}|^2]\\ &= 2\delta^2||(w_0,w_1)||_{\mathcal{H}}.\end{aligned}$$

In both cases, the system is exactly observable for $T>2$. □

3.4 The stabilization of the system (7)

In this subsection we shall discuss the exponential stability of the closed loop system (7). For this aim, we define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\begin{cases}\mathcal{A}(f,g)=(g-\alpha a(x)U(f,g), f''(x)-\alpha b(x)U(f,g)),\\ D(\mathcal{A})=\left\{\begin{array}{l}(f,g)\in\mathcal{H},\mathcal{A}(f,g)\in\mathcal{H}\mid\\ f(0)=0, f'(1)=-\beta U(f,g)\\ U(f,g)=\beta g(1)+\alpha((f,g),(a,b))_{\mathcal{H}}\end{array}\right\}.\end{cases}\tag{15}$$

Thus the closed loop system (7) can be written into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dY(t)}{dt} = \mathcal{A}Y(t), t > 0 \\ Y(0) = Y_0. \end{cases} \tag{16}$$

where $Y(t) = (p(x,t), q(x,t))$ and $Y(0) = (p(x,0), q(x,0)) \in \mathcal{H}$.

Firstly, we show the system (16) or (7) is well-posed in $\mathcal{H}$.

Proposition 12 *Let $\mathcal{A}$ be defined as (16). Then the following statements are true.*

1) $\mathcal{A}$ is a closed and densely defined linear operator in $\mathcal{H}$;

2) $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative operators. Hence $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$ which implies the system (16) is well-posed.

Proof. The density of $D(\mathcal{A})$ in $\mathcal{H}$ is obvious, we only prove the second assertion. According to the definition of $\mathcal{A}$, for $(f,g) \in D(\mathcal{A})$ and $(w,v) \in D(\mathcal{A}^*)$, we have

$$\begin{aligned}
&(\mathcal{A}(f,g),(w,v))_{\mathcal{H}} = ((f,g),\mathcal{A}^*(w,v))_{\mathcal{H}} \\
=\ & ((g - \alpha a(x)U(f,g), f'' - \alpha b(x)U(f,g)),(w,v))_{\mathcal{H}} \\
=\ & \int_0^1 (g' - \alpha a'(x)U(f,g))\overline{w'(x)}dx \\
& + \int_0^1 ((f''(x) - \alpha b(x)U(f,g))\overline{v(x)}dx \\
=\ & -\int_0^1 g(x)\overline{w''(x)}dx - \int_0^1 f'(x)\overline{v'(x)}dx \\
& -U(f,g)\overline{U(w,v)} + g(1)\overline{w'(1)} \\
=\ & -\int_0^1 f'(x)\overline{(v'(x) + \alpha a'(x)U(w,v))}dx \\
& -\int_0^1 g(x)\overline{(w''(x) + \alpha b(x)U(w,v))}dx \\
& +g(1)\overline{[w'(1) + \beta U(w,v)]}
\end{aligned}$$

Thus we get

$$\begin{cases} \mathcal{A}^*(w,v) \\ = -(v(x) + \alpha a(x)U(w,v), w''(x) + \alpha b(x)U(w,v)), \\ D(\mathcal{A}^*) = \left\{ \begin{array}{l} (w,v) \in \mathcal{H},\ \mathcal{A}^*(w,v) \in \mathcal{H} \mid \\ w'(1) = -\beta U(w,v) \\ U(w,v) = \beta v(1) \\ +\alpha((w,v),(a,b))_{\mathcal{H}} \end{array} \right\}. \end{cases} \tag{17}$$

Furthermore, for any real $F = (f,g) \in D(\mathcal{A})$, $W = (w,v) \in D(\mathcal{A}^*)$, we have

$$(\mathcal{A}F,F)_{\mathcal{H}} = -U^2(f,g) \le 0$$

$$\begin{aligned}
&(\mathcal{A}^*W,W)_{\mathcal{H}} \\
=\ & -\left(\left(v(x) + \alpha a(x)U, w''(x) + \alpha b(x)U\right),(f,g)\right)_{\mathcal{H}} \\
=\ & -[w'(1)v(1) + \alpha U \int_0^1 a'(x)w'dx \\
& +\alpha U \int_0^1 b(x)v(x)dx] \\
=\ & -U\left(\beta v(1) + \alpha\left((w,v),(a,b)\right)_{\mathcal{H}}\right) \\
=\ & -U^2(w,v) \le 0.
\end{aligned}$$

These inequalities imply that both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative operator. So $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$ (see [15]). Therefore the closed loop system (13) is well-posed. □

Remark 13 *$(f,g) \in D(\mathcal{A})$ implies that $f(x)$ is continuous differential on $[0,1]$ and $f''(x) - \alpha b(x)U \in L^2[0,1]$, the $g(x)$ is continuous on $[0,1]$ and $g(0) = 0$, and $g'(x) - \alpha a'(x)U \in L^2[0,1]$. So for any $(f,g) \in D(\mathcal{A})$, the inner product*

$$((f,g),(a,b))_{\mathcal{H}} = \int_0^1 [f'(x)\overline{a'(x)}dx + g(x)\overline{b(x)}]dx$$

is meaningful, although $(a,b) \notin \mathcal{H}$.

The following proposition gives the spectral property of $\mathcal{A}$.

Proposition 14 *Let $\mathcal{A}$ be defined as (13) and $\mu_n = (n - \frac{1}{2})\pi$. Then the following assertions hold:*

1) $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact operator on $\mathcal{H}$;

2) If $|\alpha| \ne |\beta|$, then there is no eigenvalue of $\mathcal{A}$ on the imaginary axis for any $\tau > 0$;

3) If $|\alpha| = |\beta|$, there is the following two cases:

a). Define sets

$$S_u^+(\alpha\beta > 0) = \{\tau > 0 \mid \exists n \in \mathbb{N}, s.t.\ e^{-i\mu_n\tau} = -1\},$$

$$S_u^-(\alpha\beta < 0) = \{\tau > 0 \mid \exists n \in \mathbb{N}, s.t.\ e^{-i\mu_n\tau} = 1\}$$

and $S_u = S_u^+(\alpha\beta > 0) \cup S_u^-(\alpha\beta < 0)$. If $\tau \in S_u$, then there are at least two spectral points of $\mathcal{A}$ on the imaginary axis.

b). Let

$$S^+(\alpha\beta > 0) = \left\{ \begin{array}{l} \tau > 0, \tau \notin S_u^+(\alpha\beta > 0) \mid \\ \inf_{n\in\mathbb{N}} |e^{-i\mu_n\tau} + 1| = 0 \end{array} \right\}$$

$$S^-(\alpha\beta < 0) = \left\{ \begin{array}{l} \tau > 0, \tau \notin S_u^-(\alpha\beta < 0) \mid \\ \inf_{n\in\mathbb{N}} |e^{-i\mu_n\tau} - 1| = 0 \end{array} \right\}$$

and $S = S^+(\alpha\beta > 0) \cup S^-(\alpha\beta < 0)$. If $\tau \in S$, then there is no spectrum of $\mathcal{A}$ on the imaginary axis, but the imaginary axis may be an asymptote of $\sigma(\mathcal{A})$.

Proof. The proof is completed by following four steps.

Step 1. $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact operator on $\mathcal{H}$.

For any $(w, v) \in \mathcal{H}$, we consider equation $\mathcal{A}(f, g) = (w, v)$, i.e.,

$$\begin{cases} g(x) - \alpha a(x)U(f,g) = w(x) \\ f''(x) - \alpha b(x)U(f,g) = v(x) \\ f(0) = 0 \\ f'(1) = -\beta U(f,g) \\ U(f,g) = \beta g(1) + \alpha \int_0^1 f'(x)a'(x)dx \\ \quad +\alpha \int_0^1 g(x)b(x)dx \end{cases}$$

Clearly, $g(x) = \alpha a(x)U(f, g) + w(x)$ and $f(x)$ satisfies differential equation

$$\begin{array}{l} f''(x) = \alpha b(x)U(f,g) + v(x) \\ f(0) = 0, \quad f'(1) = -\beta U(f,g). \end{array}$$

Solving the differential equation we get

$$f(x) = -\beta U(f,g)x - \mathcal{L}^{-1}v(x) - \alpha U(f,g)\mathcal{L}^{-1}b(x)$$

where

$$\mathcal{L}^{-1}v = \int_0^x dt \int_s^1 v(y)dy.$$

Inserting $g(x)$ and $f(x)$ into $U(f, g)$ yields

$$\begin{aligned} & U(f,g) = \beta g(1) + \alpha \int_0^1 g(x)b(x)dx \\ & \quad +\alpha \int_0^1 f'(x)a'(x)dx \\ = \; & \beta w(1) + \alpha\beta a(1)U(f,g) + \alpha \int_0^1 w(x)b(x)dx \\ & +\alpha^2 U(f,g) \int_0^1 a(x)b(x)dx \\ & -\alpha\beta U(f,g) \int_0^1 a'(x)dx \\ & -\alpha \int_0^1 a'(x)dx \int_x^1 v(r)dr \\ & -\alpha^2 U(f,g) \int_0^1 a'(x)dx \int_x^1 b(s)ds \\ = \; & \beta w(1) + \alpha \int_0^1 w(x)b(x)dx + \alpha \int_0^1 v(x)a(x)dx \end{aligned}$$

here we have used $a(0) = 0$. Set

$$\begin{aligned} & K(w,v) \\ & = \beta w(1) + \alpha \int_0^1 w(x)b(x)dx + \alpha \int_0^1 a(r)v(r)dr \end{aligned}$$

and $\mathcal{A}^{-1}$ can be written as

$$\begin{aligned} & \mathcal{A}^{-1}(w,v) = (f(x), g(x)) \\ = \; & \left(-\mathcal{L}^{-1}v(x), w(x)\right) \\ & -K(w,v)\left(\beta x + \alpha\mathcal{L}^{-1}b(x), -\alpha a(x)\right)\right) \end{aligned}$$

Therefore, $\mathcal{A}^{-1}$ is a compact operator on $\mathcal{H}$.

Step 2. If $|\alpha| \neq |\beta|$, then there is no eigenvalue of $\mathcal{A}$ on the imaginary axis for any $\tau > 0$, i.e., $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

According to the step 1, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ consists of all isolated eigenvalues of finite multiplicity. So we only need to consider the eigenvalue problem of $\mathcal{A}$. We shall show that $\mathcal{A}(f, g) = \lambda(f, g)$ for any $\lambda = ir, r \in \mathbb{R}$ only has unique zero solution.

Let $W = (f, g) \in D(\mathcal{A})$ satisfy equation $\mathcal{A}W = \lambda W$ for some $\lambda = ir, r \in \mathbb{R}$. Then we have from dissipative of $\mathcal{A}$

$$0 = \Re(\mathcal{A}W, W)_{\mathcal{H}} = -|U(f,g)|^2,$$

this implies $U(f, g) = 0$. Then the equation $\mathcal{A}W = \lambda W$ can be written as follows:

$$\begin{cases} g(x) = \lambda f(x) \\ f''(x) = \lambda g(x) \\ f(0) = f'(1) = 0 \\ 0 = U(f,g) = \beta g(1) + \alpha \int_0^1 f'(x)a'(x)dx \\ +\alpha \int_0^1 g(x)b(x)dx \end{cases}$$

So $f(x)$ should satisfy the equation $f''(x) = \lambda^2 f(x)$ with boundary condition $f(0) = f'(1) = 0$. The equation has nonzero solution if and only if $r = \mu_n$ for some $n \in \mathbb{N}$. We can take $f(x) = \gamma\varphi_n(x)$ and then $g(x) = i\mu_n\gamma\varphi_n(x)$ where γ is a constant.

Now we calculate $U(f, g)$ as follows

$$\begin{aligned} U(f,g) \;=\; & \beta i\mu_n\gamma\varphi_n(1) + \alpha i\mu_n\gamma \int_0^1 \varphi_n(x)b(x)dx \\ & +\alpha\gamma \int_0^1 \varphi_n'(x)a'(x)dx \end{aligned}$$

according to the expression of $a(x)$ and $b(x)$ in (10), we can get

$$\int_0^1 \varphi_n(x)b(x)dx = \varphi_n(1)\cos\mu_n\tau$$

$$\begin{aligned} & \textstyle\int_0^1 \varphi_n'(x)a'(x)dx = -\int_0^1 a(x)\varphi_n''(x)dx \\ & = \mu_n^2 \textstyle\int_0^1 a(x)\varphi_n(x)dx = \mu_n\varphi_n(1)\sin\mu_n\tau \end{aligned}$$

So

$$U(f,g) = i\mu_n\gamma\varphi_n(1)\left[\beta + \alpha e^{-i\mu_n\tau}\right]$$

Clearly, if $|\alpha| \neq |\beta|$, the equation $U(f, g) = 0$ has uniquely a solution $\gamma = 0$ that implies $f(x) = g(x) = 0$. Therefore, there is no spectrum of $\mathcal{A}$ on the imaginary axis.

Step 3. If $|\alpha| = |\beta|$ and $\tau \in S_u$, then there are at least two eigenvalues of $\mathcal{A}$ on the imaginary axis.

In fact, if $\alpha = \beta$ and $\tau \in S_u^+(\alpha\beta > 0)$, then there exists at least one $n \in \mathbb{N}$ such that $e^{-i\mu_n\tau} = -1$. In this case, we can take functions

$$f_n(x) = \frac{\varphi_n(x)}{\mu_n}, \quad g(x) = i\varphi_n(x),$$

which implies that $U(f_n, g_n) = i\alpha\varphi_n(1)\left[1 + e^{-i\mu_n\tau}\right] = 0$. Hence, $(f_n, g_n) \in D(\mathcal{A})$ and $\mathcal{A}(f_n, g_n) = i\mu_n(f_n, g_n)$. Similarly, we have $U(f_n, -g_n) = 0$ and $(f_n, -g_n) \in D(\mathcal{A})$ and $\mathcal{A}(f_n, -g_n) = -i\mu_n(-f_n, g_n)$. We can prove similar result for $\alpha = \beta$ and $\tau \in S_u^-(\alpha\beta < 0)$.

Step 4. If $|\alpha| = |\beta|$ and $\tau \in S$, then there is no spectrum of $\mathcal{A}$ on the imaginary axis, but the imaginary axis may be an asymptote of $\sigma(\mathcal{A})$.

For $|\alpha| = |\beta|$ and $\tau \in S$, there are also two cases $\alpha = \beta$ and $\alpha = -\beta$. We only consider $\alpha = \beta$ and the other case will be proved similarly. When $\alpha = \beta$, according to step 3, no spectrum of $\mathcal{A}$ exists on the imaginary axis, but the imaginary axis may be an asymptote of $\sigma(\mathcal{A})$. Here the computations are omitted. ☐

As a consequence of proposition 9 and stability of semigroup(see [16],or [17]), we have the following stability result for (7)

Corollary 15 *If $|\alpha| \neq |\beta|$ or $|\alpha| = |\beta|$ and $\tau \in S$, the system (15) is asymptotically stable.*

Finally we prove the exponential stability of the system (7) under the condition in Theorem 11.

Proposition 16 *Let $\mathcal{A}$ be defined as (15). If $|\alpha| \neq |\beta|$ or $|\alpha| = |\beta|$ and τ satisfying the condition $\min\{\inf_n |1 + e^{-i\mu_n\tau}|, \inf_n |1 - e^{-i\mu_n\tau}|\} > 0$, then the system (15) is exponentially stable.*

Proof. Let $\mathcal{A}$ be defined as (15) and $\mathcal{A}_0$ be the operator with $\alpha = \beta = 0$. Then we can write $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}_0 - U^*U$ where U is given in (6). Let $T(t)$ be the semigroup generated by $\mathcal{A}_0$ and $S(t)$ be the semigroup generated by $\mathcal{A}$. The for any $W_0 = (w_0, w_1) \in \mathcal{H}$, it holds that

$$S(t)W_0 = T(t)W_0 - \int_0^t T(t-s)U^*US(s)W_0 ds.$$

Thus for $T \geq 2$ we have

$$\begin{aligned}&\int_0^T |U(T(t)W_0|^2 dt \\ &\leq 2\int_0^T |U(S(t)W_0)|^2 ds \\ &+2\int_0^T \left|U\left(\int_0^t T(t-s)U^*U(S(s)W_0)ds\right)\right|^2 dt.\end{aligned}$$

The property of the well-posed linear system implies (see Theorem 10)

$$\begin{aligned}&\int_0^T \left|U\left(\int_0^t T(t-s)U^*U(S(s)W_0)ds\right)\right|^2 dt \\ &\leq M_T \int_0^T |U(S(s)W_0)|^2 ds,\end{aligned}$$

this leads to

$$\int_0^T |U(T(t)W_0)|^2 dt \leq (2+2M_T)\int_0^T |U(S(t)W_0)|^2 ds.$$

If $|\alpha| \neq |\beta|$ or $|\alpha| = |\beta|$ and τ such that

$$\min\{\inf_n |1 - e^{-\mu_n\tau}|, \inf_n |1 + e^{-\mu_n\tau}|\} = \delta > 0,$$

i.e., the conditions in Theorem 11 are fulfilled, then the exact observability implies

$$\begin{aligned}&(2 + 2M_T)\int_0^T |U(S(t)W_0)|^2 ds \\ &\geq \int_0^T |U(T(t)W_0)|^2 dt \geq 2\delta^2 ||W_0||^2.\end{aligned}$$

Note that for any $W_0 \in \mathcal{H}$ and for any $t > 0$, we have

$$\frac{1}{2}||S(t)W_0||^2 + \int_0^t ||U(S(s)W_0)|^2 = \frac{1}{2}||W_0||^2.$$

From this we deduce that

$$||S(T)W_0||^2 \leq \left(1 - \frac{\delta^2}{1 + M_T}\right)||W_0||^2,$$

this implies the exponential stability of (15) or (7). The proof is then complete. ☐

4 Conclusion remark

In this paper, we introduce a new control strategy for 1-d wave system with input delay in the boundary control. For any time delay $\tau > 0$, the new control strategy can stabilize the system exponentially provided that $|\alpha| \neq |\beta|$. When $|\alpha| = |\beta|$, the stability of the system with the new control law depends on the time delay. According to the proof we can get that the system is exponentially stable if $\tau \notin S \cup S_u$, and asymptotically stable if $\tau \in S$. So the new control law has a more better action in anti-time delay.

A Appendix: Proof of Lemma 6

Let $\mathcal{H} = H_E^2(0,1) \times L^2(0,1)$ where $H_E^2 = \{f \in H^2(0,1) | f(0) = f'(0) = 0\}$. Based on lemma 5, we can prove the following result.

Theorem 17 *For any $(w_0, w_1) \in \mathcal{H}$ and $u(t) \in L^2_{loc}(-\tau, \infty)$, the solution of (1) can be written as*

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x),$$

$$w_t(x,t) = \sum_{n=1}^{\infty} a_{n,t}(t)\varphi_n(x).$$

where

$$\begin{aligned} a_n(t) &= a_n^{(0)} \cos\mu_n t + a_n^{(1)} \sin\mu_n t \\ &+\varphi_n(1) \textstyle\int_0^t \sin\mu_n(t-s)\,[\alpha u(s) + \beta u(s-\tau)]\,ds, \end{aligned}$$

and $a_{n,t}(t)$ denote the derivation of $a_n(t)$ and

$$a_n^{(0)} = \int_0^1 w_0(x)\varphi_n(x)dx,\ a_n^{(1)} = \int_0^1 w_1(x)\varphi_n(x)dx.$$

Proof. For any $(w_0, w_1) \in \mathcal{H}$, we define the sequences by

$$a_n^{(0)} = \int_0^1 w_0(x)\varphi_n(x)dx,\ a_n^{(1)} = \int_0^1 w_1(x)\varphi_n(x).$$

Obviously, $\{a_n^{(0)}\}$, $\{a_n^{(0)}\} \in \ell^2$ due to $\{\varphi_n(x); n \in \mathbb{N}\}$ being an orthogonal basis for $L^2[0,1]$. Further, since $w_0 \in H^1(0,1)$, we have

$$\begin{aligned} \mu_n a_n^{(0)} &= \mu_n \textstyle\int_0^1 w_0(x)\varphi_n(x)dx \\ &= \textstyle\int_0^1 w_0'(x)\frac{\varphi_n'(x)}{\mu_n}dx \end{aligned}$$

and hence $\{\mu_n a_n^{(0)}\} \in \ell^2$.

Set

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$$

where $a_n(t) = \int_0^1 w(x,t)\varphi_n(x)dx$. Then we have

$$\begin{aligned} a_{n,tt}(t) &= \int_0^1 w_{tt}(x,t)\varphi_n(x)dx \\ &= \int_0^1 w_{xx}(x,t)\varphi_n(x)dx \\ &= w_x(1,t)\varphi_n(1) - \mu_n^2\int_0^1 w(x,t)\varphi_n(x)dx \\ &= -[\alpha u(t) + \beta u(t-\tau)]\varphi_n(1) - \mu_n^2 a_n(t) \end{aligned}$$

Then by the equation (1), we have

$$\begin{cases} a_{n,tt}(t)+\mu_n^2 a_n(t)+[\alpha u(t)+\beta u(t-\tau)]\,\varphi_n(1) = 0 \\ a_n(0) = a_n^{(0)}, \quad a_{n,t}(0) = a_n^{(1)} \end{cases} \tag{A.1}$$

Solving the ordinary differential equation (A.1) we get a solution for each $n \in \mathbb{N}$

$$\begin{aligned} a_n(t) &= a_n^{(0)} \cos\mu_n t + a_n^{(1)} \tfrac{\sin\mu_n t}{\mu_n} \\ &-\tfrac{\varphi_n(1)}{\mu_n} \textstyle\int_0^t \sin\mu_n(t-s)\,[\alpha u(s) + \beta u(s-\tau)]\,ds \end{aligned} \tag{A.2}$$

and hence

$$\begin{aligned} a_{n,t}(t) &= -a_n^{(0)}\mu_n \sin\mu_n t + a_n^{(1)} \cos\mu_n t \\ &-\varphi_n(1) \textstyle\int_0^t \cos\mu_n(t-s)\,[\alpha u(s)+\beta u(s-\tau)]\,ds. \end{aligned} \tag{A.3}$$

From above we get estimates

$$\begin{aligned} |\mu_n a_n(t)|^2 &\le 3[|\mu_n a_n^{(0)}|^2 + |a_n^{(1)}|^2 \\ &+|\varphi_n(1)|^2 \left|\textstyle\int_0^t \sin\mu_n(t-s)\,[\alpha u(s)+\beta u(s-\tau)]\,ds\right|^2] \end{aligned}$$

and

$$\begin{aligned} |a_{n,t}(t)|^2 &\le 3[|\mu_n a_n^{(0)}|^2 + |a_n^{(1)}|^2 \\ &+|\varphi_n(1)|^2 \left|\textstyle\int_0^t \cos\mu_n(t-s)\,[\alpha u(s)+\beta u(s-\tau)]\,ds\right|^2]. \end{aligned}$$

This implies $(w(x,t), w_t(x,t)) \in \mathcal{H}$. The proof is then complete. □

According to Theorem 17, we obtain the following Corollary.

Corollary 18 *For any $(w_0, w_1) \in \mathcal{H}$ and $u(t) \in L^2_{loc}(-\tau, \infty)$, the solution of (2) can be written as*

$$\widehat{w}(x,s,t) = \sum_{n=1}^{\infty} a_n(s,t)\varphi_n(x),$$

$$\widehat{w}_s(x,s,t) = \sum_{n=1}^{\infty} a_{n,s}(s,t)\varphi_n(x).$$

where

$$\begin{aligned} a_n(s,t) &= a_n(t)\cos\sqrt{\mu_n}s + a_{n,t}(t)\tfrac{\sin\sqrt{\mu_n}s}{\sqrt{\mu_n}} \\ &-\tfrac{\beta\varphi_n(1)}{\sqrt{\mu_n}} \textstyle\int_0^s \sin\sqrt{\mu_n}(s-r)u(t-\tau+r)dr \end{aligned}$$

and $a_{n,s}(s,t)$ denote the derivation of $a_n(s,t)$.

According to the Corollary 18, $p(x,t)$ and $q(x,t)$ have the following expressions respectively

$$p(x,t) = \widehat{w}(x,\tau,t)$$

$$= \sum_{n=1}^{\infty}\left[a_n(t)\cos\mu_n\tau + a_{n,t}(t)\frac{\sin\mu_n\tau}{\mu_n}\right]\varphi_n(x) - \sum_{n=1}^{\infty}\frac{\beta\varphi_n(1)}{\mu_n}\varphi_n(x)\int_{t-\tau}^{t}\sin\mu_n(t-s)u(s)ds \quad (A.4)$$

$$q(x,t) = \widehat{w}_s(x,\tau,t) = \sum_{n=1}^{\infty}[-\mu_n a_n(t)\sin\mu_n\tau + a_{n,t}(t)\cos\mu_n\tau]\varphi_n(x) - \sum_{n=1}^{\infty}\beta\varphi_n(1)\varphi_n(x)\int_{t-\tau}^{t}\cos\mu_n(t-s)u(s)ds. \quad (A.5)$$

Inserting (A.2) and (A.3) into (A.4) and (A.5) respectively yield

$$p(x,t) = \sum_{n=1}^{\infty}\left[a_n^{(0)}\cos\mu_n(t+\tau)+a_n^{(1)}\frac{\sin\mu_n(t+\tau)}{\mu_n}\right]\varphi_n(x) - \alpha\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n(x)}{\mu_n}\int_0^t\sin\mu_n(t+\tau-s)u(s)ds - \beta\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n(x)}{\mu_n}\int_{-\tau}^{t}\sin\sqrt{\mu_n}(t-s)u(s)ds \quad (A.6)$$

$$q(x,t) = \sum_{n=1}^{\infty}[-a_n^{(0)}\mu_n\sin\mu_n(t+\tau)+a_n^{(1)}\cos\mu_n(t+\tau)]\varphi_n(x) - \alpha\sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)\int_0^t\cos\mu_n(t+\tau-s)u(s)ds - \beta\sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)\int_{-\tau}^{t}\cos\mu_n(t-s)u(s)ds. \quad (A.7)$$

By simple calculation, we get

$$p_t(x,t) = \sum_{n=1}^{\infty}[-a_n^{(0)}\mu_n\sin\mu_n(t+\tau)+a_n^{(1)}\cos\mu_n(t+\tau)]\varphi_n(x) -\alpha\sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)\int_0^t\cos\mu_n(t+\tau-s)u(s)ds -\beta\sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)\int_{-\tau}^{t}\cos\mu_n(t-s)u(s)ds -\alpha u(t)\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n(x)}{\mu_n}\sin\sqrt{\mu_n}\tau;$$

$$q_t(x,t) = -\sum_{n=1}^{\infty}[a_n^{(0)}\mu_n^2\cos\mu_n(t+\tau)+a_n^{(1)}\mu_n\sin\mu_n(t+\tau)]\varphi_n(x) +\alpha\sum_{n=1}^{\infty}\varphi_n(1)\mu_n\varphi_n(x)\int_0^t\sin\mu_n(t+\tau-s)u(s)ds +\beta\sum_{n=1}^{\infty}\varphi_n(1)\mu_n\varphi_n(x)\int_{-\tau}^{t}\sin\mu_n(t-s)u(s)ds -u(t)\sum_{n=1}^{\infty}\varphi_n(1)[\alpha\cos\mu_n\tau+\beta]\varphi_n(x);$$

$$p_{xx}(x,t) = -\beta\sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)u(t) -\sum_{n=1}^{\infty}[\mu_n^2a_n^{(0)}\cos\mu_n(t+\tau)+\mu_na_n^{(1)}\sin\mu_n(t+\tau)]\varphi_n(x) +\alpha\sum_{n=1}^{\infty}\mu_n\varphi_n(1)\varphi_n(x)\int_0^t\sin\mu_n(t+\tau-s)u(s)ds +\beta\sum_{n=1}^{\infty}\mu_n\varphi_n(1)\varphi_n(x)\int_{-\tau}^{t}\sin\mu_n(t-s)u(s)ds;$$

$$p(x,0) = \sum_{n=1}^{\infty}\left[a_n^{(0)}\cos\mu_n\tau + a_n^{(1)}\frac{\sin\mu_n\tau}{\mu_n}\right]\varphi_n(x) +\beta\sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n(x)}{\mu_n}\int_{-\tau}^{0}\sin\mu_n\theta\ u(\theta)d\theta;$$

$$q(x,0) = \sum_{n=1}^{\infty}\left[-a_n^{(0)}\mu_n\sin\mu_n\tau + a_n^{(1)}\cos\mu_n\tau\right]\varphi_n(x) -\beta\sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)\int_{-\tau}^{0}\cos\mu_n\theta\ u(\theta)d\theta.$$

Now, we can define the functions

$$a(x,\theta) = \sum_{n=1}^{\infty}\frac{\varphi_n(1)\varphi_n(x)}{\mu_n}\sin\mu_n\theta;$$

$$a_1(x,\theta) = \sum_{n=1}^{\infty}\varphi_n(1)\varphi_n(x)\cos\mu_n\theta$$

and $a(x) = a(x,\tau)$ and $b(x) = a_1(x,\tau)$. We define the operators

$$E_0(w_0,w_1) = \sum_{n=1}^{\infty}\left[a_n^{(0)}\cos\mu_n\tau + a_n^{(1)}\frac{\sin\mu_n\tau}{\mu_n}\right]\varphi_n(x)$$

and

$$E_1(w_0, w_1) = \sum_{n=1}^{\infty} \left[-a_n^{(0)} \mu_n \sin \mu_n \tau + a_n^{(1)} \cos \mu_n \tau \right] \varphi_n(x)$$

Thus we have the following equations

$$\begin{cases} p_t(x,t) = q(x,t) - \alpha a(x) u(t), \quad 0 < x < 1 \\ q_t(x,t) = p_{xx}(x,t) - \alpha b(x) u(t), \\ p(0,t) = 0, \\ p_x(1,t) = -\beta u(t), \\ p(x,0) = E_0(w_0,w_1)(x) + \beta \int_{-\tau}^{0} a_0(x,s) f(s) ds, \\ q(x,0) = E_1(w_0,w_1)(x) - \beta \int_{-\tau}^{0} a_1(x,s) f(s) ds. \end{cases} \tag{A.8}$$

Clearly $a_0(x,s), a_1(x,s), a(x)$ and $b(x)$ are real functions.

Using Lemma 5, we can prove the following result.

Theorem 19 *Let $a(x), b(x), a_0(x,\theta)$ and $a_1(x,\theta)$ are defined as before. Then we have*

$$a(x) \in L^2[0,1], \quad b(x) \in (H^1[0,1])'$$

and for any $f \in L^2[-\tau, 0]$

$$\int_{-\tau}^{0} a_0(x,\theta) f(\theta) ds \in H_E^1([0,1],$$

$$\int_{-\tau}^{0} a_1(x,\theta) f(\theta) d\theta \in L^2[0,1]$$

where $H_E^1(0,1) = \{f \in H^1(0,1) | f(0) = 0\}$ and $(H^1(0,1))'$ denotes the dual space of Sobolev space $H^1(0,1)$.

Let $E_0(w_0,w_1)$ and $E_1(w_0,w_1)$ be the operators defined as before. Then $E_0 : \mathcal{H} \to H_E^2[0,1]$ and $E_1 : \mathcal{H} \to L^2[0,1]$ are bounded linear operators.

Proof. From definition of function $a(x)$ and $b(x)$ we can see the first assertion is true. For any $f \in L^2[-\tau, 0]$, we have

$$\int_{-\tau}^{0} a_0(x,\theta) f(\theta) d\theta = \sum_{n=1}^{\infty} \frac{\varphi_n(1)\varphi_n(x)}{\mu_n} \int_{-\tau}^{0} f(s) \sin \mu_n s ds$$

and

$$\int_{-\tau}^{0} a_1(x,\theta) f(\theta) d\theta = \sum_{n=1}^{\infty} \varphi_n(1)\varphi_n(x) \int_{-\tau}^{0} f(s) \cos \mu_n s ds$$

So it holds that

$$\left\| \int_{-\tau}^{0} a_0(x,\theta) f(\theta) d\theta \right\|_{H^1[0,1]}^2 = 2 \sum_{n=1}^{\infty} \left| \int_{-\tau}^{0} f(s) \sin \mu_n s ds \right|^2 < \infty$$

and

$$\left\| \int_{-\tau}^{0} a_1(x,\theta) f(\theta) d\theta \right\|_{L^2[0,1]}^2 = 2 \sum_{n=1}^{\infty} \left| \int_{-\tau}^{0} f(s) \cos \mu_n s ds \right|^2 < \infty.$$

Similarly we have

$$||E_0(w_0,w_1)||_{H^1[0,1]}^2 = \sum_{n=1}^{\infty} |\mu_n a_n^{(0)} \cos \mu_n \tau + a_n^{(1)} \sin \mu_n \tau|^2 < \infty$$

and

$$||E_1(w_0,w_1)||_{L^2[0,1]}^2 = \sum_{n=1}^{\infty} |-\mu_n a_n^{(0)} \sin \mu_n \tau + a_n^{(1)} \cos \mu_n \tau|^2 < \infty.$$

These relations prove all assertions. □

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