Exponential stability of periodic solutions for inertial Cohen-Grossberg-type BAM neural networks with time delays

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Abstract: The paper is concerned with the existence and global exponential stability of periodic solutions for inertial Cohen-Grossberg-type BAM neural networks with time delays. With variable transformation the system is transformed to first order differential equations. Some new sufficient conditions ensuring the existence and global exponential stability of periodic solutions for the system are derived by constructing suitable Lyapunov functions, using Weierstrass criteria and boundedness of the solutions. Finally, an example is given to demonstrate the obtained results.

Key–Words: Cohen-Grossberg-type BAM neural networks, inertial term, Lyapunov function, Weierstrass criteria, exponential stability

1 Introduction

Dynamical characteristics of neural networks have become the subject of intensive research in recent years. As supposed in most studies of neural networks, a mammal's brain may be exploiting dynamic attractors for its encoding and storage of associative memories rather than static attractor[1-3]. The Cohen-Grossberg-type BAM neural networks model (i.e.the BAM model that possesses Cohen-Grossberg dynamics) initially proposed by Cohen and Grossberg[4], has their promising potential for the tasks of parallel computation, associative memory and have great ability to solve difficult optimization problems. The Cohen-Grossberg-type BAM neural networks with time delays can be expressed as:

$$\begin{cases} \frac{du_i(t)}{dt} = -a_i(u_i(t))[b_i(u_i(t)) - \sum\limits_{j=1}^{m} c_{ij} f_j(v_j(t)) \\ \quad - \sum\limits_{j=1}^{m} d_{ij} f_j(v_j(t-\tau_{ji})) - I_i(t)], \\ \frac{dv_j(t)}{dt} = -e_j(v_j(t))[h_j(v_j(t)) - \sum\limits_{i=1}^{n} p_{ji} g_i(u_i(t)) \\ \quad - \sum\limits_{i=1}^{n} q_{ji} g_i(u_i(t-\sigma_{ij})) - J_j(t)], \end{cases} \tag{I}$$

We also know that it is fundamental importance to determine the convergence of the solutions of a system of differential equations to either one of a number of equilibria, or else to periodic solutions: in the context of pattern recognition, content-addressable memories are nothing more than asymptotically stable stationary solutions with basins of attraction of a positive measure[5]. Now there have been many results on the stability of solutions of Cohen-Grossberg-type BAM neural networks, see [6-21].

On the other hand, the inertia can be considered an useful tool that is added to help in the generation of chaos in neural systems. It is very useful and significant to introduce an inertial term (the influence of inductance) into the standard neural system. For example, Li et al. [22] added the inertia to a delay differential equation which can be described by

$$\ddot{x} = a\dot{x} - bx + cf(x - hx(t-\tau)).$$

and obtained obviously chaotic behavior. Wheeler and Schieve[23] added the inertia to the Hopfield effective-neuron system with a continuous-time which is shown to exhibit chaos. They explain the chaos is confirmed by Lyapunov exponents, power spectra, and phase space plots, this system is described by

$$\ddot{x_1} = -a_{11}\dot{x_1} - a_{12}x_1 + a_{13}tanh(x_1) + a_{14}tanh(x_2),$$

$$\ddot{x_2} = -b_{11}\dot{x_2} - b_{12}x_2 + b_{13}tanh(x_1) + b_{14}tanh(x_2).$$

Babcocka et al. [24] studied the electronic neural networks with added inertia and founded when the neuron couplings are of an inertial nature, the dynamics can be complex in contrast to the simpler behavior displayed when they are of the standard resistor-capacitor variety. For various values of the neuron gain and the quality factor of the couplings they find ringing about the stationary points, instability and spontaneous oscillation, intertwined basins of attraction, and chaotic response to a harmonic drive. Juhong and Jing [25]considered an inertial four-neuron delayed bidirectional associative memory model. Weak resonant double Hopf bifurcations are completely analyzed in the parameter space of the coupling weight and the coupling delay by the perturbation-incremental scheme. In [26], A kinematical description of traveling waves in the oscillations in the networks is extended to networks with inertia. When the inertia is below a critical value the state of each neuron is over-damped, properties of the networks are qualitatively the same as those without inertia. The duration of the transient oscillations then increases with inertia, and the increasing rate of the logarithm of the duration becomes more than double. When the inertia exceeds a critical value and the state of each neuron becomes under-damped, properties of the networks qualitatively change. The periodic solution is stabilized through the pitchfork bifurcation as inertia increases. More bifurcations occur so that various periodic solutions are generated, and the stability of the periodic solutions changes alternately. Further, stable oscillations generated with inertia are observed in an experiment on an analog circuit. Others, Liu et al. [27-28] investigated the Hopf bifurcation and dynamics of an inertial two-neuron system or in a single inertial neuron mode. The authors Ke and Miao [29-30] investigated stability of equilibrium point and periodic solutions in inertial BAM neural networks with time delays and unbounded delays, respectively.

To the best of our knowledge, few authors have considered the existence and exponential stability of periodic solutions for the Cohen-Grossberg-type BAM neural networks, which is very important in theories and applications. We consider the following Cohen-Grossberg-type BAM neural networks with inertia

$$\begin{cases} \frac{d^2u_i(t)}{dt^2} = -\alpha_i\frac{du_i(t)}{dt} - a_i(u_i(t))[b_i(u_i(t)) \\ -\sum\limits_{j=1}^{m} c_{ij}f_j(v_j(t)) - \sum\limits_{j=1}^{m} d_{ij}f_j(v_j(t-\tau_{ji})) - I_i(t)], \\ \frac{d^2v_j(t)}{dt^2} = -\beta_j\frac{dv_j(t)}{dt} - e_j(v_j(t))[h_j(v_j(t)) \\ -\sum\limits_{i=1}^{n} p_{ji}g_i(u_i(t)) - \sum\limits_{i=1}^{n} q_{ji}g_i(u_i(t-\sigma_{ij})) - J_j(t)], \end{cases} \tag{1}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$, where the second derivative are called inertial term of system (1); $\alpha_i > 0$ and $\beta_j > 0$ are constants; $u_i(t)$ and $v_j(t)$ are the state of the ith neurons from the neural field F_U and the jth neurons from the neural field F_V at time t, respectively; $a_i(u_i(t))$and $e_j(v_j(t))$ represent amplification function of the jth neurons and the ith neurons at time t, respectively; $b_i(u_i(t))$and $h_j(v_j(t))$ are appropriately behaved function of the jth neurons and the ith neurons at time t, respectively; c_{ij}, d_{ij}, p_{ji} and q_{ji} are constants, and denote the connection strengths; f_j and g_i denote the activation function. τ_{ji} and σ_{ij} denote correspond to the transmission delays and satisfy $0 \le \tau_{ji} \le \tau$ and $0 \le \sigma_{ij} \le \sigma$; I_i and J_j denote the external inputs on the ith neurons from the neural field F_U and the jth neurons from the neural field F_V, respectively.

The initial values of system (1) are

$$\begin{cases} u_i(s) = \varphi_{ui}(s),\ \frac{du_i(s)}{dt} = \psi_{ui}(s), -\tau \le s \le 0, \\ v_j(s) = \varphi_{vj}(s),\ \frac{dv_j(s)}{dt} = \psi_{vj}(s), -\sigma \le s \le 0, \end{cases} \tag{2}$$

where $\varphi_{ui}(s), \psi_{ui}(s), \varphi_{vj}(s)$ and $\psi_{vj}(s)$ are bounded and continuous function.

From the viewpoints of mathematics and physics, the system (1) is a class of nonlinear second-order dynamical system where $\alpha_i > 0$ is a damping coefficient, then the system (I) can be considered as a model overdamped (i.e. the damp tend to infinite). In practical application it is necessary to consider the existence and stability of periodic solutions of system with damp (or weak damp).

This paper is organized as follows. Model description and some preliminaries are given in Section 2. In Section 3, the sufficient conditions are derived for the existence and exponential stability of periodic solutions for inertial Cohen-Grossberg-type BAM neural networks with time delays. In Section 4, an illustrative example is given to show the effectiveness of the proposed theory. Finally conclusions are drawn in section 5.

2 Model description and preliminaries

Throughout this paper, we make the following assumptions where $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.
(H_1) The activation functions $f_j(\cdot), g_i(\cdot)$ are bounded and satisfy Lipschitz condition, i.e., there exist constant $L_j > 0, \bar{f}_j > 0, N_i > 0, \bar{g}_i > 0$, such that

$$|f_j(v_1) - f_j(v_2)| \le L_j|v_1 - v_2|, \quad |f_j(x)| \le \bar{f}_j;$$
$$|g_i(v_1) - g_i(v_2)| \le N_i|v_1 - v_2|, \quad |g_i(x)| \le \bar{g}_i;$$

for $\forall v_1, v_2, x \in R$.
(H_2) $I_i(\cdot)$ and $J_j(\cdot)$ are continuously periodic functions defined on $t \in [0, \infty)$ with common period $\omega > 0$, and satisfy

$$0 \le |I_i(t)| \le \bar{I}_i, \quad 0 \le |J_j(t)| \le \bar{J}_j,$$

where $\bar{I}_i \geq 0,\ \bar{J}_j \geq 0$ are constant.
(H$_3$) $a_i(\cdot), e_j(\cdot)$ are continuously differentiable functions and satisfy

$$|a_i'(x)| \leq a_i^*, \quad 0 < \underline{a}_i \leq a_i(x) \leq \bar{a}_i, \quad \forall x \in R;$$
$$|e_j'(x)| \leq e_j^*, \quad 0 < \underline{e}_j \leq e_j(x) \leq \bar{e}_j, \quad \forall x \in R;$$

where $a_i^*, e_j^*, \underline{a}_i, \underline{e}_j, \bar{a}_i, \bar{e}_j > 0$ are constant.
(H$_4$) $b_i(\cdot), h_j(\cdot)$are continuously differentiable functions and satisfy

$$b_i(0) = 0, \quad 0 \leq \underline{b}_i \leq b_i'(x) \leq \bar{b}_i, \quad \forall x \in R;$$
$$h_j(0) = 0, \quad 0 \leq \underline{h}_j \leq h_j'(x) \leq \bar{h}_j, \quad \forall x \in R;$$

where $\underline{b}_i, \underline{h}_j, \bar{b}_i, \bar{h}_j > 0$ are constant.
(H$_5$) $a_i(\cdot), b_j(\cdot), e_j(\cdot), h_j(\cdot)$ are continuously differentiable functions and satisfy

$$0 < \underline{K}_i \leq (a_i(x)b_i(x))' \leq \bar{K}_i, \forall x \in R;$$
$$0 < \underline{T}_i \leq (e_j(x)h_j(x))' \leq \bar{T}_j, \forall x, y \in R;$$

where $\bar{K}_i, \bar{T}_j, \underline{K}_i, \underline{T}_j > 0$ are constant.

Introducing variable transformation

$$\begin{cases} y_i(t) = \frac{du_i(t)}{dt} + u_i(t),\ i = 1, 2, \cdots, n, \\ z_j(t) = \frac{dv_j(t)}{dt} + v_j(t),\ j = 1, 2, \cdots, m. \end{cases}$$

then (1) and (2) can be rewritten as

$$\begin{cases} \frac{du_i(t)}{dt} = -u_i(t) + y_i(t), \\ \frac{dy_i(t)}{dt} = -(1-\alpha_i)u_i(t) - (\alpha_i - 1)y_i(t) \\ \quad -a_i(u_i(t))[b_i(u_i(t)) - \sum\limits_{j=1}^{m} c_{ij} f_j(v_j(t)) \\ \quad - \sum\limits_{j=1}^{m} d_{ij} f_j(v_j(t-\tau_{ji})) - I_i(t)], \\ \frac{dv_j(t)}{dt} = -v_j(t) + z_j(t), \\ \frac{dz_j(t)}{dt} = -(1-\beta_j)v_j(t) - (\beta_j - 1)z_j(t) \\ \quad -e_j(v_j(t))[h_j(v_j(t)) - \sum\limits_{i=1}^{n} p_{ji} g_i(u_i(t)) \\ \quad - \sum\limits_{i=1}^{n} q_{ji} g_i(u_i(t-\sigma_{ij})) - J_j(t)], \end{cases} \tag{3}$$

and

$$\begin{cases} u_i(s) = \varphi_{ui}(s), \frac{du_i(s)}{dt} = \psi_{ui}(s), \quad -\tau \leq s \leq 0, \\ y_i(s) = \varphi_{ui}(s) + \psi_{ui}(s) \doteq \bar{\varphi}_{ui}(s), -\tau \leq s \leq 0, \\ v_j(s) = \varphi_{vj}(s), \frac{dv_j(s)}{dt} = \psi_{vj}(s), \quad -\sigma \leq s \leq 0, \\ z_j(s) = \varphi_{vj}(s) + \psi_{vj}(s) \doteq \bar{\varphi}_{vj}(s), -\sigma \leq s \leq 0, \end{cases} \tag{4}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.
Let

$$U_i(t) = \begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix}, V_j(t) = \begin{pmatrix} v_j(t) \\ z_j(t) \end{pmatrix},$$

system (3) can becomes

$$\frac{dU_i(t)}{dt} = -A_i U_i(t) + P \begin{pmatrix} 0 \\ F_i(u_i(t), v_j(t)) \end{pmatrix}, \tag{5}$$
$$\frac{dV_j(t)}{dt} = -B_j V_j(t) + P \begin{pmatrix} 0 \\ G_j(u_i(t), v_j(t)) \end{pmatrix}, \tag{6}$$

where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_i = \begin{pmatrix} 1 & -1 \\ 1-\alpha_i & \alpha_i - 1 \end{pmatrix},$$
$$B_j = \begin{pmatrix} 1 & -1 \\ 1-\beta_j & \beta_j - 1 \end{pmatrix},$$

$$F_i(u_i(t), v_j(t)) = -a_i(u_i(t))[b_i(u_i(t)) - \sum_{j=1}^{m} c_{ij} f_j(v_j(t)) - \sum_{j=1}^{m} d_{ij} f_j(v_j(t-\tau_{ji})) - I_i(t)],$$

$$G_j(u_i(t), v_j(t)) = -e_j(v_j(t))[h_j(v_j(t)) - \sum_{i=1}^{n} p_{ji} g_i(u_i(t)) - \sum_{i=1}^{n} q_{ji} g_i(u_i(t-\sigma_{ij})) - J_j(t)],$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.
Let

$$u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T,$$
$$v(t) = (v_1(t), v_2(t), \cdots, v_m(t))^T,$$
$$u^*(t) = (u_1^*(t), u_2^*(t), \cdots, u_n^*(t))^T,$$
$$v^*(t) = (v_1^*(t), v_2^*(t), \cdots, v_m^*(t))^T.$$

Definition 1. *Let $(u^{*T}(t), v^{*T}(t))^T$ be an $\omega-$ periodic solution of system (1) with initial value*

$$u_i^*(s) = \varphi_{ui}^*(s), \quad \frac{du_i^*(s)}{dt} = \psi_{ui}^*(s), \quad -\tau \leq s \leq 0,$$
$$v_j^*(s) = \varphi_{vj}^*(s), \quad \frac{dv_j^*(s)}{dt} = \psi_{vj}^*(s), \quad -\sigma \leq s \leq 0.$$

For every solution $(u^T(t), v^T(t))^T$ of system (1) with any initial value

$$u_i(s) = \varphi_{ui}(s), \quad \frac{du_i(s)}{dt} = \psi_{ui}(s), \quad -\tau \leq s \leq 0,$$
$$v_j(s) = \varphi_{vj}(s), \quad \frac{dv_j(s)}{dt} = \psi_{vj}(s), \quad -\sigma \leq s \leq 0,$$

If there exist constants $\eta > 0$ and $M > 1$, such that

$$\sum_{i=1}^{n}(u_i(t) - u_i^*(t))^2 + \sum_{j=1}^{m}(v_j(t) - v_j^*(t))^2 \leq M e^{-\eta t}[\|\varphi_u - \varphi_u^*\|^2 + \|\varphi_v - \varphi_v^*\|^2], t \geq 0,$$

then periodic solution $(u^{*T}(t), v^{*T}(t))^T$ *is said to be globally exponentially stable, where*

$$\|\varphi_u - \varphi_u^*\|^2 = \sup_{-\tau\le t\le 0}\sum_{i=1}^{n}|\varphi_{ui}(t)-\varphi_{ui}^*(t)|^2,$$
$$\|\varphi_v - \varphi_v^*\|^2 = \sup_{-\sigma\le t\le 0}\sum_{j=1}^{m}|\varphi_{vj}(t)-\varphi_{vj}^*(t)|^2.$$

3 Main results

In this section, we can derive some sufficient conditions which ensure the globally exponential stability and existence of periodic solutions for system (1) by constructing a suitable Lyapunov function and some analysis techniques, such as Weierstrass criteria of series, the method of magnifying and shrinking of inequality, etc.

Theorem 1 *For system (1), under the hypotheses* $(\mathrm{H}_1)-(\mathrm{H}_5)$, $u_i(t), u_i'(t)$, $v_j(t), v_j'(t)$ *are bounded, for* $i=1,2,\cdots,n, j=1,2,\cdots,m, t\ge 0$.

Proof: It follows from (1) that

$$\begin{aligned}\frac{d^2|u_i(t)|}{dt^2} &= -\alpha_i\frac{d|u_i(t)|}{dt} - sgn(u_i(t))a_i(u_i(t))b_i(u_i(t))\\ &+sgn(u_i(t))a_i(u_i(t))[\sum_{j=1}^{m}c_{ij}f_j(v_j(t)) + \sum_{j=1}^{m}d_{ij}f_j(v_j(t-\tau_{ij}))+I_i(t)]\\ &\le -\alpha_i\frac{d|u_i(t)|}{dt} - \underline{K}_i|u_i(t)| + \bar{a}_i[\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)+\bar{I}_i].\end{aligned} \tag{7}$$

From (7), we can obtain

$$|u_i(t)| \le C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t} + \frac{1}{a_i}[\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)+\bar{I}_i], \tag{8}$$

where $\lambda_{1,2}=\frac{-\alpha_i\pm\sqrt{\alpha_i^2-4\underline{K}_i}}{2}, C_1, C_2$ are any real constants. Since $\alpha_i>0$, we have $Re(\lambda_1)<0$, $Re(\lambda_2)<0$, formula (8) shows that all $u_i(t)$ are bounded for $i=1,2,\cdots,n, t\ge 0$. We may assume that $|u_i(t)|\le R_i$, $R_i>0$ are constants, $i=1,2,\cdots,n$.

On the other hand, from (1) we also can obtain

$$\begin{aligned}\frac{du_i(t)}{dt} &= e^{-\alpha_i t}\frac{du_i(0)}{dt} - e^{-\alpha_i t}\int_0^t e^{\alpha_i s}a_i(u_i(s))\\ &[b_i(u_i(s)) - \sum_{j=1}^{m}c_{ij}f_j(u_j(s))\\ &- \sum_{j=1}^{m}d_{ij}f_j(u_j(s-\tau_{ij})) - I_i(s)]\,\mathrm{d}s,\end{aligned} \tag{9}$$

From (9), we have

$$\begin{aligned}|\frac{du_i(t)}{dt}| &\le |\varphi_{ui}(0)|\\ &+\frac{1}{\alpha_i}[\bar{K}_iR_i + \bar{a}_i\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)+\bar{I}_i],\end{aligned} \tag{10}$$

Formula (10) shows that all $u_i'(t)$ are bounded for $i=1,2,\cdots,n, t\ge 0$. Similar to the above method, we can obtain $v_j(t), v_j'(t)$are bounded, $j=1,2,\cdots,m, t\ge 0$. □

Theorem 2 *Under the hypotheses* $(\mathrm{H}_1)-(\mathrm{H}_5)$, *if*

$$\alpha_i - \bar{K}_i > 0, \quad \beta_j - \bar{T}_j > 0,$$

$$-2+\alpha_i-\underline{K}_i + a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|) + \sum_{j=1}^{m}\bar{e}_jN_i(|p_{ji}|+|q_{ji}|) + a_i^*\bar{I}_i < 0,$$

$$2-\alpha_i-\underline{K}_i + a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|) + \sum_{j=1}^{m}\bar{a}_iL_j(|c_{ij}|+|d_{ij}|) + a_i^*\bar{I}_i < 0,$$

$$-2+\beta_j-\underline{T}_j + e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|) + \sum_{i=1}^{n}\bar{a}_iL_j(|c_{ij}|+|d_{ij}|) + e_j^*\bar{J}_j < 0,$$

$$2-\beta_j-\underline{T}_j + e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|) + \sum_{i=1}^{n}\bar{e}_jN_i(|p_{ji}|+|q_{ji}|) + e_j^*\bar{J}_j < 0,$$

for $i=1,2,\cdots,n, j=1,2,\cdots,m$, *then system (1) has one* $\omega-$ *periodic solution, which is globally exponentially stable.*

Proof: Let $(u^{*T}(t), v^{*T}(t))^T$ be an solution of system (1) with initial value

$$u_i^*(s)=\varphi_{ui}^*(s),\ \frac{du_i^*(s)}{dt}=\psi_{ui}^*(s),\ -\tau\le s\le 0,$$
$$v_j^*(s)=\varphi_{vj}^*(s),\ \frac{dv_j^*(s)}{dt}=\psi_{vj}^*(s),\ -\sigma\le s\le 0.$$

$(u^T(t), v^T(t))$ be an solution of system (1) with initial value

$$u_i(s)=\varphi_{ui}(s),\ \frac{du_i(s)}{dt}=\psi_{ui}(s),\ -\tau\le s\le 0,$$
$$v_j(s)=\varphi_{vj}(s),\ \frac{dv_j(s)}{dt}=\psi_{vj}(s),\ -\sigma\le s\le 0.$$

$$y_i^*(t)=\frac{du_i^*(t)}{dt}+u_i^*(t), \quad z_j^*(t)=\frac{dv_j^*(t)}{dt}+v_j^*(t),$$

$$\bar{U}_i(t)=\begin{pmatrix}u_i(t)-u_i^*(t)\\ y_i(t)-y_i^*(t)\end{pmatrix}, \quad \bar{V}_j(t)=\begin{pmatrix}v_j(t)-v_j^*(t)\\ z_j(t)-z_j^*(t)\end{pmatrix},$$

for $i=1,2\cdots,n, j=1,2\cdots,m$.
From (5), we have

$$\frac{d\bar{U}_i(t)}{dt} = -A_i\bar{U}_i(t) + P\begin{pmatrix} 0 \\ \bar{F}_i(\bar{u}_i(t),\bar{v}_j(t))\end{pmatrix}, \quad (11)$$

where

$$\bar{F}_i(\bar{u}_i(t),\bar{v}_j(t)) = -a_i(u_i(t))[b_i(u_i(t)) - \sum_{j=1}^{m} c_{ij}f_j(v_j(t)) - \sum_{j=1}^{m} d_{ij}f_j(v_j(t-\tau_{ji})) - I_i(t)] + a_i(u_i^*(t))[b_i(u_i^*(t)) - \sum_{j=1}^{m} c_{ij}f_j(v_j^*(t)) - \sum_{j=1}^{m} d_{ij}f_j(v_j^*(t-\tau_{ji})) - I_i(t)].$$

Left multiplying both sides of (11) with

$$\bar{U}_i^T = (u_i(t)-u_i^*(t), y_i(t)-y_i^*(t)),$$

we get

$$\begin{aligned}
&\bar{U}_i^T\tfrac{d\bar{U}_i}{dt} = -(u_i(t)-u_i^*(t))^2 + (1-\alpha_i)(y_i(t)-y_i^*(t))^2 \\
&+\alpha_i(u_i(t)-u_i^*(t))(y_i(t)-y_i^*(t)) + (y_i(t)-y_i^*(t))\cdot \\
&\big\{-[a_i(u_i(t))b_i(u_i(t)) - a_i(u_i^*(t))b_i(u_i^*(t))] \\
&+a_i(u_i(t))\sum_{j=1}^{m} c_{ij}[f_j(v_j(t)) - f_j(v_j^*(t))] \\
&+a_i(u_i(t))\sum_{j=1}^{m} d_{ij}[f_j(v_j(t-\tau_{ji})) - f_j(v_j^*(t-\tau_{ji}))] \\
&+[a_i(u_i(t)) - a_i(u_i^*(t))]\cdot \\
&[\sum_{j=1}^{m} c_{ij}f_j(v_j^*(t)) + \sum_{j=1}^{m} d_{ij}f_j(v_j^*(t-\tau_{ji}))] \\
&+[a_i(u_i(t)) - a_i(u_i^*(t))]I_i(t)\big\} \\
&\le -(u_i(t)-u_i^*(t))^2 + (1-\alpha_i)(y_i(t)-y_i^*(t))^2 \\
&+[\alpha_i - \tfrac{a_i(u_i(t))b_i(u_i(t))-a_i(u_i^*(t)}{u_i(t)-u_i^*(t)}] \\
&\times (u_i(t)-u_i^*(t))(y_i(t)-y_i^*(t)) \\
&+|y_i(t)-y_i^*(t)|\cdot\big\{\bar{a}_i\sum_{j=1}^{m}|c_{ij}|L_j|v_j(t)-v_j^*(t)| \\
&\quad+\bar{a}_i\sum_{j=1}^{m}|d_{ij}|L_j|v_j(t-\tau_{ji}) - v_j^*(t-\tau_{ji})| \\
&\quad+a_i^*|u_i(t)-u_i^*(t)|\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|) \\
&+a_i^*\bar{I}_i|u_i(t)-u_i^*(t)|\big\}.
\end{aligned} \quad (12)$$

where

$$\alpha_i - \bar{K}_i \le \alpha_i - \frac{a_i(u_i(t))b_i(u_i(t)) - a_i(u_i^*(t)}{u_i(t)-u_i^*(t)} \le \alpha_i - \underline{K}_i,$$

If

$$\alpha_i - \bar{K}_i > 0,$$

it follows from (12) that

$$\begin{aligned}
&\bar{U}_i^T\tfrac{d\bar{U}_i}{dt} \le \tfrac{1}{2}\big[-2+\alpha_i-\underline{K}_i \\
&\quad+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|) + a_i^*\bar{I}_i\big](u_i(t)-u_i^*(t))^2 \\
&+\tfrac{1}{2}\big[2-\alpha_i-\underline{K}_i + a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|) \\
&\quad+\bar{a}_i\sum_{j=1}^{m}L_j(|c_{ij}|+|d_{ij}|) + a_i^*\bar{I}_i\big](y_i(t)-y_i^*(t))^2 \\
&+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|c_{ij}|L_j(v_j(t)-v_j^*(t))^2 \\
&+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|d_{ij}|L_j(v_j(t-\tau_{ji}) - v_j^*(t-\tau_{ji}))^2.
\end{aligned} \quad (13)$$

Similar to the above method, if

$$\beta_j - \bar{T}_j > 0,$$

we can obtain

$$\begin{aligned}
&\bar{V}_i^T\tfrac{d\bar{V}_i}{dt} \le \tfrac{1}{2}\big[-2+\beta_j-\underline{T}_j \\
&\quad+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|) + e_i^*\bar{J}_j\big](v_j(t)-v_j^*(t))^2 \\
&\quad+\tfrac{1}{2}\big[2-\beta_j-\underline{T}_j + e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ij}|+|q_{ij}|) \\
&\quad+\bar{e}_j\sum_{i=1}^{n}N_i(|p_{ji}|+|q_{ji}|) + e_j^*\bar{J}_j\big](z_j(t)-z_j^*(t))^2 \\
&\quad+\tfrac{\bar{e}_j}{2}\sum_{i=1}^{n}|p_{ji}|N_i(u_i(t)-u_i^*(t))^2 \\
&\quad+\tfrac{\bar{e}_j}{2}\sum_{i=1}^{n}|q_{ji}|N_i(u_i(t-\sigma_{ij}) - u_i^*(t-\sigma_{ij}))^2.
\end{aligned} \quad (14)$$

We consider the Lyapunov functional:

$$V(t) = V_1(t) + V_2(t),$$

where

$$\begin{aligned}
&V_1(t) = \sum_{i=1}^{n}\big\{\tfrac{\|\bar{U}_i(t)\|^2}{2}e^{2\varepsilon t} \\
&+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|d_{ij}|L_j\int_{t-\tau_{ji}}^{t}(v_j(s)-v_j^*(s))^2e^{2\varepsilon(s+\tau_{ji})}\,\mathrm{d}s\big\},
\end{aligned} \quad (15)$$

$$\begin{aligned}
&V_2(t) = \sum_{j=1}^{m}\big\{\tfrac{\|\bar{V}_j(t)\|^2}{2}e^{2\varepsilon t} \\
&+\tfrac{\bar{e}_j}{2}\sum_{i=1}^{n}|q_{ji}|N_i\int_{t-\sigma_{ij}}^{t}(u_i(s)-u_i^*(s))^2e^{2\varepsilon(s+\sigma_{ij})}\,\mathrm{d}s\big\},
\end{aligned} \quad (16)$$

$\varepsilon > 0$ is a small number. By (13), we have the upper right Dini-derivative $D^+V_1(t)$ of $V_1(t)$ along the solution of (11)

$$\begin{aligned}
D^+V_1(t) &= \sum_{i=1}^{n}\big\{\bar{U}_i^T\tfrac{d\bar{U}_i(t)}{dt}e^{2\varepsilon t}+\varepsilon e^{2\varepsilon t}\|\bar{U}_i(t)\|^2\\
&\quad+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|d_{ij}|L_j[(v_j(t)-v_j^*(t))^2e^{2\varepsilon(t+\tau_{ji})}\\
&\quad-(v_j(t-\tau_{ji})-v_j^*(t-\tau_{ji}))^2e^{2\varepsilon t}]\big\}\\
&\le \sum_{i=1}^{n}e^{2\varepsilon t}\big\{\varepsilon\|\bar{U}_i(t)\|^2+\tfrac{1}{2}[-2+\alpha_i-\underline{K}_i\\
&\quad a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i\big](u_i(t)-u_i^*(t))^2\\
&\quad+\tfrac{1}{2}\big[2-\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\quad+\bar{a}_i\sum_{j=1}^{m}L_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i\big](y_i(t)-y_i^*(t))^2\\
&\quad+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|c_{ij}|L_j(v_j(t)-v_j^*(t))^2\\
&\quad+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|d_{ij}|L_j(v_j(t)-v_j^*(t))^2e^{2\varepsilon\tau_{ji}}\big\}\\
&\le \tfrac{1}{2}\sum_{i=1}^{n}e^{2\varepsilon t}\big\{[2\varepsilon-2+\alpha_i-\underline{K}_i\\
&\quad a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i\big](u_i(t)-u_i^*(t))^2\\
&\quad+\big[2\varepsilon+2-\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\quad+\bar{a}_i\sum_{j=1}^{m}L_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i\big](y_i(t)-y_i^*(t))^2\\
&\quad+\bar{a}_i\sum_{j=1}^{m}L_j(|c_{ij}|+|d_{ij}|e^{2\varepsilon\tau_{ji}})(v_j(t)-v_j^*(t))^2\big\}.
\end{aligned}\tag{17}$$

Similar to the above method, by (14) we have the upper right Dini-derivative $D^+V_2(t)$ of $V_2(t)$.

$$\begin{aligned}
D^+V_2(t) &\le \tfrac{1}{2}\sum_{j=1}^{m}e^{2\varepsilon t}\big\{[2\varepsilon-2+\beta_j-\underline{T}_j\\
&\quad+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ij}|+|q_{ij}|)+e_j^*\bar{J}_j\big](v_j(t)-v_j^*(t))^2\\
&\quad+\big[2\varepsilon+2-\beta_j-\underline{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ij}|+|q_{ij}|)\\
&\quad+\bar{e}_j\sum_{i=1}^{n}N_i(|p_{ji}|+|q_{ji}|)+e_j^*\bar{J}_j\big](z_j(t)-z_j^*(t))^2\\
&\quad+\bar{e}_j\sum_{i=1}^{n}N_i(|p_{ji}|+|q_{ji}|e^{2\varepsilon\sigma_{ij}})(u_i(t)-u_i^*(t))^2\big\}.
\end{aligned}\tag{18}$$

It follows from (17), (18) that

$$\begin{aligned}
D^+V(t) &= D^+V_1(t)+D^+V_2(t)\\
&\le \tfrac{1}{2}\sum_{i=1}^{n}e^{2\varepsilon t}\big\{[2\varepsilon-2+\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\quad+\sum_{j=1}^{m}\bar{e}_jN_i(|p_{ji}|+|q_{ji}|e^{2\varepsilon\sigma})+a_i^*\bar{I}_i](u_i(t)-u_i^*(t))^2\\
&\quad+\big[2\varepsilon+2-\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\quad+\bar{a}_i\sum_{j=1}^{m}L_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i](y_i(t)-y_i^*(t))^2\big\}\\
&\quad+\tfrac{1}{2}\sum_{j=1}^{m}e^{2\varepsilon t}\big\{[2\varepsilon-2+\beta_i-\bar{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|)\\
&\quad+\sum_{i=1}^{n}\bar{a}_iL_j(|c_{ij}|+|d_{ij}|e^{2\varepsilon\tau})+e_j^*\bar{J}_j](v_j(t)-v_j^*(t))^2\\
&\quad+\big[2\varepsilon+2-\beta_j-\underline{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|)\\
&\quad+\bar{e}_j\sum_{i=1}^{n}N_i(|p_{ji}|+|q_{ji}|)+e_j^*\bar{J}_j](z_j(t)-z_j^*(t))^2\big\}.
\end{aligned}\tag{19}$$

From condition of Theorem 2, we can choose a small $\varepsilon>0$ such that

$$\begin{aligned}
&2\varepsilon-2+\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\qquad+\sum_{j=1}^{m}\bar{e}_jN_i(|p_{ji}|+|q_{ji}|e^{2\varepsilon\sigma})+a_i^*\bar{I}_i\le 0,\\
&2\varepsilon+2-\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\qquad+\bar{a}_i\sum_{j=1}^{m}L_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i\le 0,\\
&2\varepsilon-2+\beta_i-\underline{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|)\\
&\qquad+\sum_{i=1}^{n}\bar{a}_iL_j(|c_{ij}|+|d_{ij}|e^{2\varepsilon\tau})+e_j^*\bar{J}_j\le 0,\\
&2\varepsilon+2-\beta_j-\underline{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|)\\
&\qquad+\bar{e}_j\sum_{i=1}^{n}N_i(|p_{ji}|+|q_{ji}|)+e_j^*\bar{J}_j\le 0,
\end{aligned}$$

for $i=1,2\cdots,n,j=1,2\cdots,m$.

From (19), we get $D^+V(t)\le 0$, and so $V(t)\le V(0)$, for all $t\ge 0$. From (15) and (16), we have

$$\begin{aligned}
V(t) &\ge \sum_{i=1}^{n}\tfrac{\|\bar{U}_i(t)\|^2}{2}e^{2\varepsilon t}+\sum_{j=1}^{m}\tfrac{\|\bar{V}_j(t)\|^2}{2}e^{2\varepsilon t}\\
&= \sum_{i=1}^{n}\tfrac{e^{2\varepsilon t}}{2}[(u_i(t)-u_i^*(t))^2+(y_i(t)-y_i^*(t))^2]\\
&\quad+\sum_{j=1}^{m}\tfrac{e^{2\varepsilon t}}{2}[(v_j(t)-v_j^*(t))^2+(z_j(t)-z_j^*(t))^2].
\end{aligned}\tag{20}$$

$$\begin{aligned}
V(0) &= \sum_{i=1}^{n}\big\{\tfrac{\|\bar{U}_i(0)\|^2}{2}\\
&\quad+\tfrac{\bar{a}_i}{2}\sum_{j=1}^{m}|d_{ij}|L_j\cdot\int_{-\tau_{ji}}^{0}(v_j(s)-v_j^*(s))^2e^{2\varepsilon(s+\tau_{ji})}\,\mathrm{d}s\big\}\\
&\quad+\sum_{j=1}^{m}\big\{\tfrac{\|\bar{V}_j(0)\|^2}{2}\\
&\quad+\tfrac{\bar{e}_j}{2}\sum_{i=1}^{n}|q_{ji}|N_i\cdot\int_{-\sigma_{ij}}^{0}(u_i(s)-u_i^*(s))^2e^{2\varepsilon(s+\sigma_{ij})}\,\mathrm{d}s\big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\{\tfrac{(\varphi_{ui}(0)-\varphi_{ui}^*(0))^2}{2}+\tfrac{(\bar\varphi_{ui}(0)-\bar\varphi_{ui}^*(0))^2}{2}\\
&+\tfrac{\bar a_i}{2}\sum_{j=1}^{m}|d_{ij}|L_j\cdot\int_{-\tau_{ji}}^{0}(\varphi_{vj}(s)-\varphi_{vj}^*(s))^2e^{2\varepsilon(s+\tau_{ji})}\,\mathrm{d}s\}\\
&+\sum_{j=1}^{m}\{\tfrac{(\varphi_{vj}(0)-\varphi_{vj}^*(0))^2}{2}+\tfrac{(\bar\varphi_{vj}(0)-\bar\varphi_{vj}^*(0))^2}{2}\\
&+\tfrac{\bar e_j}{2}\sum_{i=1}^{n}|q_{ji}|N_i\cdot\int_{-\sigma_{ij}}^{0}(\varphi_{ui}(s)-\varphi_{ui}^*(s))^2e^{2\varepsilon(s+\sigma_{ij})}\,\mathrm{d}s\}\\
&\le \tfrac{\|\varphi_u-\varphi_u^*\|^2}{2}+\tfrac{\|\bar\varphi_u-\bar\varphi_u^*\|^2}{2}\\
&\quad+\tfrac{\tau}{2}e^{2\varepsilon\tau}\sum_{i=1}^{n}\bar a_i\max_{1\le j\le m}\{|d_{ij}|L_j\}\|\varphi_v-\varphi_v^*\|^2\\
&\quad+\tfrac{\|\varphi_v-\varphi_v^*\|^2}{2}+\tfrac{\|\bar\varphi_v-\bar\varphi_v^*\|^2}{2}\\
&\quad+\tfrac{\sigma}{2}e^{2\varepsilon\sigma}\sum_{j=1}^{m}\bar e_j\max_{1\le i\le n}\{|q_{ji}|N_i\}\|\varphi_u-\varphi_u^*\|^2\\
&= \tfrac12[1+\sigma e^{2\varepsilon\sigma}\sum_{j=1}^{m}\bar e_j\max_{1\le i\le n}\{|q_{ji}|N_i\}]\|\varphi_u-\varphi_u^*\|^2\\
&\quad+\tfrac12[1+\tau e^{2\varepsilon\tau}\sum_{i=1}^{n}\bar a_i\max_{1\le j\le m}\{|d_{ij}|L_j\}]\|\varphi_v-\varphi_v^*\|^2\\
&+\tfrac{\|\bar\varphi_u-\bar\varphi_u^*\|^2}{2}+\tfrac{\|\bar\varphi_v-\bar\varphi_v^*\|^2}{2},
\end{aligned}\tag{21}
$$

Since $V(0)\ge V(t)$, from (20) and (21), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}\tfrac{e^{2\varepsilon t}}{2}[(u_i(t)-u_i^*(t))^2+(y_i(t)-y_i^*(t))^2]\\
&\quad+\sum_{j=1}^{m}\tfrac{e^{2\varepsilon t}}{2}[(v_j(t)-v_j^*(t))^2+(z_j(t)-z_j^*(t))^2])\\
&\le \tfrac12[1+\sigma e^{2\varepsilon\sigma}\sum_{j=1}^{m}\bar e_j\max_{1\le i\le n}\{|q_{ji}|N_i\}]\|\varphi_u-\varphi_u^*\|^2\\
&+\tfrac12[1+\tau e^{2\varepsilon\tau}\sum_{i=1}^{n}\bar a_i\max_{1\le j\le m}\{|d_{ij}|L_j\}]\|\varphi_v-\varphi_v^*\|^2\\
&+\tfrac{\|\bar\varphi_u-\bar\varphi_u^*\|^2}{2}+\tfrac{\|\bar\varphi_v-\bar\varphi_v^*\|}{2}.
\end{aligned}\tag{22}
$$

By multiplying both sides of (22) with $2e^{-2\varepsilon t}$, we get

$$
\begin{aligned}
&\sum_{i=1}^{n}[(u_i(t)-u_i^*(t))^2+(y_i(t)-y_i^*(t))^2]\\
&+\sum_{j=1}^{m}[(v_j(t)-v_j^*(t))^2+(z_j(t)-z_j^*(t))^2]\\
&\le e^{-2\varepsilon t}.\\
&\{[1+\sigma e^{2\varepsilon\sigma}\sum_{j=1}^{m}\bar e_j\max_{1\le i\le n}\{|q_{ji}|N_i\}]\|\varphi_u-\varphi_u^*\|^2\\
&+[1+\tau e^{2\varepsilon\tau}\sum_{i=1}^{n}\bar a_i\max_{1\le j\le m}\{|d_{ij}|L_j\}]\|\varphi_v-\varphi_v^*\|^2\\
&\qquad+\|\bar\varphi_u-\bar\varphi_u^*\|^2+\|\bar\varphi_v-\bar\varphi_v^*\|^2\}\\
&\le e^{-2\varepsilon t}\cdot\{M^*[\|\varphi_u-\varphi_u^*\|^2+\|\varphi_v-\varphi_v^*\|^2]\\
&+\|\bar\varphi_u-\bar\varphi_u^*\|^2+\|\bar\varphi_v-\bar\varphi_v^*\|^2\}
\end{aligned}\tag{23}
$$

for all $t\ge0$, where

$$
\begin{aligned}
M^*=\max\{[1+\sigma e^{2\varepsilon\sigma}\sum_{j=1}^{m}\bar e_j\max_{1\le i\le n}\{|q_{ji}|N_i\},\\
1+\tau e^{2\varepsilon\tau}\sum_{i=1}^{n}\bar a_i\max_{1\le j\le m}\{|d_{ij}|L_j\}\}.
\end{aligned}
$$

Let $M=M^*+\frac{\|\bar\varphi_u-\bar\varphi_u^*\|^2+\|\bar\varphi_v-\bar\varphi_v^*\|^2}{\|\varphi_u-\varphi_u^*\|^2+\|\varphi_v-\varphi_v^*\|^2}>1$, from (23), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}[(u_i(t)-u_i^*(t))^2+(y_i(t)-y_i^*(t))^2]\\
&\quad+\sum_{j=1}^{m}[(v_j(t)-v_j^*(t))^2+(z_j(t)-z_j^*(t))^2]\\
&\le Me^{-2\varepsilon t}(\|\varphi_u-\varphi_u^*\|^2.+\|\varphi_v-\varphi_v^*\|^2),
\end{aligned}\tag{24}
$$

for all $t\ge0$. Then $(u^{*T}(t),v^{*T}(t))^T$ is globally exponentially stable.

Since $I_i(t),J_j(t)$ are continuously periodic functions defined on $t\in[0,\infty)$ with common period $\omega>0$, then for any natural number k, $u_i(t+k\omega)$ and $v_j(t+k\omega)$ are the solution of (1). Thus, from (24) there exist constant $N>0$ and $\delta>0$, such that

$$|u_i(t+(k+1)\omega)-u_i(t+k\omega)|\le Ne^{-\delta(t+k\omega)},\tag{25}$$

$$|v_j(t+(k+1)\omega)-v_j(t+k\omega)|\le Ne^{-\delta(t+k\omega)},\tag{26}$$

for $i=1,2,\dots,n,\ \ ,j=1,2,\dots,m,\ \ t>0.$

It is noted that for any natural number p,

$$
\begin{aligned}
&u_i(t+(p+1)\omega)\\
&=u_i(t)+\sum_{k=0}^{p}(u_i(t+(k+1)\omega)-u_i(t+k\omega)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|u_i(t+(p+1)\omega)|\\
&\le|u_i(t)|+\sum_{k=0}^{p}|u_i(t+(k+1)\omega)-u_i(t+k\omega)|.
\end{aligned}\tag{27}
$$

Since $u_i(t)$ is bounded, using Weierstrass criteria, it follows (25) and (27) that $\{u(t+p\omega)\}$ uniformly converges to a continuous function $u^*(t)=(u_1^*(t),u_2^*(t),\cdots,u_n^*(t))$ on any compact set of R.

Similarly, since $v_j(t)$ is bounded, from (26) that $\{v(t+p\omega)\}$ uniformly converges to a continuous function $v^*(t)=(v_1^*(t),v_2^*(t),\cdots,v_n^*(t))$ on any compact set of R.

When $u_i(t),u_i'(t),v_j(t)$ and $v_j'(t)$ are bounded, similarly, they can be proved that $\{y(t+p\omega)\},\{z(t+p\omega)\}$ uniformly converge to continuous function $y^*(t)=(y_1^*(t),y_2^*(t),\cdots,y_n^*(t))$ and $z^*(t)=(z_1^*(t),z_2^*(t),\cdots,z_m^*(t))$ on any compact set of R, respectively.

Now we will show that $(u^{*T}(t),v^{*T}(t))^T$ is the $\omega-$ periodic solution of system (1). First, $u^*(t),u^*(t)$ are $\omega-$ periodic function, since

$$u^*(t+\omega)=\lim_{p\to\infty}u(t+(p+1)\omega)=u^*(t),$$

$$v^*(t+\omega)=\lim_{p\to\infty}v(t+(p+1)\omega)=v^*(t).$$

Second, we prove that $(u^{*T}(t), v^{*T}(t))^T$ is a solution of system (1).

In fact, since $I_i(t+p\omega) = I_i(t), J_j(t+p\omega) = J_j(t)$, and

$$\begin{cases} \frac{du_i(t+p\omega)}{dt} = -u_i(t+p\omega) + y_i(t+p\omega), \\ \frac{dy_i(t+p\omega)}{dt} = -(1-\alpha_i)u_i(t+p\omega) \\ \quad -(\alpha_i - 1)y_i(t+p\omega) - a_i(u_i(t+p\omega))\cdot \\ \quad [b_i(u_i(t+p\omega)) - \sum\limits_{j=1}^{m} c_{ij} f_j(v_j(t+p\omega)) \\ \quad - \sum\limits_{j=1}^{m} d_{ij} f_j(v_j(t+p\omega - \tau_{ji})) - I_i(t)], \\ \frac{dv_j(t+p\omega)}{dt} = -v_j(t+p\omega) + z_j(t+p\omega), \\ \frac{dz_j(t+p\omega)}{dt} = -(1-\beta_j)v_j(t+p\omega) \\ \quad -(\beta_j - 1)z_j(t+p\omega) - e_j(v_j(t+p\omega))\cdot \\ \quad [h_j(v_j(t+p\omega)) - \sum\limits_{i=1}^{n} p_{ji} g_i(u_i(t+p\omega)) \\ \quad - \sum\limits_{i=1}^{n} q_{ji} g_i(u_i(t+p\omega - \sigma_{ij})) - J_j(t)], \end{cases} \tag{28}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.

Since $\{u(t+p\omega)\}$ and $\{v(t+p\omega)\}$ uniformly converge to continuous function

$u^*(t) = (u_1^*(t), u_2^*(t), \cdots, u_n^*(t))$,

and

$v^*(t) = (v_1^*(t), v_2^*(t), \cdots, v_m^*(t))$,

respectively. $\{y(t+p\omega)\}$ and $\{z(t+p\omega)\}$ uniformly converge to a continuous function

$y^*(t) = (y_1^*(t), y_2^*(t), \cdots, y_n^*(t))$,

and

$z^*(t) = (z_1^*(t), z_2^*(t), \cdots, z_m^*(t))$,

respectively. Under the hypotheses $(\mathrm{H}_1) - (\mathrm{H}_5)$, (28) implies that $\{\frac{du_i(t+p\omega)}{dt}\}, \{\frac{dy_i(t+p\omega)}{dt}\}, \{\frac{dv_j(t+p\omega)}{dt}\}, \{\frac{dz_j(t+p\omega)}{dt}\}$ uniformly converge to continuous functions on any compact set of R, respectively. Thus, let $p \to \infty$, we obtain

$$\begin{cases} \frac{du_i^*(t)}{dt} = -u_i^*(t) + y_i^*(t), \\ \frac{dy_i^*(t)}{dt} = -(1-\alpha_i)u_i^*(t) - (\alpha_i - 1)y_i^*(t) \\ \quad -a_i(u_i^*(t))[b_i(u_i^*(t)) - \sum\limits_{j=1}^{m} c_{ij} f_j(v_j^*(t)) \\ \quad - \sum\limits_{j=1}^{m} d_{ij} f_j(v_j^*(t - \tau_{ji})) - I_i(t)], \\ \frac{dv_j^*(t)}{dt} = -v_j^*(t) + z_j^*(t), \\ \frac{dz_j^*(t)}{dt} = -(1-\beta_j)v_j^*(t) - (\beta_j - 1)z_j^*(t) \\ \quad -e_j(v_j^*(t))[h_j(v_j^*(t)) - \sum\limits_{i=1}^{n} p_{ji} g_i(u_i^*(t)) \\ \quad - \sum\limits_{i=1}^{n} q_{ji} g_i(u_i^*(t - \sigma_{ij})) - J_j(t)], \end{cases} \tag{29}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.

That is $(u^{*T}(t), v^{*T}(t))$ is a periodic solution of system (1). From (24), we have

$$\begin{aligned} &\sum_{i=1}^{n} |u_i(t) - u_i^*(t)|^2 + \sum_{j=1}^{m} |v_j(t) - v_j^*(t)|^2 \\ &\quad \le M e^{-2\varepsilon t}(\|\varphi_u - \varphi_u^*\|^2 + \|\varphi_v - \varphi_v^*\|^2), \end{aligned}$$

for all $t \ge 0$, then system (1) has one $\omega-$ periodic solution, which is globally exponentially stable. □

4 Numerical example

In this section, we give an example for showing our results.

Example 1 Consider the following inertial BAM neural networks with time delay

$$\begin{cases} \frac{d^2u_i(t)}{dt^2} = -\alpha_i \frac{du_i(t)}{dt} - a_i(u_i(t))[b_i(u_i(t))) \\ \quad - \sum\limits_{j=1}^{2} c_{ij} f_j(v_j(t)) - \sum\limits_{j=1}^{2} d_{ij} f_j(v_j(t - \tau_{ji})) - I_i(t)], \\ \frac{d^2v_j(t)}{dt^2} = -\beta_j \frac{dv_j(t)}{dt} - e_j(v_j(t))[h_j(v_j(t)) \\ \quad - \sum\limits_{i=1}^{2} p_{ji} g_i(u_i(t)) - \sum\limits_{i=1}^{2} q_{ji} g_i(u_i(t - \sigma_{ij})) - J_j(t)], \end{cases} \tag{30}$$

for $i = 1, 2, j = 1, 2$, where

$$\begin{aligned} &f_j(x) = g_i(x) = \frac{1}{2}(|x+1| - |x-1|), \\ &a_i(x) = e_j(x) = \frac{\pi + \arctan x}{8}, \\ &b_i(x) = h_j(x) = x, \\ &I_i(t) = 0.1 \sin 2t,\ J_j(t) = 0.1 \cos 2t, \\ &\alpha_i = 2,\ \beta_j = 1.9,\ c_{ij} = d_{ij} = p_{ji} = q_{ji} = 0.005. \end{aligned}$$

Obviously, $f_j(x), g_i(x)$ satisfies the condition of hypotheses (H_1) and $L_j = N_i = \bar{f}_j = \bar{g}_i = 1$; $I_i(t), J_j(t)$ satisfies the condition of hypotheses (H_2) and $\bar{I} = \bar{J} = 0.1$, $a_i(x), e_j(x), b_i(x), h_j(t)$ satisfies the condition of hypotheses $(\mathrm{H}_3) - (\mathrm{H}_5)$ and

$$\begin{array}{ll} a_i^* = e_j^* = 1, & \bar{a}_i = \bar{e}_j = \frac{3\pi}{16}, \\ \underline{a}_i = \underline{e}_j = \frac{\pi}{16}, & \\ \underline{b}_i = \underline{h}_j = 0, & \bar{b}_i = \bar{h}_j = 1, \\ \bar{K}_i = \bar{T}_j = \frac{3\pi+1}{16}, & \underline{K}_i = \underline{T}_j = \frac{\pi-1}{16} \end{array}$$

By simple calculation

$$\alpha_i - \bar{K}_i > 0,\ \beta_j - \bar{T}_j > 0,$$

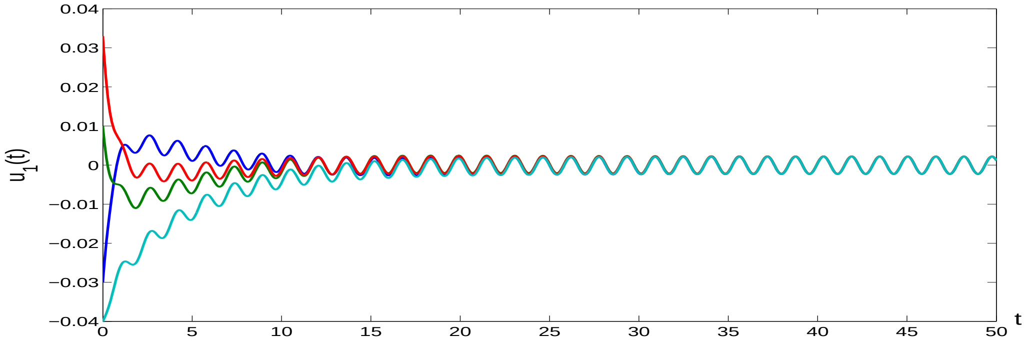

Fig.1. Transient response of state variables $u_1(t)$ of Example 4.1

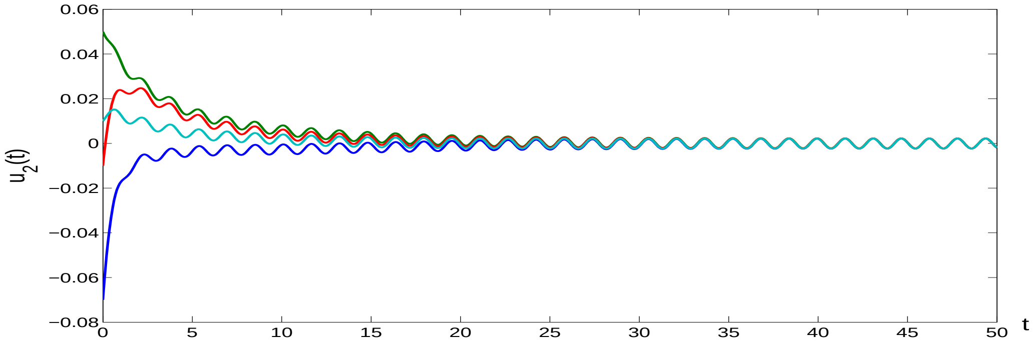

Fig.2. Transient response of state variables $u_2(t)$ of Example 4.1

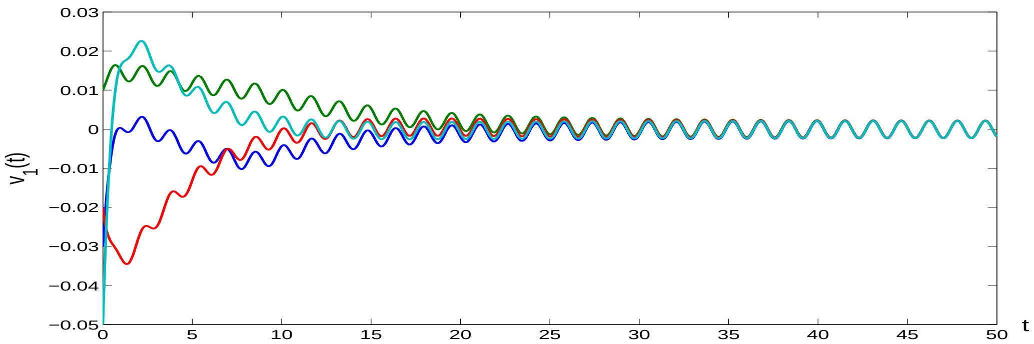

Fig.3. Transient response of state variables $v_1(t)$ of Example 4.1

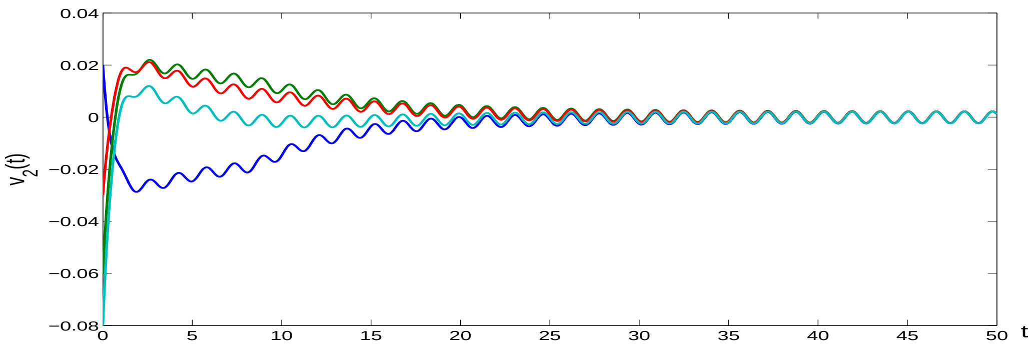

Fig.4. Transient response of state variables $v_2(t)$ of Example 4.1

$$
\begin{aligned}
&-2+\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\qquad+\sum_{j=1}^{m}\bar{e}_jN_i(|p_{ji}|+|q_{ji}|)+a_i^*\bar{I}_i<0,\\
&2-\alpha_i-\underline{K}_i+a_i^*\sum_{j=1}^{m}\bar{f}_j(|c_{ij}|+|d_{ij}|)\\
&\qquad+\sum_{j=1}^{m}\bar{a}_iL_j(|c_{ij}|+|d_{ij}|)+a_i^*\bar{I}_i<0,\\
&-2+\beta_j-\underline{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|)\\
&\qquad+\sum_{i=1}^{n}\bar{a}_iL_j(|c_{ij}|+|d_{ij}|)+e_j^*\bar{J}_j<0,\\
&2-\beta_j-\underline{T}_j+e_j^*\sum_{i=1}^{n}\bar{g}_i(|p_{ji}|+|q_{ji}|)\\
&\qquad+\sum_{i=1}^{n}\bar{e}_jN_i(|p_{ji}|+|q_{ji}|)+e_j^*\bar{J}_j<0,
\end{aligned}
$$

Then, the conditions of Theorem 2 hold. It follows from Theorem 2 that this system (30) has an unique π-periodic solution, and all other solutions of system (30) exponentially converge to it as $t \to +\infty$.

For numerical simulation, the following four cases are given:

Case 1 with the initial state

$$
\begin{aligned}
&[\varphi_{u1}(0),\psi_{u1}(0),\varphi_{v1}(0),\psi_{v1}(0),\\
&\quad\varphi_{u2}(0),\psi_{u2}(0),\varphi_{v2}(0),\psi_{v2}(0)]\\
&=[-0.03;0.03;-0.07;0.04;\\
&\quad-0.03;0.04;0.02;-0.08];
\end{aligned}
$$

Case 2 with the initial state

$$
\begin{aligned}
&[\varphi_{u1}(0),\psi_{u1}(0),\varphi_{v1}(0),\psi_{v1}(0),\\
&\quad\varphi_{u2}(0),\psi_{u2}(0),\varphi_{v2}(0),\psi_{v2}(0)]\\
&=[0.01;-0.04;0.05;0.03;\\
&\quad0.01;0.02;-0.06;0.08];
\end{aligned}
$$

Case 3 with the initial state

$$
\begin{aligned}
&[\varphi_{u1}(0),\psi_{u1}(0),\varphi_{v1}(0),\psi_{v1}(0),\\
&\varphi_{u2}(0),\psi_{u2}(0),\varphi_{v2}(0),\psi_{v2}(0)]\\
&=[0.033;-0.04;-0.01;0.07;\\
&\quad-0.02;-0.06;-0.03;0.06];
\end{aligned}
$$

Case 4 with the initial state

$$
\begin{aligned}
&[\varphi_{u1}(0),\psi_{u1}(0),\varphi_{v1}(0),\psi_{v1}(0),\\
&\quad\varphi_{u2}(0),\psi_{u2}(0),\varphi_{v2}(0),\psi_{v2}(0)]\\
&=[-0.04;-0.03;0.01;0.02;\\
&\quad-0.05;0.1;-0.08;0.09].
\end{aligned}
$$

Figs.1 - Figs.4 depict the time responses of state variables of $u_1(t), u_2(t), v_1(t), v_2(t)$ of system (30), respectively.

Evidently, this consequence is coincident with the results of Theorem 2.

5 Conclusion

In this paper, we give the theorem to ensure the existence and the exponential stability of the periodic solution for Cohen-Grossberg-type BAM neural networks. Novel existence and stability conditions are stated in simple algebraic forms so that their verification and applications are straightforward and convenient. Finally an example illustrate the effectiveness.

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