Boundary Problems for Stationary Neutron Transport Equations Solved by Homotopy Perturbation Method

OLGA MARTIN
Department of Applied Mathematics
Polytechnic University of Bucharest
Splaiul Independentei 313
ROMANIA
omartin_ro@yahoo.com

Abstract: This paper proposes an algorithm based on the homotopy perturbation method (HPM) for the solving the one-dimensional neutron transport equation with suitable multipoint boundary conditions (MPBC). The homotopy perturbation method is a coupling of traditional perturbation method and the homotopy function from topology, which continuously deforms the given problem to another that can be easily solved. The new version of homotopy method, upon which our algorithm is built, yields rapid convergence of the solution series to exact solution. Usually only two iterations lead to high accuracy solutions. Illustrative numerical examples are provided to prove the efficiency of the proposed algorithm for integral differential equations accompanied by the multipoint boundary conditions.

Key-Words: Integral-differential equation, homotopy perturbation method, multipoint boundary value problem

1. Introduction

The integral - differential equations describe many phenomena in different fields of mechanical and nuclear engineering, chemistry, astronomy, biology, economics, potential theory and electrostatics. The resolution of boundary problems for these equations is the subject of several recent papers in which the authors have approached in most cases numerical methods: the finite element method, Monte Carlo, truncated series of Chebyshev polynomials, the fictitious domain method, S_N method, [3], [8], [11] - [13]. An exact solution of this integral-differential equation was found only in the particular cases. Generally, these are obtained with the help of the methods of mathematical analysis, abstract functional analysis and the spectral methods.

In recent years, the basic ideas of the homotopy, which is a concept of the topology and differential geometry, were used to obtain the approximate solutions for a wide class of differential, integral and integral – differential equations. We mention here the homotopy perturbation method (HPM) proposed by He in 1998 and the homotopy analysis method (HAM) proposed by Liao in 1992. The perturbation methods approximate the solution of given problem by a series of small parameters. Unfortunately, the majority of non-linear problems have no small parameters and an unsuitable choice of these parameters can lead to bad effects. The new homotopy perturbation technique (HPM) embeds a parameter p that ranges from zero to one. When the embedding parameter is zero, we get a linear equation and if it is equal to one, we get the original transport equation. This embedding parameter that belongs to the interval [0, 1] can be considered as a small parameter.

In this study, the homotopy perturbation method [1], [4]-[7], [9]-[10] is implemented in a new form, such that we can to solve an integral- differential equation in which the unknown function depends on two variables. Multipoint boundary value problems (MBVP) of the ordinary differential and integral-differential equations occurred in the areas of applied mathematics, fluid dynamics, plasma physics, biological sciences, chemical and mechanical engineering, especially, on the theoretical aspects. For example, He have used HPM for solving the boundary problem for partial differential equation [6], Shakeri and Dehghan for the delay differential equation that arise in biology and engineering.

In this paper we present the forms of source function, which lead to the exact solutions of the proposed multipoint boundary value problems. There will be two types of the multipoint boundary value problems (MBVP).

The first refers to the particle transport in the homogeneous and isotropic medium for a plane geometry, in which the value of the solution obtained by HPM into one end of its domain is a linear combination of the values in some given points of this domain.

In the second case, the neutron transport equation into a non-homogeneous medium with isotropic scattering is studied using HPM and the MBVP techniques.

The numerical examples that will be presented show that our algorithm can successfully applied and in the case of the integral – differential equations whose solutions have a finite number of the discontinuity points.

2. Problem formulation

2.1 Homogeneous isotropic medium

Let us consider the integral-differential equation of transport theory for the stationary case and isotropic medium [2]:

$$\mu \frac{\partial \varphi(x,\mu)}{\partial x} + \sigma(x)\varphi(x,\mu) = \frac{\sigma_s(x)}{2}\int_{-1}^{1}\varphi(x,\mu')d\mu' + f(x,\mu) \tag{1}$$

$$\forall (x,\mu) \in D_1 \times D_2 = [0,1]\times[-1,1],$$

$$D_2 = D_2' \bigcup D_2'' = [-1,0]\bigcup[0,1]$$

with the following boundary conditions:

$$\varphi(0,\mu) = a,\ \ \varphi(1,\mu) = \sum_{i=1}^{m}\alpha_i \varphi(\xi_i,\mu) + \beta, \quad \mu \in D_2, \xi_i \in D_1 \tag{2}$$

where

- $\varphi(x, \mu)$ is the density of neutrons, which migrate in a direction that makes an angle α with the x axis and $\mu = \cos\alpha$;
- $\sigma_s(x)$ is the scattering coefficient,
- $\sigma_a(x)$ is the absorption coefficient and $\sigma(x) = \sigma_s(x) + \sigma_a(x)$;
- $f(x, \mu)$ is a given radioactive source function.

Now, we split the equation (1) in two equations, using the following notations:

$$\varphi^+ = \varphi(x,\mu) \text{ if } \mu > 0; \varphi^- = \varphi(x,\mu) \text{ if } \mu < 0 \tag{3}$$

Denoting $\mu'' = -\mu'$, the following integral becomes

$$\int_{-1}^{0}\varphi(x,\mu')d\mu' = \int_{0}^{1}\varphi(x,-\mu)d\mu = \int_{0}^{1}\varphi^- d\mu$$

In view of (3), the equation (1) can be written in the forms

$$\mu\frac{\partial \varphi^+}{\partial x} + \sigma\varphi^+ = \frac{\sigma_s}{2}\int_0^1(\varphi^+ + \varphi^-)d\mu' + f^+ \tag{4}$$

$$-\mu\frac{\partial \varphi^-}{\partial x} + \sigma\varphi^- = \frac{\sigma_s}{2}\int_0^1(\varphi^+ + \varphi^-)d\mu' + f^- \tag{5}$$

Adding and subtracting the equations (4) – (5) and introducing the notations:

$$u = \frac{1}{2}\left(\varphi^+ + \varphi^-\right),\ \ v = \frac{1}{2}\left(\varphi^+ - \varphi^-\right)$$
$$g = \frac{1}{2}\left(f^+ + f^-\right),\ \ r = \frac{1}{2}\left(f^+ - f^-\right) \tag{6}$$

we obtain the system

$$\mu\frac{\partial v}{\partial x} + \sigma u = \sigma_s\int_0^1 u\,d\mu + g \qquad (a)$$
$$\mu\frac{\partial u}{\partial x} + \sigma v = r \qquad (b) \tag{7}$$

Determining v from the second equation of (7) and using the first equation, we rewrite the problem (7) - (8) in the following form

$$-\frac{\mu^2}{\sigma}\frac{\partial^2 u}{\partial x^2} + \sigma u = \sigma_s \int_0^1 u d\mu + g - \frac{\mu}{\sigma}\frac{\partial r}{\partial x} \qquad (9)$$

$$u(0,\mu) = a, \; u(1,\mu) = \sum_{i=1}^{m} \alpha_i u(\xi_i, \mu) + \beta, \qquad (10)$$
$$\mu \in D_2, \xi_i \in D_1$$

To solve the problem (9) - (10) for an homogeneous medium with $\sigma(x) = \sigma_s(x) = 1$ (non-absorption medium), we construct the homotopy

$$H(Y,p) = (1-p)(\mu^2 \frac{\partial^2 Y}{\partial x^2}(x,\mu) - y_0(x,\mu)) +$$
$$+ p(\mu^2 \frac{\partial^2 Y(x,\mu)}{\partial x^2} - Y(x,\mu) + \qquad (11)$$
$$+ \int_0^1 Y(x,\mu')d\mu' - S(x,\mu)) = 0$$

where $p \in [0, 1]$ is an embedding parameter, y_0 is the initial guesses function for $Y(x, \mu)$ and

$$S(x,\mu) = \mu \frac{\partial r(x,\mu)}{\partial x} - g(x,\mu) \qquad (12)$$

Obviously,

$$H(Y,0) = \mu^2 \frac{\partial^2 Y}{\partial x^2}(x,\mu) - y_0(x,\mu) = 0$$
$$H(Y,1) = A(Y) - S(x,\mu) = 0 \qquad (13)$$

where the operator A is of the form

$$A(Y) = \mu^2 \frac{\partial^2 Y}{\partial x^2} - Y + \int_0^1 Y(x,\mu')d\mu' \qquad (14)$$

So, if the parameter p increases from 0 to 1, the solution $Y(x, \mu)$ of the equation (11) is "deformed" continuously from the function Y_0 to the solution of the problem (9) – (10). Now, by applying the classical perturbation technique, we assume that this solution can be expressed as a power series in p

$$Y(x,\mu) = Y_0(x,\mu) + Y_1(x,\mu)p + Y_2(x,\mu)p^2 + ... \qquad (15)$$

When $p \to 1$, the sum of (14) becomes the solution of equation (9):

$$A(Y(x,\mu)) = S(x,\mu) \qquad (16)$$

Next we rewrite the equation (11) as

$$\mu^2 \frac{\partial^2 Y}{\partial x^2} - y_0 + p(y_0 - Y + \int_0^1 Y d\mu - S) = 0 \qquad (17)$$

and substitute (15) in (17). Equating the like powers of p, we get the following set of integral-differential equations and boundary conditions:

$$p^0: \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = y_0(x,\mu)$$
$$(18)$$
$$Y_0(0,\mu) = a, Y_0(1,\mu) = \sum_{i=1}^{m} \alpha_i Y_0(\xi_i,\mu) + \beta$$

$$p^1: \quad \mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -y_0(x,\mu) + Y_0(x,\mu) -$$
$$-\int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu) \qquad (19)$$
$$Y_1(0,\mu) = 0, Y_1(1,\mu) = \sum_{i=1}^{m} \alpha_i Y_0(\xi_i,\mu)$$

$$p^2: \quad \mu^2 \frac{\partial^2 Y_2(x,\mu)}{\partial x^2} = Y_1(x,\mu) - \int_0^1 Y_1(x,\mu')d\mu$$

$$Y_2(0,\mu)=0, Y_2(1,\mu)=\sum_{i=1}^{m}\alpha_i Y_0(\xi_i,\mu) \tag{20}$$

....

The above nonlinear equations can be easily solved and the components $Y_n(x, \mu)$ can be completely determined using the boundary conditions.
Finally, the approximate solution for $Y(x, \mu)$ is obtained as

$$u(x,\mu) = \lim_{N\to\infty} \omega_N(x,\mu) \tag{21}$$

where

$$\omega_N(x,\mu) = \sum_{n=0}^{N-1} Y_n(x,\mu) \tag{22}$$

2.2 Non-homogenous medium

Let us consider the integral-differential equation (9)

$$-\frac{\mu^2}{\sigma(x)}\frac{\partial^2 u(x,\mu)}{\partial x^2} + \sigma(x)u(x,\mu) = \sigma_s(x)\int_0^1 u(x,\mu)d\mu - S(x,\mu) \quad \forall (x,\mu)\in[0,1]\times[0,1] \tag{23}$$

with the scattering coefficient (pure scattering) defined by

$$\sigma_s(x) = \sigma(x) = \begin{cases} \sigma_1, & x\in[0,c] \\ \sigma_2, & x\in(c,1] \end{cases} \tag{24}$$

The boundary conditions will be chosen now of the form

$$u(0,\mu)=0, \quad \frac{\partial u}{\partial x}(0,\mu) = b \text{ if } x\in[0,c]$$

$$u(1,\mu)=0, \quad \frac{\partial u}{\partial x}(1,\mu) = d \text{ if } x\in[c,1] \tag{25}$$

For each subinterval of D_1 we will construct a homotopy of the type (17). Substituting (15) in (17) and equating the coefficients of p with the same power, we will solve the Cauchy problems presented below.

I. For $x \in [0, c]$:

$$p^0: \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = y_0(x,\mu)$$

$$Y_0(0,\mu)=0, \quad Y_0'(0,\mu)=b \tag{26}$$

$$p^1: \quad \mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -y_0(x,\mu) + Y_0(x,\mu) - \int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu) \tag{27}$$

$$Y_1(0,\mu)=0, Y_1'(0,\mu)=0$$

$$p^2: \quad \mu^2 \frac{\partial^2 Y_2(x,\mu)}{\partial x^2} = Y_1(x,\mu) - \int_0^1 Y_1(x,\mu')d\mu'$$

$$Y_2(0,\mu)=0, Y_2'(0,\mu)=0$$

....

(28)

II. For $x \in (c, 1]$:

$$p^0 : \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = y_0(x,\mu) \qquad (29)$$
$$Y_0(1,\mu) = 0, \quad Y_0^{'}(1,\mu) = d$$

$$p^1 : \quad \mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -y_0(x,\mu) + Y_0(x,\mu) -$$
$$-\int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu) \qquad (30)$$

$$Y_1(1,\mu) = 0, Y_1^{'}(1,\mu) = 0$$

$$p^2 : \quad \mu^2 \frac{\partial^2 Y_2(x,\mu)}{\partial x^2} = Y_1(x,\mu) - \int_0^1 Y_1(x,\mu')d\mu'$$
$$Y_2(1,\mu) = 0, Y_2^{'}(1,\mu) = 0$$
....

(31)

Numerical example will prove that this method is rapid convergence, being needed only two terms of the series (15) for each subintervals of D_1 to get in the common point $x = 0.5$ values very close to one of the other for the continuous exact solutions on the spatial interval [0, 1].

3. Numerical examples

3.1. Homogeneous medium

In this subsection we present three source functions, odd with respect to μ. This leads for $p = 1$ to the rapid converged series (15) to the exact solution of given transport problems.

***Example* 1**

Let us consider the integral-differential equation in a homogeneous isotropic medium:

$$\mu \frac{\partial \varphi(x,\mu)}{\partial x} + \varphi(x,\mu) = \frac{1}{2}\int_{-1}^{1} \varphi(x,\mu')d\mu' + f(x,\mu) \quad (32)$$

where

$$f(x,\mu) = 3\mu x^2 \quad \Rightarrow$$

$$S(x,\mu) = 6\mu^2 x$$

The equation (9) becomes of the form

$$\mu^2 \frac{\partial^2 u(x,\mu)}{\partial x^2} - u(x,\mu) + \int_0^1 u(x,\mu')d\mu' = S(x,\mu) \qquad (33)$$

with a solution that verifies the conditions

$$u(0,\mu) = 0$$
$$u(1,\mu) = 3u\left(\frac{1}{3},\mu\right) + 3u\left(\frac{2}{3},\mu\right)$$

The exact solution of this MBVP is

$$u_e(x,\mu) = x^3$$

Choosing the function $y_0(x,\mu) = \mu^2$, we have the problems:

$$p^0 : \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = \mu^2 \qquad (34)$$

$$Y_0(0,\mu) = 0,$$
$$Y_0(1,\mu) = 3Y_0\left(\frac{1}{3},\mu\right) - 3Y_0\left(\frac{2}{3},\mu\right)$$

with the solution:

$$Y_0(x,\mu) = \frac{x^2}{2} - \frac{x}{6} \tag{35}$$

$$p^1 : \mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -\mu^2 + Y_0(x,\mu) - \int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu) \tag{36}$$

$$Y_1(0,\mu) = 0,$$
$$Y_1(1,\mu) = 3Y_1\left(\frac{1}{3},\mu\right) + 3Y_1\left(\frac{2}{3},\mu\right)$$

with the solution:

$$Y_1(x,\mu) = -\frac{x^2}{2} + x^3 + \frac{x}{6} \tag{37}$$

The sum of the first two terms of the series (15) leads to the exact solution:

$$Y(x,\mu) = Y_0(x,\mu) + Y_1(x,\mu) = u_e(x,\mu) \tag{38}$$

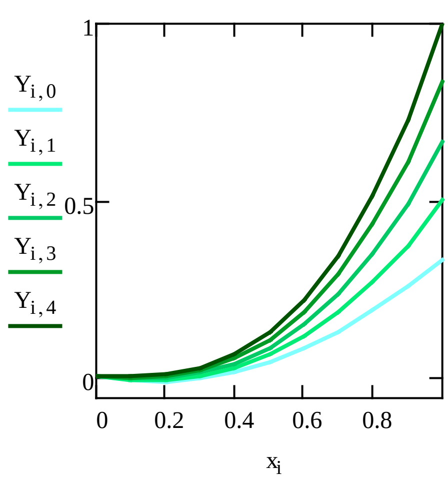

Fig. 3

Figure 3 shows the process of the "deformation" of $Y(x,\mu)$ from $Y_0(x,\mu)$ to the exact solution of given problem function:

$$Y(x_i,\mu) = Y_0(x_i,\mu) + pY_1(x_i,\mu), \quad x_i \in [0,1],$$

where $p = 1$.

Notations used are as follows:

$$Y_{i,j} = Y_0(x_i,\mu) + p_j Y_1(x_i,\mu),$$
$$p_j = j/4, \ j = 0,1,2,3,4. \tag{39}$$

***Example* 2**

The source function is now of the form

$$f(x,\mu) = \frac{\pi\mu}{2}\cos\left(\frac{\pi x}{2}\right) \Rightarrow$$
$$S(x,\mu) = -\frac{\pi^2\mu^2}{4}\sin\left(\frac{\pi x}{2}\right) \tag{40}$$

and the solution of (33) verifies the conditions

$$u(0,\mu) = 0$$
$$u(1,\mu) = u\left(\frac{1}{3},\mu\right) + \frac{1}{\sqrt{2}}u\left(\frac{1}{2},\mu\right) \tag{41}$$

The exact solution of this MBVP is

$$u_e(x,\mu) = \sin\left(\frac{\pi x}{2}\right)$$

Choosing the function $y_0(x,\mu) = \mu^2$, we have the problems:

$$p^0 : \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = \mu^2 \tag{42}$$

$$Y_0(0,\mu)=0,$$

$$Y_0(1,\mu)=Y_0\left(\frac{1}{3},\mu\right)+\frac{1}{\sqrt{2}}Y_0\left(\frac{1}{2},\mu\right)$$

with the solution:

$$Y_0(x,\mu)=0.5x^2-1.12x$$

Then

$$p^1:\quad \mu^2\frac{\partial^2 Y_1(x,\mu)}{\partial x^2}=-\mu^2+Y_0(x,\mu)-$$
$$-\int_0^1 Y_0(x,\mu')d\mu'+S(x,\mu)$$

$$Y_1(0,\mu)=0,$$
$$Y_1(1,\mu)=Y_1\left(\frac{1}{3},\mu\right)+\frac{1}{\sqrt{2}}Y_1\left(\frac{1}{2},\mu\right) \tag{43}$$

with the solution:

$$Y_1(x,\mu)=-0.5x^2+1.12x+\sin\left(\frac{\pi x}{2}\right) \tag{44}$$

The sum of the first two terms of the series (15) leads to the exact solution:

$$Y(x,\mu)=Y_0(x,\mu)+Y_1(x,\mu)=u_e(x,\mu) \tag{45}$$

The "deformation" of Y from Y_0 to u_e is shown in the Figure 4, where we used the same notations (39).

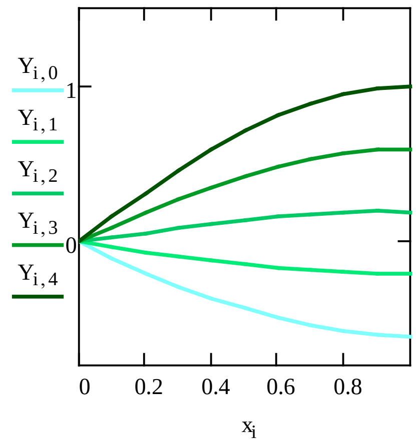

Fig. 4

***Example* 3**

Let us consider the integral-differential equation (32), where

$$f(x,\mu)=\frac{\pi\mu}{2}\cos\left(\frac{\pi x}{2}\right)\quad\Rightarrow$$

$$S(x,\mu)=-\frac{\pi\mu^2}{4}\sin\left(\frac{\pi x}{2}\right)$$

The solution of the equation (32) will be subject to the following conditions:

$$u(0,\mu)=2\mu^2+1,$$
$$u(1,\mu)=2u(1/4,\mu)-u(3/4,\mu)+1 \tag{46}$$

The exact solution of this MBVP is

$$u_e(x,\mu)=2\mu^2+\cos(2\pi x) \tag{47}$$

If the function $y_0(x,\mu)=\mu^2$, we have the problems:

$$p^0: \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = \mu^2 \tag{48}$$

$$Y_0(0,\mu) = 2\mu^2 + 1,$$

$$Y_0(1,\mu) = 2Y_0(1/4,\mu) - Y_0(3/4,\mu) + 1$$

with the solution:

$$Y_0(x,\mu) = x^2/2 + x \cdot 9/40 + 2\mu^2 + 1 \tag{49}$$

$$p^1: \quad \mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -\mu^2 + Y_0(x,\mu) - \\ -\int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu)$$

$$\begin{aligned} &Y_1(0,\mu) = 0, \\ &Y_1(1,\mu) = 2Y_1(1/4,\mu) - Y_1(3/4,\mu) \end{aligned} \tag{50}$$

with the solution:

$$Y_1(x,\mu) = -x^2/2 + \cos(2\pi x) - x \cdot 9/40 \tag{51}$$

The sum of the first two terms of the series (15) lead to the exact solution:

$$Y(x,\mu) = Y_0(x,\mu) + Y_1(x,\mu) = u_e(x,\mu) \tag{52}$$

3.2 Non-homogenous medium

In a non-absorption medium we define the scattering coefficient of the form

$$\sigma_s(x) = \sigma(x) = \begin{cases} 1, & x \in [0,c] \\ 1/2, & x \in (c,1] \end{cases} \tag{53}$$

and

$$f(x,\mu) = \begin{cases} -2\pi\sin(2\pi x) + 2\mu^2 - 2/3, & x \in [0,1/2] \\ -2(1-x)\mu^3 + \dfrac{(1-x)^2(6\mu^2-1)}{12}, & x \in (1/2,1] \end{cases}$$

The solutions for the two sub-intervals will be presented below.

I. $x \in [0, 1/2]$. In equation

$$\mu^2 \frac{\partial^2 u(x,\mu)}{\partial x^2} - u(x,\mu) + \int_0^1 u(x,\mu')d\mu' = S(x,\mu),$$

$$S(x,\mu) = 2\mu^4 + x^2\left(\frac{1}{3} - \mu^2\right),$$

$$u(0,\mu) = 0, u'(0,\mu) = 0$$

$$u_e(x,\mu) = (1-x)^2 \mu.^2$$

$$\begin{aligned} &p^0: \quad \mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = \mu^2, \\ &Y_0(0,\mu) = Y_0'(0,\mu) = 0 \end{aligned} \tag{54}$$

with the solution: $Y_0(x,\mu) = x^2/2$.

$$p^1: \quad \mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -\mu^2 + Y_0(x,\mu) - \\ -\int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu) \tag{55}$$

$$Y_1(0,\mu) = 0, Y_0'(0,\mu) = 0$$

with the solution:

$$\mu^2 Y_1(x,\mu) = -x^2\mu^2/2 + \mu^4 x^2 + \frac{x^4}{12}\left(\frac{1}{3} - \mu^2\right)$$

Also, summing only the first two terms of the series (15) (p = 1), we get

$$\mu^2 Y(x,\mu) = \mu^2 Y_0(x,\mu) + \mu^2 Y_1(x,\mu) = \\ = \mu^2 u_e(x,\mu) + R_1(x,\mu), \quad (56)$$

where $u_e(x,\mu) = \mu^2 x^2$ and $R_1 = \frac{x^4}{12}\left(\frac{1}{3} - \mu^2\right)$.

II. $\underline{x \in [1/2, 1]}$. In this case (23) becomes

$$2\mu^2 \frac{\partial^2 u(x,\mu)}{\partial x^2} - \frac{u(x,\mu)}{2} + \frac{1}{2}\int_0^1 u(x,\mu')d\mu' = S(x,\mu)$$

$$S(x,\mu) = 4\mu^4 + \left(\frac{1}{3} - \mu^2\right)\frac{(1-x)^2}{2}$$

with the boundary conditions:

$$u(1,\mu) = 0,\ u'(1,\mu) = 0 \\ u_e(x,\mu) = (1-x)^2\mu \quad (57)$$

If we choose $y_0(x, \mu) = 2\mu^2$, we have

$$p^0: \quad 2\mu^2 \frac{\partial^2 Y_0(x,\mu)}{\partial x^2} = 2\mu^2,\ Y_0(1,\mu) = Y_0'(1,\mu) = 0$$

with the solution: $Y_0(x,\mu) = x^2/2 - x + 1/2$. (58)

$$p^1: 2\mu^2 \frac{\partial^2 Y_1(x,\mu)}{\partial x^2} = -2\mu^2 + Y_0(x,\mu) - \\ - \int_0^1 Y_0(x,\mu')d\mu' + S(x,\mu),$$

$$Y_1(1,\mu) = Y_1'(1,\mu) = 0$$

with the solution:

$$2\mu^2 Y_1(x,\mu) = -x^2\mu^2 + 2\mu^4 x^2 + \frac{(1-x)^4}{24}\left(\frac{1}{3} - \mu^2\right) + \\ + \left(2\mu^2 - 4\mu^4\right)x + 2\mu^4 - \mu^2 \quad (59)$$

Also, summing only the first two terms (58) and (59) of the series (15) (p = 1), we get

$$2\mu^2 Y(x,\mu) = 2\mu^2 Y_0(x,\mu) + 2\mu^2 Y_1(x,\mu) = \\ = 2\mu^2 u_e(x,\mu) + R_2(x,\mu)$$

where the rest function R_2 is of the form

$$R_2(x,\mu) = \frac{(1-x)^4}{24}\cdot\left(\frac{1}{3} - \mu^2\right) \quad (60)$$

Next, we define the error function of μ, which corresponds to x = 1/2 :

$$er(\mu_k) = er_k,\ er_k = \left| R_1(1/2,\mu_k) - R_2(1/2,\mu_k) \right| \quad (61)$$

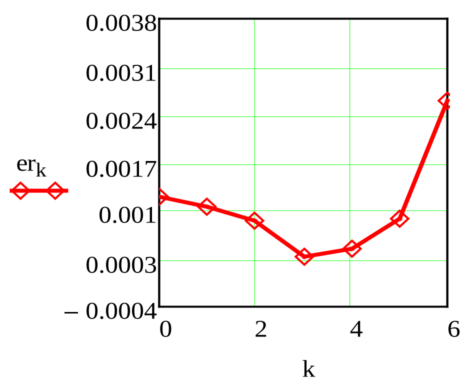

Fig. 5

The variation of er is shown in the Fig. 5, where the seven values of μ_k :

$$\mu_k \in \{1/6, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}, \quad k = \overline{0,6}$$

are marked with a symbol.

In Fig. 6 are presented the graphs of the unknown function, $u_{i,k} = u(x_i, \mu_k)$, $x_i = ih$, $h = 1/20$, $i = 1,..,20$,

$$\mu_k \in \{1/6, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}, \quad k = \overline{0,6} .$$

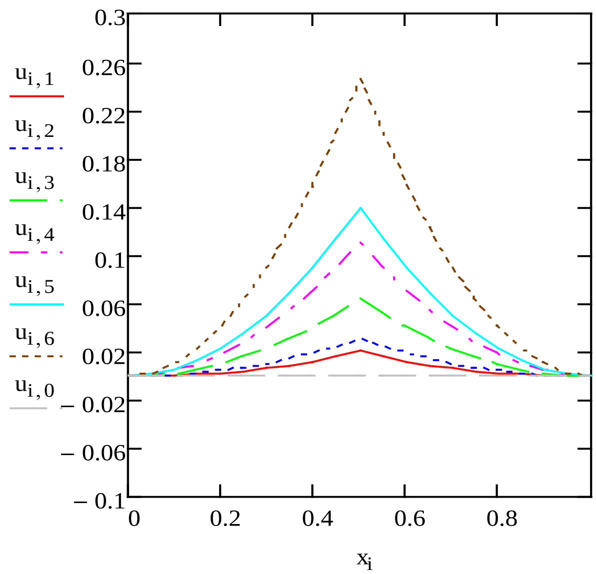

Fig. 6

In practical applications it should be mention that the value of the density is equal to zero when the direction of the movement of particles makes an angle of 90 degrees to the axis *Ox*. Significant values of the function u will be obtained for $\mu\in[0.25, 1]$, so the angles $\alpha\in[0^0, 75^0]$.

4. Conclusions

Using HPM we have obtained the series solutions for the integral-differential equations accompanied by multipoint boundary conditions. The numerical examples prove that we get a good approximation of the exact solution considering only the first two terms of the series. This success was due to the form that we choose for the initial guess function y_0. Unlike other existing works in literature, where the integral-differential equations are solved for an unknown functions that depends on a variable, in this paper we present an algorithm for a MBVP in which the solution depends on two variabiles.

In the future we will extend this method to the transport problems in which the solution has a finite number of discontinuities.

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