Positive Periodic Solutions of A Generalized Gilpin-Ayala Competitive System With Time Delays

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Abstract: In this paper, we investigate a generalized Gilpin-Ayala competition system which is more general and more realistic than the classical Lotka-Volterra competition system. By the fixed-point theorem and differential mean value, some sufficient conditions guaranteeing the existence, uniqueness and exponential stability of positive periodic solutions for a generalized Gilpin-Ayala competition system with time delays are given. Two illustrative examples are also given in the end to show the effectiveness of our results.

Key–Words: exponential stability;Gilpin-Ayala competition system; periodic solution; fixed-point theorem

1 Introduction

The role of spatial heterogeneity and dispersal in the dynamics of populations has been an important research subject. There are many works about it in the literature[1-3]. While many models are mainly based on Lotka-Volterra systems. Since that time, many different forms of the Lotka-Volterra competition model have been studied (see, for example, [4], [5]). However, the Lotka-Volterra competition model is linear (i.e. the rate of change in the size of each species is a linear function of sizes of the interacting species) and this property is considered as a disadvantage of this model. In 1973, Gilpin and Ayala [6] claimed that a little more complicated model was needed in order to obtain more realistic solutions, so they proposed a few competition models, for example,

$$\frac{dN_i(t)}{dt} = r_i N_i(t)[1 - (\frac{N_i}{k_i})^{\theta_i} - a_{ij}\frac{N_j}{k_i}],$$

$i, j = 1, 2, \cdots, d.$ where θ_i are the parameters which modify the classical Lotka-Volterra model and they represent a nonlinear measure of interspecific interference $(i = 1, 2, \ldots, d)$. It was noticed that the Gilpin-Ayala model has even some properties which do not exist in the Lotka-Volterra model [7].

In recent years, many researchers have studied the global stability and other dynamical behaviors of the Gilpin-Ayala competition model, see [5,6,8,9].

At the same time, many different forms of the Lotka-Volterra competition model have been studied. For example, [10] considered the n-species Lotka-Volterra systems

$$\frac{dx_i(t)}{dt} = x_i(t)[r_i - \sum_{j=1}^{n} a_{ij}x_j(t)]. \qquad (A1)$$

[11] investigate the Positive almost periodic solutions of Lotka-Volterra recurrent neural networks by the following delayed differential equations:

$$\begin{aligned}\frac{dx_i(t)}{dt} &= x_i(t)[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\ &\quad - \sum_{j=1}^{n} b_{ij}x_j(t-\tau_{ij}(t))].\end{aligned} \qquad (A2)$$

Moreover, several important results for periodic solutions of Gilpin-Ayala competition model have been obtained in Refs. [7,12-15]. For example, The authors in [12,14] have investigated existence and attractivity of periodic solution for Gilpin-Ayala competition system.

However, to our knowledge, few papers have been published on the exponential stability of positive periodic solutions for a generalized Gilpin-Ayala competitive system. In this paper, we will investigate the globally exponential stability of positive periodic solutions for the following generalized Gilpin-Ayala competitive system with time delays

$$\begin{aligned}\tfrac{dx_i(t)}{dt} &= x_i(t)[r_i(t) - a_{ii}(t)x_i^{\theta_i}(t) \\ &- \sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij})],\end{aligned} \tag{1}$$

for $i = 1, 2, \cdots, n$, where $x_i(t), r_i$ and a_{ii} are the population size at a time t, the intrinsic exponential growth rate and the carrying capacity in the absence of competition, respectively, for the $i-th$ species; $\theta_i > 0$ are the parameters that modify the classical Lotka-Volterra model; $a_{ij}(i \neq j = 1, 2\cdots, n)$ and $b_{ij}(i, j = 1, 2\cdots, n)$ represent the effect of interspecific interaction, respectively; τ_{ij} is time delay of at the time t.

The initial conditions of system (1) are given by

$$x_i(s) = \phi_i(s), s \in [-\tau, 0], \tag{2}$$

where $i = 1, 2, \ldots .n$, $\tau = \max\limits_{1\leq i\leq n, 1\leq j\leq m}\{\tau_{ij}\}$, $\phi_i(s) > 0$ are bounded and continuous on $[-\tau, 0]$.

2 Preliminaries

In order to establish the existence, uniqueness and exponential stability of positive periodic solution for system (1), we give assumptions.

- (H) : For each $i, j = 1, 2, \ldots, n, r_i(t),\quad a_{ij}(t)$, $b_{ij}(t)$ are $\omega-$ periodic continuously functions and satisfy

$$0 < \underline{r}_i \leq r_i(t) \leq \bar{r}_i,$$
$$0 < \underline{a}_{ij} \leq a_{ij}(t) \leq \bar{a}_{ij},$$
$$0 \leq \underline{b}_{ij} \leq b_{ij}(t) \leq \bar{b}_{ij}.$$

Definition 1 *Let $x^*(t) = (x_1^*(t), x_2^*(t), \cdots x_n^*(t)))^T$ be an $\omega-$ periodic solution of system (1) with initial value*

$$\psi = (\psi_1(t), \psi_2(t), \cdots, \psi_n(t))^T.$$

If there exist constants $\alpha > 0$ and $M > 0$, for every solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ of system (1) with initial value

$$\varphi = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T,$$

such that

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \leq Me^{-\alpha t}\|\psi - \varphi\|, t > 0,$$

then $\omega-$ periodic solution $x^*(t)$ is said to be exponentially stable, where

$$\|\psi - \varphi\| = \sup_{-\tau\leq t\leq 0} \sum_{i=1}^{n} |\psi_i(t) - \varphi_i(t)|.$$

Let

$$d_i^{\theta_i} = \frac{\bar{r}_i}{\underline{a}_{ii}},$$

$$A_i = \underline{r}_i - \sum_{j=1,j\neq i}^{n} \bar{a}_{ij}d_j - \sum_{j=1}^{n} \bar{b}_{ij}d_j,$$

$$e_i^{\theta_i} = \frac{A_i}{\bar{a}_{ii}},$$

$$d = \max_{1\leq i\leq n}\{d_i\},\ \ i = 1, 2, \cdots, n.$$

$$\bar{r} = \min_{1\leq i\leq n}\{\bar{r}_i\}, \qquad \bar{a} = \max_{1\leq i\leq n}\{\bar{a}_{ii}\},$$
$$q_i = e^{-\int_0^\omega r_i(u)du}, \quad q = \max_{1\leq i\leq n}\{q_i\},$$

$$R_i(t, s) = \frac{e^{-\int_t^s r_i(u)du}}{1 - e^{-\int_0^\omega r_i(u)du}}, \quad s \in [t, t+\omega].$$

Lemma 2 *Under hypotheses (H), if for any given initial value $\varphi_i(0) > 0$, then there is a unique positive solution $x(t)$ of system (1) with satisfy initial value , for $t \geq 0$.*

Proof. From (1), we can obtain

$$\begin{aligned}x_i(t) &= x_i(0)\exp\{\textstyle\int_0^t [r_i(s) - a_{ii}(s)x_i^{\theta_i}(s) \\ &- \sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(s) - \sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds\},\end{aligned}$$

where $i = 1, 2, \cdots, n$.

With the initial value $\varphi_i(0) > 0$, we know

$$\begin{aligned}x_i(t) &= \varphi_i(0)\exp\{\textstyle\int_0^t [r_i(s) - a_{ii}(s)x_i^{\theta_i}(s) \\ &- \sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(s) - \sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds\},\end{aligned}$$

where $i = 1, 2, \cdots, n$. Obviously, this solution is a unique positive solution of system (1).

Lemma 3 *Under hypotheses (H), if $A_i > 0, 0 < e_i \leq \varphi_i(0) \leq d_i, i = 1, 2, \cdots, n$, let $x(t)$ be an positive solution of system (1) with satisfy initial value (2), then $x(t)$ is bounded, and*

$$e_i \leq x_i(t) \leq d_i,\ \ i = 1, 2, \cdots, n.$$

Proof. From (1), if $x(t)$ is an positive solution, we have

$$\frac{dx_i(t)}{dt} \leq x_i(t)[r_i(t) - a_{ii}(t)x_i^{\theta_i}(t)],$$

or

$$x_i^{-(\theta_i+1)}(t)\frac{dx_i(t)}{dt} - r_i(t)x_i^{-\theta_i}(t) + a_{ii}(t) \leq 0. \tag{3}$$

Let $z_i(t) = x_i^{-\theta_i}(t)$. We have

$$\frac{dz_i(t)}{dt} + \theta_i r_i(t) z_i(t) - \theta_i a_{ii}(t) \geq 0,$$

or

$$\frac{dz_i(t)}{dt} + \theta_i \bar{r}_i z_i(t) \geq \theta_i \underline{a}_{ii}. \tag{4}$$

From (4), we have

$$\frac{d}{dt}(e^{\theta_i \bar{r}_i t} z_i(t)) \geq e^{\theta_i \bar{r}_i t} \theta_i \underline{a}_{ii},$$

thus we can obtain

$$z_i(t) \geq [z_i(0) - \frac{\underline{a}_{ii}}{\bar{r}_i}] e^{-\theta_i \bar{r}_i t} + \frac{\underline{a}_{ii}}{\bar{r}_i}. \tag{5}$$

If $\varphi_i(0) \leq d_i$, from (5) we have

$$x_i^{\theta_i}(t) \leq \frac{1}{[\varphi_i^{-\theta_i}(0) - \frac{\underline{a}_{ii}}{\bar{r}_i}] e^{-\theta_i \bar{r}_i t} + \frac{\underline{a}_{ii}}{\bar{r}_i}} \leq \frac{\bar{r}_i}{\underline{a}_{ii}},$$

i.e.,

$$x_i(t) \leq (\frac{\bar{r}_i}{\underline{a}_{ii}})^{\frac{1}{\theta_i}} = d_i, \ \ i = 1, 2, \cdots, n.$$

On the other hand, when $x_i(t) \leq d_i$, from (1)

$$\frac{dx_i(t)}{dt} \geq x_i(t)[\underline{r}_i - \bar{a}_{ii} x_i^{\theta_i}(t) - \sum_{j=1, j\neq i}^{n} \bar{a}_{ij} d_j - \sum_{j=1}^{n} \bar{b}_{ij} d_j],$$

or

$$x_i^{-(\theta_i+1)}(t) \frac{dx_i(t)}{dt} - A_i x_i^{-\theta_i}(t) + \bar{a}_{ii} \geq 0. \tag{6}$$

Let $z_i(t) = x_i^{-\theta_i}(t)$. We have

$$\frac{dz_i(t)}{dt} + \theta_i A_i z_i(t) - \theta_i \bar{a}_{ii} \leq 0. \tag{7}$$

If $e_i \leq \varphi_i(0)$, from (7) we can obtain

$$z_i(t) \leq [z_i(0) - \frac{\bar{a}_{ii}}{A_i}] e^{-\theta_i A_i t} + \frac{\bar{a}_{ii}}{A_i} \leq \frac{\bar{a}_{ii}}{A_i}. \tag{8}$$

From (8), we have

$$x_i^{\theta_i}(t) \geq \frac{A_i}{\bar{a}_{ii}},$$

i.e.,

$$x_i(t) \geq e_i, \ \ i = 1, 2, \cdots, n.$$

We obtain

$$e_i \leq x_i(t) \leq d_i, \ \ i = 1, 2, \cdots, n.$$

3 Main results

In this section, we will derive some sufficient conditions which ensure the existence, uniqueness and the exponential stability of positive periodic solution for system (1).

Let

$$X = \{x = (x_1(t), x_2(t), \cdots, x_n(t))^T \in C(R, R^n) :$$

$$x(t+\omega) = x(t), t \in R\}$$

with the norm defined by

$$|x_i| = \max_{t\in[0,\omega]} \{|x_i(t)|\},$$

$$\|x\| = \max_{1\leq i\leq n} \{|x_i|\}.$$

Define the cone P in X by

$$P = \{x = (x_1(t), x_2(t), \cdots, x_n(t))^T \in X :$$

$$0 < x_i(t) \leq (\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{\theta_i}}, t \in [0, \omega], i = 1, 2, \cdots n\}.$$

Theorem 4 *Under the hypotheses (H), and all of the conditions in Lemma 2 are satisfied, if*

$$\frac{\omega}{1-q_i}\{(\theta_i+1)\bar{a}_{ii}\frac{(1-q)\bar{r}}{\bar{a}} + \sum_{j=1,j\neq i}^{n}[(\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{\theta_i}} + (\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{\theta_j}}]\bar{a}_{ij} + \sum_{j=1}^{n}[(\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{\theta_i}} + (\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{\theta_j}}]\bar{b}_{ij}\} < 1,$$

for $i = 1, 2, \cdots, n$, then system (1) has one positive $\omega-$ periodic solution.

Proof. Let the map ψ be defined by

$$(\psi x)(t) = ((\psi x)_1, (\psi x)_2, \cdots, (\psi x)_n)^T,$$

where $x \in P$, $\quad t \in R$, for$i = 1, 2, \cdots, n$,

$$(\psi x)_i(t) = \int_t^{t+\omega} R_i(t,s) x_i(s)[a_{ii}(s) x_i^{\theta_i}(s) + \sum_{j=1,j\neq i}^{n} a_{ij}(s) x_j(s) + \sum_{j=1}^{n} b_{ij}(s) x_j(s-\tau_{ij})] ds. \tag{9}$$

Since

$$R_i(t+\omega, s+\omega) = R_i(t,s),$$

$$\frac{q_i}{1-q_i} \leq R_i(t,s) \leq \frac{1}{1-q_i},$$

where $i=1,2,\cdots,n$.For any $x\in P$, it is easy to see that $\psi x\in C(R,R^n)$. From (9), we obtain

$$\begin{aligned}(\psi x)_i(t+\omega) &= \int_{t+\omega}^{t+2\omega} R_i(t+\omega,s)x_i(s)[a_{ii}(s)x_i^{\theta_i}(s) \\ &\quad + \sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(s)+\sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds \\ &= \int_t^{t+\omega} R_i(t+\omega,z+\omega)x_i(z+\omega)[a_{ii}(z \\ &\quad +\omega)x_i^{\theta_i}(z+\omega)+\sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(z+\omega) \\ &\quad +\sum_{j=1}^{n} b_{ij}(z+\omega)x_j(z+\omega-\tau_{ij})]dz \\ &= \int_t^{t+\omega} R_i(t,s)x_i(s)[a_{ii}(s)x_i^{\theta_i}(s) \\ &\quad +\sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(s)+\sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds \\ &= (\psi x)_i(t).\end{aligned}$$

and

$$\begin{aligned}(\psi x)_i(t) &\geq \frac{q_i}{1-q_i}\int_t^{t+\omega} x_i(s)[a_{ii}(s)x_i^{\theta_i}(s) \\ &\quad +\sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(s)+\sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds \\ &= \frac{q_i}{1-q_i}\int_0^{\omega} x_i(s)[a_{ii}(s)x_i^{\theta_i}(s)+\sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(s)) \\ &\quad +\sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds>0.\end{aligned}$$

$$\begin{aligned}(\psi x)_i(t) &\leq \frac{1}{1-q_i}\int_t^{t+\omega} x_i(s)[a_{ii}(s)x_i^{\theta_i}(s) \\ &\quad +\sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(s)+\sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij})]ds \\ &= \frac{1}{1-q_i}\int_0^{\omega} x_i(s)[a_{ii}(s)x_i^{\theta_i}(s)+\sum_{j=1,j\neq i}^{n} a_{ij}(s)x_j(s)) \\ &\quad +\sum_{j=1}^{n} b_{ij}(s)x_j(s-\tau_{ij}))]ds \\ &\leq \frac{\omega}{1-q_i}(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}[\bar a_{ii}\frac{(1-q)\bar r}{\bar a} \\ &\quad +\sum_{j=1,j\neq i}^{n}(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}\bar a_{ij}+\sum_{j=1}^{n}(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}\bar b_{ij}] \\ &\leq (\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}.\end{aligned}$$

Hence we obtain $\psi x\in P$.

Next we will prove ψ is contraction mapping. For any $x,y\in P$, we have

$$\begin{aligned}&(\psi x)_i(t)-(\psi y)_i(t) \\ &= \int_t^{t+\omega} R_i(t,s)\{a_{ii}(s)(x_i^{\theta_i+1}(s)-y_i^{\theta_i+1}(s) \\ &\quad +\sum_{j=1,j\neq i}^{n} a_{ij}(s)[x_i(s)x_j(s)-y_i(s)y_j(s)] \\ &\quad +\sum_{j=1}^{n} b_{ij}(s)[x_i(s)x_j(s-\tau_{ij})-y_i(s)y_j(s-\tau_{ij})]\}ds \\ &= \int_t^{t+\omega} R_i(t,s)\{(\theta_i+1)a_{ii}(s)(\xi_i^{\theta_i}(x_i(s)-y_i(s)) \\ &\quad +\sum_{j=1,j\neq i}^{n} a_{ij}(s)[x_i(s)(x_j(s)-y_j(s))+y_j(s)(x_i(s) \\ &\quad -y_i(s))]+\sum_{j=1}^{n} b_{ij}(s)[x_i(s)(x_j(s-\tau_{ij})-y_j(s-\tau_{ij})) \\ &\quad +y_j(s-\tau_{ij})(x_i(s)-y_i(s))]]\}ds,\end{aligned} \tag{10}$$

where ξ_i lies between $x_i(t)$ and $y_i(t)$.

From (10), we can obtain

$$\begin{aligned}&\|(\psi x)-(\psi y)\| = \max_{1\leq i\leq n}\max_{t\in[0,\omega]}|(\psi x)_i(t)-(\psi y)_i(t)| \\ &\leq \max_{1\leq i\leq n}\max_{t\in[0,\omega]}\frac{1}{1-q_i}\int_t^{t+\omega}\{(\theta_i+1)a_{ii}(s)\xi_i^{\theta_i}|x_i(s) \\ &\quad -y_i(s)|+\sum_{j=1,j\neq i}^{n} a_{ij}(s)[x_i(s)|x_j(s)-y_j(s)| \\ &\quad +y_j(s)|x_i(s)-y_i(s)|]+\sum_{j=1}^{n} b_{ij}(s)[x_i(s)|x_j(s-\tau_{ij}) \\ &\quad -y_j(s-\tau_{ij})|+y_j(s-\tau_{ij})|x_i(s)-y_i(s)|]\}ds \\ &\leq \max_{1\leq i\leq n}\max_{t\in[0,\omega]}\frac{1}{1-q_i}\int_0^{\omega}\{(\theta_i+1)\bar a_{ii}\frac{(1-q)\bar r}{\bar a}|x_i(s) \\ &\quad -y_i(s)|+\sum_{j=1,j\neq i}^{n}\bar a_{ij}[(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}|x_j(s)-y_j(s)| \\ &\quad +(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}|x_i(s)-y_i(s)|]+\sum_{j=1}^{n}\bar b_{ij}[(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}} \\ &\quad \cdot|x_j(s-\tau_{ij})-y_j(s-\tau_{ij})|+(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}|x_i(s)-y_i(s)|]\}ds \\ &\leq \max_{1\leq i\leq n}\frac{1}{1-q_i}\int_0^{\omega}\{(\theta_i+1)\bar a_{ii}\frac{(1-q)\bar r}{\bar a} \\ &\quad +\sum_{j=1,j\neq i}^{n}[(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}+(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}]\bar a_{ij} \\ &\quad +\sum_{j=1}^{n}[(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}+(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}]\bar b_{ij}\}\|x-y\|ds \\ &\leq \max_{1\leq i\leq n}\frac{\omega}{1-q_i}\{(\theta_i+1)\bar a_{ii}\frac{(1-q)\bar r}{\bar a} \\ &\quad +\sum_{j=1,j\neq i}^{n}[(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}+(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}]\bar a_{ij} \\ &\quad +\sum_{j=1}^{n}[(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_i}}+(\frac{(1-q)\bar r}{\bar a})^{\frac{1}{\theta_j}}]\bar b_{ij}\}\|x-y\| \\ &< \|x-y\|.\end{aligned}$$

Thus, ψ is contraction mapping, by fixed-point theorem, it follows that there exist uniqueness an fixed point $x^*(t)$ satisfying

$$\psi(x^*)(t)=x^*(t).$$

Now we will show that $x^*(t)$ is the positive $\omega-$ periodic solution of (1). From (9), we obtain

$$\begin{aligned}&\frac{d(\psi x)_i(t)}{dt} \\ &= r_i(t)(\psi x)_i(t)+R_i(t,t+\omega)x_i(t+\omega) \\ &\quad [a_{ii}(t+\omega)x_i^{\theta_i}(t+\omega)+\sum_{j=1,j\neq i}^{n} a_{ij}(t+\omega)x_j(t+\omega)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} b_{ij}(t+\omega)x_j(t+\omega-\tau_{ij})]\\
&-R_i(t,t)x_i(t)[a_{ii}(t)x_i^{\theta_i}(t)+\sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(t)\\
&+\sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij})]\\
&=r_i(t)(\psi x)_i(t)+(\tfrac{q_i}{1-q_i}-\tfrac{1}{1-q_i})x_i(t)[a_{ii}(t)x_i^{\theta_i}(t)\\
&+\sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(t)+\sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij})]\\
&=r_i(t)(\psi x)_i(t)-x_i(t)[a_{ii}(t)x_i^{\theta_i}(t)+\sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(t)\\
&+\sum_{j=1}^{n} b_{ij}(t)x_j(t-\tau_{ij})].
\end{aligned}
$$

We have

$$
\begin{aligned}
&\tfrac{dx_i^*(t)}{dt}=r_i(t)x_i^*(t)-x_i^*(t)[a_{ii}(t)(x_i^*(t))^{\theta_i}\\
&+\sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j^*(t)+\sum_{j=1}^{n} b_{ij}(t)x_j^*(t-\tau_{ij})],
\end{aligned}
$$

and

$$x_i^*(t+\omega)=x_i^*(t).$$

From the above, we can see that system (1) has one positive $\omega-$ periodic solution.

Theorem 5 *Under the hypotheses (H), and all of the conditions in Theorem 4 are satisfied, if $\theta_i \geq 1$ and*

$$-\underline{a}_{ii}(\frac{A_i}{\bar{a}_{ii}})^{\frac{\theta_i-1}{\theta_i}}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}+\sum_{j=1}^{n}\bar{b}_{ji}<0,$$

$i=1,2,\cdots,n$.Then system (1) has a unique positive $\omega-$ periodic solution which is globally exponentially stable.

Proof. By using Theorem 4, system (1) has a unique positive $\omega-$ periodic solution. In the following we will prove the unique positive $\omega-$ periodic solution is globally exponentially stable. Let $\bar{x}(t)=(\bar{x}_1(t),\bar{x}_2(t),\cdots,\bar{x}_n(t))^T$ be an positive $\omega-$ periodic solution of system (1) with initial value

$$\bar{x}_i(s)=\bar{\varphi}_i(s),\ \ e_i\leq\bar{\varphi}_i(0)\leq d_i,\quad -\tau\leq s\leq 0,$$

Let $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T$ be an any solution of system (1) with initial value

$$x_i(s)=\varphi_i(s),\ \ e_i\leq\varphi_i(0)\leq d_i,\quad -\tau\leq s\leq 0.$$

From Lemma 3, we have

$$x_i(t)>0,\bar{x}_i(t)>0,i=1,2,\cdots,n.$$

Let

$$y_i(t)=\ln x_i(t),\quad \bar{y}_i(t)=\ln\bar{x}_i(t),i=1,2,\ldots,n.$$

From (1), we can obtain

$$
\begin{aligned}
&\tfrac{dy_i(t)}{dt}=r_i(t)-a_{ii}(t)e^{\theta_i y_i(t)}\\
&-\sum_{j=1,j\neq i}^{n} a_{ij}(t)e^{y_j(t)}-\sum_{j=1}^{n} b_{ij}(t)e^{y_j(t-\tau_{ij})}.
\end{aligned}\tag{11}
$$

$$
\begin{aligned}
&\tfrac{d\bar{y}_i(t)}{dt}=r_i(t)-a_{ii}(t)e^{\theta_i\bar{y}_i(t)}\\
&-\sum_{j=1,j\neq i}^{n} a_{ij}(t)e^{\bar{y}_j(t)}-\sum_{j=1}^{n} b_{ij}(t)e^{\bar{y}_j(t-\tau_{ij})}.
\end{aligned}\tag{12}
$$

From (11) and (12), we have

$$
\begin{aligned}
&\tfrac{d(y_i(t)-\bar{y}_i(t))}{dt}\\
&=-a_{ii}(t)(e^{\theta_i y_i(t)}-e^{\theta_i\bar{y}_i(t)})-\sum_{j=1,j\neq i}^{n} a_{ij}(t)(e^{y_j(t)}\\
&-e^{\bar{y}_j(t)})-\sum_{j=1}^{n} b_{ij}(t)(e^{y_j(t-\tau_{ij})}-e^{\bar{y}_j(t-\tau_{ij})}).
\end{aligned}\tag{13}
$$

If $\theta_i\geq 1$, we can obtain

$$
\begin{aligned}
&sgn(y_i(t)-\bar{y}_i(t))(e^{\theta_i y_i(t)}-e^{\theta_i\bar{y}_i(t)})\\
&\geq e^{(\theta_i-1)\bar{y}_i(t)}|e^{y_i(t)}-e^{\bar{y}_i(t)}|.
\end{aligned}\tag{14}
$$

for $i=1,2,\cdots,n,t>0$.

We consider the Lyapunov functional:

$$
\begin{aligned}
&V(t)=\sum_{i=1}^{n}[e^{\varepsilon t}|y_i(t)-\bar{y}_i(t)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\int_{t-\tau_{ij}}^{t} e^{\varepsilon(s+\tau_{ij})}|e^{y_j(s)}-e^{\bar{y}_j(s)}|ds]
\end{aligned}\tag{15}
$$

where $\varepsilon>0$ is a small number.

Calculating the upper right Dini-derivative $D^+V(t)$ of $V(t)$ along the solution of (13), using (14) we have

$$
\begin{aligned}
&D^+V(t)=\sum_{i=1}^{n}\{\varepsilon e^{\varepsilon t}|y_i(t)-\bar{y}_i(t)|+e^{\varepsilon t}sgn(y_i(t)\\
&-\bar{y}_i(t))\tfrac{d(y_i(t)-\bar{y}_i(t))}{dt}+\sum_{j=1}^{n}\bar{b}_{ij}[e^{\varepsilon(t+\tau_{ij})}|e^{y_j(t)}-e^{\bar{y}_j(t)}|\\
&-e^{\varepsilon t}|e^{y_j(t-\tau_{ij})}-e^{\bar{y}_j(t-\tau_{ij})}|]\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{\varepsilon t}\sum_{i=1}^{n}\{\varepsilon|y_i(t)-\bar{y}_i(t)|+sgn(y_i(t)-\bar{y}_i(t))\\
&\cdot[-a_{ii}(t)(e^{\theta_i y_i(t)}-e^{\theta_i\bar{y}_i(t)})-\sum_{j=1,j\neq i}^{n} a_{ij}(t)(e^{y_j(t)}\\
&-e^{\bar{y}_j(t)})-\sum_{j=1}^{n} b_{ij}(t)(e^{y_j(t-\tau_{ij})}-e^{\bar{y}_j(t-\tau_{ij})})]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\bar{b}_{ij}[e^{\varepsilon\tau_{ij}}|e^{y_j(t)}-e^{\bar{y}_j(t)}|-|e^{y_j(t-\tau_{ij})}-e^{\bar{y}_j(t-\tau_{ij})}|]\}\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\{\varepsilon|y_i(t)-\bar{y}_i(t)|-\underline{a}_{ii}e^{(\theta_i-1)\bar{y}_i(t)}|e^{y_i(t)}\\
&-e^{\bar{y}_i(t)}|+\sum_{j=1,j\neq i}^{n}\bar{a}_{ij}|e^{y_j(t)}-e^{\bar{y}_j(t)}|+\sum_{j=1}^{n}\bar{b}_{ij}\\
&\cdot|e^{y_j(t-\tau_{ij})}-e^{\bar{y}_j(t-\tau_{ij})}|]+\sum_{j=1}^{n}\bar{b}_{ij}[e^{\varepsilon\tau_{ij}}|e^{y_j(t)}\\
&-e^{\bar{y}_j(t)}|-|e^{y_j(t-\tau_{ij})}-e^{\bar{y}_j(t-\tau_{ij})}|]\}\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\{\varepsilon|y_i(t)-\bar{y}_i(t)|-\underline{a}_{ii}e^{(\theta_i-1)\bar{y}_i(t)}\\
&\cdot|e^{y_i(t)}-e^{\bar{y}_i(t)}|+\sum_{j=1,j\neq i}^{n}\bar{a}_{ij}|e^{y_j(t)}-e^{\bar{y}_j(t)}|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}e^{\varepsilon\tau_{ij}}|e^{y_j(t)}-e^{\bar{y}_j(t)}|\}\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\{\tfrac{\varepsilon}{\xi_i}-\underline{a}_{ii}(\tfrac{A_i}{\bar{a}_{ii}})^{\frac{\theta_i-1}{\theta_i}}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}\\
&+e^{\varepsilon\tau}\sum_{j=1}^{n}\bar{b}_{ji}\}|x_i(t)-\bar{x}_i(t)|\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\{\tfrac{\varepsilon}{e_i}-\underline{a}_{ii}(\tfrac{A_i}{\bar{a}_{ii}})^{\frac{\theta_i-1}{\theta_i}}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}\\
&+e^{\varepsilon\tau}\sum_{j=1}^{n}\bar{b}_{ji}\}|x_i(t)-\bar{x}_i(t)|
\end{aligned}
\tag{16}
$$

where ξ_i lies between $x_i(t)$ and $\bar{x}_i(t)$.

From condition of Theorem 5, we can choose a small $\varepsilon > 0$ such that

$$\frac{\varepsilon}{e_i}-\underline{a}_{ii}(\frac{A_i}{\bar{a}_{ii}})^{\frac{\theta_i-1}{\theta_i}}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}+e^{\varepsilon\tau}\sum_{j=1}^{n}\bar{b}_{ji}\le 0,$$

for $i = 1, 2, \cdots, n$.

From (16), we get $D^+V(t) \le 0$, then $V(t) \le V(0)$, for all $t \ge 0$.

From (15), we have

$$
\begin{aligned}
V(t)&\ge\sum_{i=1}^{n}e^{\varepsilon t}|y_i(t)-\bar{y}_i(t)|\\
&=e^{\varepsilon t}\sum_{i=1}^{n}\tfrac{1}{\xi_i}|x_i(t)-\bar{x}_i(t)|.
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&V(0)=\sum_{i=1}^{n}[|y_i(0)-\bar{y}_i(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\int_{-\tau_{ij}}^{0}e^{\varepsilon(s+\tau_{ij})}|e^{y_j(s)}-e^{\bar{y}_j(s)}|ds]\\
&=\sum_{i=1}^{n}[\tfrac{1}{\xi_i}|\varphi_i(0)-\bar{\varphi}_i(0)|\\
&+\sum_{j=1}^{n}\bar{b}_{ij}\int_{-\tau_{ij}}^{0}e^{\varepsilon(s+\tau_{ij})}|\varphi_j(s)-\bar{\varphi}_j(s)|ds]\\
&\le\max_{1\le i\le n}\{\tfrac{1}{e_i}+\sum_{j=1}^{n}\bar{b}_{ji}\int_{-\tau}^{0}e^{\varepsilon(s+\tau)}ds\}\|\varphi-\bar{\varphi}\|.
\end{aligned}
\tag{18}
$$

Since $V(0) \ge V(t)$, from (17) and (18), we obtain

$$
\begin{aligned}
&\tfrac{1}{d}e^{\varepsilon t}\sum_{i=1}^{n}|x_i(t)-\bar{x}_i(t)|\le e^{\varepsilon t}\sum_{i=1}^{n}\tfrac{1}{\xi_i}|x_i(t)-\bar{x}_i(t)|\\
&\le\max_{1\le i\le n}\{\tfrac{1}{e_i}+\sum_{j=1}^{n}\bar{b}_{ji}\int_{-\tau}^{0}e^{\varepsilon(s+\tau)}ds\}\|\varphi-\bar{\varphi}\|.
\end{aligned}
\tag{19}
$$

By multiplying both sides of (19) with $de^{-\varepsilon t}$, we get

$$\sum_{i=1}^{n}|x_i(t)-\bar{x}_i(t)|\le Me^{-\varepsilon t}\|\varphi-\bar{\varphi}\|,\quad t>0. \tag{20}$$

where $M = d\max_{1\le i\le n}\{\frac{1}{e_i}+\sum_{j=1}^{n}\bar{b}_{ji}\int_{-\tau}^{0}e^{\varepsilon(s+\tau)}ds\}$.

By Definition 1, system (1) has one positive $\omega-$ periodic solution which globally exponentially stable.

Theorem 6 *Under the hypotheses (H), and all of the conditions in Theorem 4 are satisfied, if $0 < \theta_i < 1$, and*

$$-\theta_i\underline{a}_{ii}(\frac{A_i}{\bar{a}_{ii}})^{\frac{\theta_i-1}{\theta_i}}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}+\sum_{j=1}^{n}\bar{b}_{ji}<0,$$

for $i = 1, 2, \cdots, n$, then system (1) has one positive $\omega-$ periodic solution which globally exponentially stable.

If $0 < \theta_i < 1$, using Taylor expansion, we have

$$x_i^{\theta_i}-\bar{x}_i^{\theta_i}=\theta_i\bar{x}_i^{(\theta_i-1)}(x_i-\bar{x}_i)+o(\|x-\bar{x}\|^2),$$

where $o(\|x-\bar{x}\|^2)$ is second-order infinitely small of $\|x-\bar{x}\|$. Similar to the proof of Theorem 5, we can obtain result of Theorem 6, proof is omitted.

Remark 7 *For system (1), when $\theta_i = 1, b_{ij} = 0$, we obtain the Lotka-Volterra system $(A1)$; when $\theta_i = 1$, we obtain the Lotka-Volterra recurrent neural networks $(A2)$.*

From Theorem 4 and Theorem 5, we may obtain the following Corollary 8, Corollary 9.

Corollary 8 *Under the hypotheses (H), and all of the conditions in Lemma 2 are satisfied, if*

$$-\underline{a}_{ii}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}<0, i=1,2,\cdots,n,$$

$$\frac{2\omega}{1-q_i}\frac{(1-q)\bar{r}}{\bar{a}}\{\bar{a}_{ii}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ij}\}<1, i=1,2,\cdots,n,$$

then system $(A1)$ has a unique positive $\omega-$ periodic solution which globally exponentially stable.

Corollary 9 *Under the hypotheses* (H), *and all of the conditions in Lemma 2 are satisfied, if*

$$-\underline{a}_{ii}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}+\sum_{j=1}^{n}\bar{b}_{ji}<0, i=1,2,\cdots,n,$$

$$\frac{2\omega}{1-q_i}\frac{(1-q)\bar{r}}{\bar{a}}\{\bar{a}_{ii}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ij}+\sum_{j=1}^{n}\bar{b}_{ij}\}<1,$$

$i=1,2,\cdots,n$, *then system* $(A2)$ *has a unique positive* $\omega-$ *periodic solution which globally exponentially stable.*

4 Examples

In the section, we give two examples for showing our results.

Example 10 *Consider the following a generalized Gilpin-Ayala competitive system with time delay*$(n=2)$

$$\begin{cases}\frac{dx_1(t)}{dt}=x_1(t)[r_1(t)-a_{11}(t)x_1^{\theta_1}(t)\\ \qquad -a_{12}(t)x_2(t)-\sum\limits_{j=1}^{2}b_{1j}(t)x_j(t-\tau_{1j})],\\ \frac{dx_2(t)}{dt}=x_2(t)[r_2(t)-a_{22}(t)x_2^{\theta_2}(t)\\ \qquad -a_{21}(t)x_1(t)-\sum\limits_{j=1}^{2}b_{2j}(t)x_j(t-\tau_{2j})],\end{cases}\tag{21}$$

for $i=1,2, j=1,2$, *where*

$$\theta_i=2,\quad r_i(t)=(\sin 8t+2)/9,$$

$$a_{ii}(t)=\cos 8t+4,$$

$$a_{ij}(t)=\frac{1}{96}(\cos 8t+2)(i\neq j),$$

$$b_{ij}(t)=\frac{1}{96}(\sin 8t+1), i,j=1,2.$$

We select

$$\omega=\pi/4,\ \ \underline{r}_i=1/9,\ \ \bar{r}_i=1/3,$$

$$\bar{r}=\min_{1\leq i\leq 2}\{\bar{r}_i\}=1/3,\ \ q_i=e^{\frac{-\pi}{18}},$$

$$q=\max_{1\leq i\leq 2}\{q_i\}=e^{\frac{-\pi}{18}},$$

$$\underline{a}_{ii}=3,\ \ \bar{a}_{ii}=5,\ \ \bar{a}_{ij}=\frac{1}{32}(i\neq j),$$

$$\bar{a}=\max_{1\leq i\leq 2}\{\bar{a}_{ii}\}=5,\ \ \bar{b}_{ij}=\frac{1}{48},$$

$$d_i=(\frac{\bar{r}_i}{\underline{a}_{ii}})^{1/2}=1/3, i,j=1,2.$$

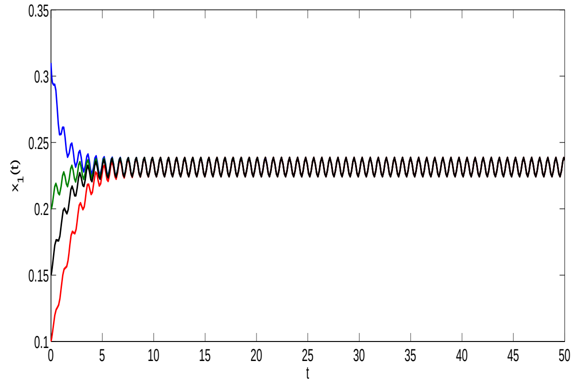

Fig.1. Transient response of state variables $x_1(t)$ of Example 4.1

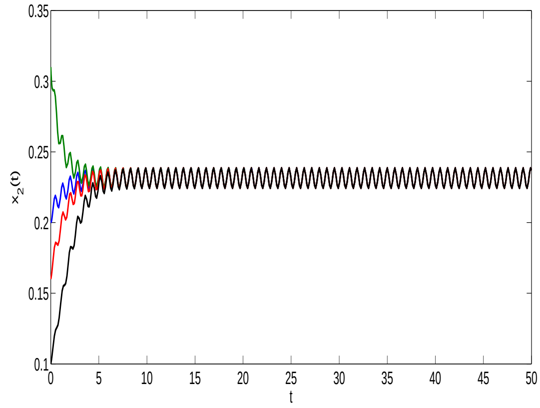

Fig.2. Transient response of state variables $x_2(t)$ of Example 4.1

For numerical simulation, let $\tau_{11}=0.15, \tau_{12}=0.26, \tau_{21}=0.3, \tau_{22}=0.1$, the following four cases are given:

case 1: with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.3,0.2]$;

case 2 with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.2,0.3]$;

case 3 with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.1,0.16]$;

case 4 with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.15,0.1]$.

Figs. 1-2 depict the time responses of state variables of $x_1(t)$ and $x_2(t)$ of system in example 10, respectively.

On the other hand, by calculation, we have the following results

$$A_i=\underline{r}_i-\sum_{j=1,j\neq i}^{2}\bar{a}_{ij}d_j-\sum_{j=1}^{2}\bar{b}_{ij}d_j=\frac{25}{288}>0,$$

$$e_i=(\frac{A_i}{\bar{a}_{ii}})^{\frac{1}{2}}=\frac{\sqrt{10}}{24}\leq\varphi_i(0)\leq d_i=1/3,$$

$$-\underline{a}_{ii}(\frac{A_i}{\bar{a}_{ii}})^{\frac{1}{2}}+\sum_{j=1,j\neq i}^{2}\bar{a}_{ji}+\sum_{j=1}^{2}\bar{b}_{ji}=\frac{7-12\sqrt{10}}{96}<0,$$

$$\frac{\omega}{1-q_i}(\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{2}}\{3\bar{a}_{ii}(\frac{(1-q)\bar{r}}{\bar{a}})^{\frac{1}{2}}$$

$$+2\sum_{j=1,j\neq i}^{2}\bar{a}_{ij}+2\sum_{j=1}^{2}\bar{b}_{ij}\}<\frac{1.21\pi}{4}<1, i=1,2.$$

It follows from Theorem 4 and Theorem 5 that this system has one unique $\pi/4$- periodic solution, and all other solutions of system exponentially converge to it as $t\to+\infty$.

Example 11 *For system (21), let*

$$\theta_i=1/2,\quad r_i(t)=(\sin t+2)/18,$$

$$a_{ii}(t)=\cos t+2,$$

$$a_{ij}(t)=\frac{1}{6}(\cos t+2)(i\neq j),$$

$$b_{ij}(t)=\frac{1}{6}(\sin t+2),\ i,j=1,2.$$

We select

$$\omega=2\pi,\quad \underline{r}_i=1/18,\quad \bar{r}_i=1/6,$$

$$\bar{r}=\min_{1\le i\le 2}\{\bar{r}_i\}=1/6,\quad q_i=e^{-\frac{2\pi}{9}},$$

$$q=\max_{1\le i\le 2}\{q_i\}=e^{-\frac{2\pi}{9}},$$

$$\underline{a}_{ii}=1,\ \ \bar{a}_{ii}=3,\ \ \bar{a}_{ij}=\frac{1}{2}(i\neq j),$$

$$\bar{a}=\max_{1\le i\le 2}\{\bar{a}_{ii}\}=3,\ \ \bar{b}_{ij}=\frac{1}{2},$$

$$d_i=(\frac{\bar{r}_i}{\underline{a}_{ii}})^2=1/36, i,j=1,2.$$

For numerical simulation, let $\tau_{11}=0.04,\tau_{12}=0.03,\tau_{21}=0.02,\tau_{22}=0.01$, the following four cases are given:

case 1 with the initial state $[\varphi_1(0),\varphi_2(0)]=[[0.002;0.0045]$;

case 2 with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.004;0.0025]$;

case 3 with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.001;0.0014]$;

case 4 with the initial state $[\varphi_1(0),\varphi_2(0)]=[0.003;0.003]$.

Figs. 3-4 depict the time responses of state variables of $x_1(t)$ and $x_2(t)$ of system in example 11, respectively.

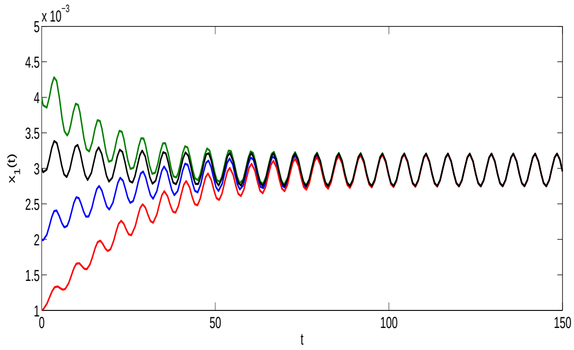

Fig.3. Transient response of state variables $x_1(t)$ of Example 4.2

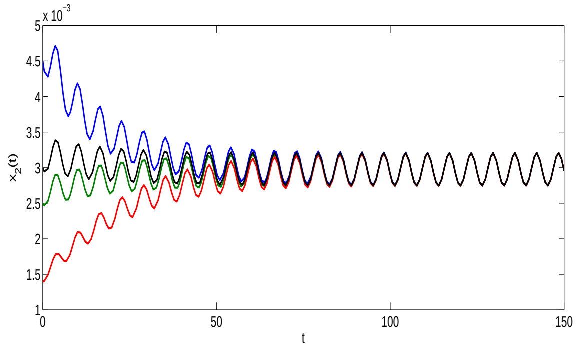

Fig.4. Transient response of state variables $x_2(t)$ of Example 4.2

On the other hand, by calculation, we have the following results

$$A_i=\underline{r}_i-\sum_{j=1,j\neq i}^{2}\bar{a}_{ij}d_j-\sum_{j=1}^{2}\bar{b}_{ij}d_j=\frac{1}{72}>0,$$

$$e_i=(\frac{A_i}{\bar{a}_{ii}})^2=\frac{1}{216^2}\le\varphi_i(0)\le d_i=1/36$$

$$-\frac{\underline{a}_{ii}}{2}(\frac{A_i}{\bar{a}_{ii}})^{-1}+\sum_{j=1,j\neq i}^{n}\bar{a}_{ji}+\sum_{j=1}^{n}\bar{b}_{ji}=-106.5<0.$$

$$\frac{\omega}{1-q_i}\{\frac{3}{2}\bar{a}_{ii}\frac{(1-q)\bar{r}}{\bar{a}}+2(\frac{(1-q)\bar{r}}{\bar{a}})^2\sum_{j=1,j\neq i}^{2}\bar{a}_{ij}$$

$$+2(\frac{(1-q)\bar{r}}{\bar{a}})^2\sum_{j=1}^{2}\bar{b}_{ij}\}<\frac{31\pi}{162}<1, i=1,2.$$

It follows from Theorem 4 and Theorem 6 that this system has one unique 2π- periodic solution, and all other solutions of system exponentially converge to it as $t\to+\infty$.

5 Conclusions

Since the periodic solutions for system is very important in theories and applications. In this paper, we give theorems to ensure the existence and the exponential stability of the positive periodic solution for a generalized Gilpin-Ayala competition system. In exceptional circumstances, some sufficient conditions guaranteeing the existence, uniqueness and exponential stability of positive periodic solutions for system $(A1)$ and $(A2)$ are given. Novel existence and stability conditions are stated in simple algebraic forms so that their verification and applications are straightforward and convenient. Two examples are given to show the effectiveness of the results.

Acknowledgements: This work is supported by the Natural Science Foundation of Zhejiang Province (No.Y6100096).

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