

Global Dynamics of an SEIRS Epidemic Model with Constant Immigration and Immunity

Li Juan Zhang
Institute of disaster prevention
Basic Course Department
Sanhe, Hebei 065201
P. R. CHINA
Lijuan262658@126.com

Yingqiu Li Qingqing Ren Zhenxiang Huo
Institute of disaster prevention
Basic Course Department
Sanhe, Hebei 065201
P. R. CHINA

Abstract: An SEIRS model for disease transmission that includes immigration of the infective, susceptible, exposed, and recovered has been constructed and analyzed. For the reason that the immunity of the recovered is temporary, a proportion δ_1 of recovered will come back to susceptible. We also consider vaccine injection to the susceptible with a proportion c . The model also incorporates a population size dependent contact rate and a disease-related death. As the infected fraction cannot be eliminated from the population, this kind of model has only one unique endemic equilibrium that is globally asymptotically stable. In a special case where the new members of immigration are all susceptible, the model shows a threshold phenomenon. In order to prove the global asymptotical stability of the endemic equilibrium, we change our system to a three-dimensional asymptotical autonomous system with limit equation. Finally, we discussed syphilis as a case to predict the development in China. Computer simulation shows that the model can reflect the dynamic and immigration behaviour for disease transmission.

Key-Words: SEIRS model, population size dependent contact rate, syphilis, compound matrix

1 Introduction

The incidence of a disease is the number of new cases per unit time, and it plays an important role in the study of mathematical epidemiology. Thieme and Castillo-Chavez [1] argued that the general form of a population-size-dependent incidence should be written as $\beta\lambda(N)\frac{SI}{N}$, where S is the number of susceptible at time t , I the number of infective at time t , and N the total population size at time t , then $S + I \leq N$, β is the probability per unit time of transmitting a disease between two individuals in contact. And $\lambda(N)$ is the probability for an individual to take part in a contact. In many articles $\lambda(N)$ is also called the contact rate, $\beta\lambda(N) \triangleq C(N)$ which is the average number of adequate contacts of an individual per unit time is said to be an adequate contact rate. An adequate contact is a contact which is sufficient for transmission of the infection from an infective to a susceptible. Many contact rate forms are used in the incidence term in deterministic epidemic models described by differential equation. For example the standard incidence $\frac{\lambda SI}{N}$ starts with the assumption that adequate contact rate is constant λ . The bilinear incidence $\lambda SI = \beta N \frac{S}{N} I$ implies that the adequate contact rate is βN that is linearly proportional to the total population size N . This would be realistic when the total population size

N is not too large because the number of adequate contacts made by an individual per unit time should increase as the total population size N increases. On the contrary, if population size is quite large, because the number of adequate contacts made by an infective per unit time should be limited, or grows less rapidly as N increases, the linear contact rate βN is not available and the constant adequate contact rate λ may be more realistic. Hence the two above-mentioned adequate contact rates are actually two extreme cases for the total population size N being very small and very large.

Many expressions have also been used for the incidence term. Anderson [2] argued that if $C(N) = \bar{\lambda}N^\delta$ is an appropriate contact rate, then the incidence is $C(N) = \bar{\lambda}N^\delta SI/N$. The data in Anderson article [2] for human diseases in communities with population sizes from 1000 to 400000 imply that δ is between 0.03 and 0.07. That also shows that the incidence give by $C(N)SI/N$ with a contact rate such as $C(N) = \bar{\lambda}N^{0.05}$ would be even better.

In analogy to Hollings [3] derivation of predator's functional response to the amount of prey, the contact function $C(N)$ should be $\frac{\beta a N}{1 + b N}$. Heesterbeek and Metz [4] derived an effective contact rate of the form $C(N) = \frac{\beta b N}{1 + b N + \sqrt{1 + 2b N}}$, which is modelling the formation of the short-time social complexes.

All above-mentioned adequate contact rate forms satisfy the following two assumptions:

(H₁) $C(N)$ is a nonnegative continuous function as $N \geq 0$ and is continuous differential as $N > 0$;

(H₂) $D(N) = C(N)/N$ is a non-increasing continuously differentiable function as $N > 0$, $D(0) \neq 0$ and $C'(N) + |D'(N)| \neq 0$.

Many authors have done researches on SEIR models of epidemics transmission recently. Greenhalgh [4] considered SEIR models that incorporate density dependence in the death rate. Cooke and van den Driessche [5] introduced and studied the SEIRS models with two delays. Recently, Greenhalgh [6] studied Hopf bifurcations in models of the SEIRS type, with density-dependent contact rate and death rate. Li and Muldowney [7] and Lietai [8], studied global dynamics of the SEIR models with a nonlinear incidence rate and with a standard incidence, respectively. Li et al. [9] analyzed the global dynamics of an SEIR model with vertical transmission and a bilinear incidence. Research on epidemic models of SEIR or SEIRS types with the general population-size-dependent adequate contact rate $C(N)$ satisfying the assumptions (H₁) and (H₂) are scarce in the published reports. Zhang [10] has studied an SEIR model with population size dependent contact rate and new flow of immigration, but has not considered the failure of immunity and the vaccination. But it is well known that they are very important in reality.

In this manuscript, we construct and study the SEIRS epidemic model with a general contact rate and immigration of distinct compartments. Especially, to establish the global stability of the endemic equilibrium, we reduced the model to a three-dimensional asymptotical autonomous differential system with a limit system. $C(N)/I$ is a general population-size-dependent function satisfying the assumptions (H₁) (H₂). We assume a constant proportion of new members into the population per unit time, of which a fraction q is exposed, a fraction p is infective, and a fraction b is recovered. So the fraction $1 - p - q - b$ is susceptible, where p, q, b are nonnegative constants with $0 \leq p + q + b \leq 1$. Assume that natural deaths occur at a rate proportional to the population size N . Then the natural death rate term is μN , μ is the natural death rate constant. Under our assumptions, the infection cannot be eliminated because there is a constant new infected individuals moving in, when $1 - q - p > 0$. In order to eradicate the disease, it would be necessary to isolate the fraction of arriving infected individuals. Our model has some features in common with models that include vertical transmission, but vertical transmission models normally include a new infected proportional, which are already in the population and thus may have a disease-free equilibrium [11, 12]. We

also make some simulation with the syphilis. With our conclusion, we can make some prediction. This can be used to some disease control department and so on.

2 Problem Formulation

2.1 Model Formulation

The total population size $N(t)$ is divided into four distinct epidemic subclasses (compartments) of individuals which are susceptible, exposed, infectious, and recovered (with temporary immunity), with the denoted $S(t)$, $E(t)$, $I(t)$, and $R(t)$ respectively. q, p and b are all nonnegative constant, and $0 \leq p + q + b \leq 1$. The parameters μ, ε, γ are all positive constants. α is a nonnegative constant and represents the death rate because of disease (the disease-related death rate), we assume it is small. μ is the rate constant for natural death, γ is the rate constant for recovery, and ε is the rate constant at which the exposed individuals become infective, so that $\frac{1}{\varepsilon}$ is the mean latent period. The recovered individuals are assumed to acquire temporary immunity, so there is transfer from class R back to class S with a proportion δ_1 . The positive constant A is the constant recruitment rate into the population, so that A/μ represents a carrying capacity, or maximum possible population size, rather than the population size N . $C(N)$ satisfying the assumptions (H₁) and (H₂) is the adequate contact rate. Then the SEIRS model with the adequate contact rate and immigration of different compartment individuals is derived on the basis of the basic assumptions. By using the transfer diagram, it is described by the following system of differential equations

$$\begin{aligned} S' &= (1 - p - q - b)A - \beta\lambda(N)\frac{S}{N}I \\ &\quad - \mu S + \delta_1 R - cS \\ E' &= qA + \beta\lambda(N)\frac{S}{N}I - \mu E - \varepsilon E \\ I' &= pA + \varepsilon E - \mu I - \alpha I - \gamma I \\ R' &= bA + \gamma I - \mu R + cS - R\delta_1 \\ N(t) &= S(t) + E(t) + I(t) + R(t). \end{aligned} \quad (1)$$

We study (1) in the closed set:

$$\Gamma = \{(S, E, I, R) \in R_+^4 : 0 \leq S + E + I + R = N \leq A/\mu\}$$

R_+^4 denotes the nonnegative cone of R_4 , including its lower dimensional faces. It can be verified that Γ is positively invariant with respect to (1). $\partial\Gamma, \overset{\circ}{\Gamma}$ denote the boundary and the interior of Γ respectively.

The continuity of the right side of (1) on $N > 0$ implies that solutions exist in Γ and solutions remain in Γ , they are continuous for all $t > 0$ (see [13]). If $D(N) = \frac{\beta\lambda(N)}{N} = \frac{C(N)}{N}$, $D(N) \in C'[0, \frac{A}{\mu}]$, then the right side of (1) is globally Lipschitz in Γ so that the initial value problem has a unique solution for all $t \geq 0$ that depends continuously on the data. If $D'(0)$ is bounded, then $D'(N)$ is still bounded on $[0, A/\mu]$. Then in the interior of the region Γ , the right side of (1) is locally Lipschitz, as a Lipschitz constant exists for every closed bounded subset of the interior of Γ . Thus for any point in the interior of Γ , the initial value problem has a unique solution for all $t \geq 0$. If $0 < p + q + b < 1$, the solutions starting at points on the boundary of Γ enter the interior of Γ . If $p + q + b = 0$, solutions starting on the S-axis approach $P_0 = (\frac{A(\mu + \delta_1)}{\mu(\mu + \delta_1 + c)}, 0, 0, \frac{Ac}{\mu(\mu + \delta_1 + c)})$, but all the other solutions starting at points on the boundary of Γ enter the interior of Γ . If $p + q = 1 - b$, for all solutions starting at points on the boundary of Γ , they enter the interior of Γ or stay on $\partial\Gamma$. So the initial value problem is well posed in a close set Γ .

2.2 Equilibrium

The equilibrium of system (1) can be found by setting the right sides of the four equations of (1) equal to zero, giving the algebraic system

$$\begin{aligned} (1 - p - b - q)A - \text{SID}(N) - \mu S \\ + \delta_1 R - cS = 0 \\ qA + D(N)SI = (\mu + \varepsilon)E \\ pA + \varepsilon E = (\mu + \alpha + \gamma)I \\ bA + \gamma I = \mu R - cS + \delta_1 R. \end{aligned} \tag{2}$$

Adding all equations in (2), we obtain

$$I = \frac{1}{\alpha}(A - \mu N) \tag{3}$$

Let $\delta = \mu + \alpha + \gamma$, $\omega = \mu + \varepsilon$ then R and E can be expressed in terms of N :

$$R = \frac{bA}{\mu + \delta_1} + \frac{\gamma(A - \mu N)}{\alpha(\mu + \delta_1)} + \frac{\alpha c(1 - q - b - p)A}{(\mu + \delta_1)(\alpha\mu + D(N)A - D(N)\mu N + \alpha\delta_1 + \alpha c)} \tag{4}$$

$$\begin{aligned} E &= \frac{\delta}{\varepsilon}I - \frac{pA}{\varepsilon}(A - \mu N) - \frac{pA}{\varepsilon} \\ S &= \frac{\alpha(1 - p - b - q)A}{\alpha\mu + D(N)(A - \mu N) + \alpha\delta_1 + \alpha c} \end{aligned} \tag{5}$$

Substituting (4) and (5) into the second equation in(2), we have

$$\begin{aligned} \frac{1}{\alpha}(A - \mu N)[D(N)\frac{\alpha(1 - p - b - q)A}{\alpha\mu + D(N)(A - \mu N) + \alpha\delta_1 + \alpha c} \\ - \frac{(\mu + \varepsilon)\delta}{\varepsilon}] + \frac{pA(\mu + \varepsilon)}{\varepsilon} + qA = 0 \end{aligned} \tag{6}$$

The existence and number of equilibrium correspond to those of the roots of the equation (6) in the interval $[0, \frac{A}{\mu}]$.

$$R_0 = \beta\lambda(A/\mu)\frac{\varepsilon}{\varepsilon + \mu}\frac{1}{\mu + \gamma + \alpha} = \frac{\varepsilon C(A/\mu)}{\delta\omega} \tag{7}$$

Theorem 1 Suppose $p + q + b = 0$ or new members of immigration are all susceptible.

The point

$$P_0 = \left(\frac{A(\mu + \delta_1)}{\mu(\mu + \delta_1 + c)}, 0, 0, \frac{Ac}{\mu(\mu + \delta_1 + c)} \right)$$

is the disease-free equilibrium of the system (1). It is stable when $R_0 \leq 1$ and unstable when $R_0 > 1$

1. When $R_0 < 1$, the solutions of the system (1) starting sufficiently close to P_0 in Γ move away from P_0 except those starting on the invariant S-axis which approach P_0 along the axis.

2. When $R_0 > 1$ system (1) has a unique interior (endemic) equilibrium and its coordinates satisfy (3)-(5)if and only if R_0 is between $\frac{\mu}{\mu + \delta_1 + c}$ and $\frac{C(A/\mu)}{\alpha}$.

Proof. When $p + q + b = 0$, that is $p = q = b = 0$, the equation (6) becomes:

$$\begin{aligned} G(N) &= \left[\frac{D(N)\alpha A}{\alpha(\mu + \delta_1 + c) + D(N)(A - \mu N)} \right. \\ &\quad \left. - \frac{(\mu + \varepsilon)\delta}{\varepsilon} \right] \frac{1}{\alpha}(A - \mu N) \\ &= 0. \end{aligned}$$

We can see that either $A - \mu N = 0$, then $I = 0$, and we can deduce

$$E = 0, I = 0, R = \frac{cA}{\mu(\mu + c + \delta_1)}, S = \frac{A\mu + A\delta_1}{\mu(\mu + c + \delta_1)}.$$

When $R_0 \leq 1$, we set $L = \varepsilon E + \omega I$, then its derivative along the solutions of (1) is

$$L' = \delta\omega I(R_0 \frac{S}{N} \frac{C(N)}{C(A/\mu)} - 1) \leq 0.$$

That implies that all paths in Γ approach the largest positively invariant subset of the set M where $L' = 0$.

The reason is the *Lyapunov-Lasell* theorem [18]. M is the set where $I = 0$. At the same time $E = 0$.

$$S' = (A + \delta_1 R) - (\mu + c)S$$

$$S = \frac{A + \delta_1 R}{\mu + c} + [S(0) - \frac{A + \delta_1 R}{\mu + c}]e^{-(\mu+c)S}$$

$$S \rightarrow \frac{A + \delta_1 R}{\mu + c}$$

We can conclude that all solutions of (1) will approach the disease-free equilibrium.

When $R_0 > 1$ then L' can be written as:

$$L' = \delta\omega I(R_0 \frac{S}{A/\mu} \frac{D(N)}{D(A/\mu)} - 1).$$

So $L' > 0$ for S which is sufficiently close to A/μ , except the invariant S-axis.

Solutions starting sufficiently close to P_0 move away neighbourhood of P_0 , except those starting on the S-axis which approach P_0 . From the assumptions (H_1) and (H_2) , the function $G(N)$ is decreasing, so that the equation $G(N) = 0$ has at most one unique root. We let

$$F(N) = D(N) \frac{\alpha A}{\alpha\mu + D(N)(A - \mu N) + \alpha\delta_1 + \alpha c} - \frac{(\mu + \varepsilon)\delta}{\varepsilon}$$

$$= \frac{A\alpha C(N)}{N\alpha(\mu + \delta_1 + c) + C(N)(A - \mu N)} - \frac{1}{R_0} \frac{C(A/\mu)}{\alpha}$$

$$\lim_{N \rightarrow 0} F(N) = 1 - \frac{1}{R_0} \frac{C(A/\mu)}{\alpha}$$

$$\lim_{N \rightarrow A/\mu} F(N) = C(A/\mu) \left(\frac{\mu}{\mu + \delta_1 + c} - \frac{1}{R_0} \right).$$

If and only if R_0 is between $\frac{\mu}{\mu + \delta_1 + c}$ and $C(A/\mu)/\alpha$, that is for

$$\frac{\mu}{\mu + \delta_1 + c} < R_0 < C(A/\mu)/\alpha$$

or

$$C(A/\mu)/\alpha < R_0 < \frac{\mu}{\mu + \delta_1 + c},$$

equation (6) has only one positive root in the interval $N_1 \in (0, A/\mu)$. That is the system has one endemic equilibrium $P_1 = (S_1, E_1, I_1, R_1)$ except disease equilibrium P_0 , and it satisfies:

$$S_1 = \frac{\alpha(1 - p - q - b)A}{\alpha\mu + D(N_1)(A - \mu N_1) + \alpha\delta_1 + \alpha c}$$

$$I_1 = \frac{1}{\alpha}(A - \mu N_1)$$

$$E_1 = \frac{\delta}{\alpha\varepsilon}(A - \mu N_1) - \frac{pA}{\varepsilon}$$

$$R_1 = \frac{\alpha c(1 - p - q - b)A}{(\mu + \delta_1)(\alpha\mu + D(N_1)(A - \mu N_1) + \alpha\delta_1 + \alpha c)} + \frac{bA}{\mu + \delta_1} + \frac{\gamma(A - \mu N_1)}{\alpha(\mu + \delta_1)}$$

is coordinates of P_1 satisfy(3)-(5)when $p = 0, q = 0, b = 0$, where $N = N_1$.

Theorem 2 When $p + q + b = 1$, that is all of new members of immigration are not susceptible, system (1) has a unique boundary equilibrium,

$$P_2 = \left(0, \frac{qA}{\mu + \varepsilon}, \frac{pA + qA\varepsilon/\mu + \varepsilon}{\mu + \alpha + \gamma}, \frac{bA}{\mu + \delta_1} + \frac{pA\gamma}{\delta(\mu + \delta_1)} + \frac{qA\omega\gamma}{\omega\delta(\mu + \delta_1)} \right)$$

and it is globally asymptotically stable.

Proof. Let $0 < p + q < 1 - b$ in system (1), it is easy to see that P_2 is the unique equilibrium of the system(1). The global asymptotical stability of the point P_2 can be proved by using the function $L = S$. \square

Theorem 3 When $0 < p + q < 1 - b$ system (1) only has equilibrium $P^* = (S^*, E^*, I^*, R^*)$ and P^* is in the feasible region of Γ .

Proof. It is easy to see that system (1) has no disease-free equilibrium, the equation (6) becomes

$$\frac{1}{\alpha}(A - \mu N) \left[D(N) \frac{\alpha(1 - p - q - b)A}{\alpha\mu + D(N)(A - \mu N) + \alpha\delta_1 + \alpha c} - \frac{(\mu + \varepsilon)\delta}{\varepsilon} \right] + \frac{pA(\mu + \varepsilon)}{\varepsilon} + qA = 0$$

to simplify we can see

$$G(N) = \frac{1}{\alpha}(A - \mu N) \left[\frac{C(N)(1 - p - q - b)A\alpha}{(A - \mu N)C(N) + N\alpha(\mu + \delta_1 + c)} - \frac{\omega\delta}{\varepsilon} \right] + \frac{pA\omega}{\varepsilon} + qA = 0$$

From the assumptions (H_1) - (H_2) , the function $G(N)$ is decreasing, so the equation has at most a unique root. And we have

$$\lim_{N \rightarrow 0} G(N) = (1 - p - q - b)A - \frac{\omega\delta A}{\alpha\varepsilon} + \frac{pA\omega}{\varepsilon} = \frac{A}{\varepsilon\alpha} [\mu(\alpha - \varepsilon - c) - \varepsilon\gamma - \varepsilon\alpha b] < 0,$$

$$\lim_{\mu \rightarrow A/\mu} G(N) = \frac{pA\omega}{\varepsilon} + qA > 0.$$

So the equation $G(N) = 0$ exists only one positive root N^* in $(0, A/\mu)$.

Now we can find the only equilibrium $P^* = (S^*, E^*, I^*, R^*)$ satisfying

$$\begin{aligned}
 I^* &= \frac{1}{\alpha}(A - \mu N^*) \\
 S^* &= \frac{\alpha(1 - p - q - b)A}{\alpha\mu + D(N^*)(A - \mu N^*) + \alpha\delta_1 + \alpha c} \\
 E^* &= \frac{\delta}{\alpha\varepsilon}(A - \mu N^*) - \frac{pA}{\varepsilon} \\
 R^* &= \frac{\alpha c(1 - p - q - b)A}{(\mu + \delta_1)(\alpha\mu + D(N^*)A - D(N^*)\mu N^* + \alpha\delta_1 + \alpha c)} \\
 &\quad + \frac{\gamma(A - \mu N^*)}{\alpha(\mu + \delta_1)} + \frac{bA}{\mu + \delta_1}
 \end{aligned}$$

2.3 Global stability of the Equilibrium

In this section, we establish that all solutions of (1) in the interior of Γ converge to P^* if $0 < p + q < 1 - b$ and to P_1 if $p + q + b = 0$ and if R_0 is between $\frac{\mu}{\mu + \delta_1 + c}$ and $C(A/\mu)/\alpha$ respectively. For this purpose, we first introduce the change of variable that can reduce the four-dimensional autonomous system (1) into a three-dimensional asymptotical autonomous system with a limit system, then prove that the equilibrium of this limit system corresponding to P_1 and P^* are globally asymptotically stable, by using the geometric approach to global stability problems in Li and Muldowney [14]. This implies the main conclusion in Corollary 3.1 of the third section.

For system (1), let $\delta_1 = 0$ then the equation can be:

$$\begin{aligned}
 S' &= (1 - p - q - b)A - \beta\lambda(N)\frac{S}{N}I - \mu S - cS \\
 E' &= qA + \beta\lambda(N)\frac{S}{N}I - \mu E - \varepsilon E \\
 I' &= pA + \varepsilon E - \mu I - \alpha I - \gamma I
 \end{aligned} \tag{8}$$

N, R can be obtained from $N' = A - \mu N - \alpha I$ and $R = N - S - E - I$. The feasible region of (8) is $T = \{(S, E, I) \in R_+^3 : 0 \leq S + E + I \leq A/\mu\}$ is positively invariant with respect to (8), where R_+^3 denotes the nonnegative cone of R^3 , including its lower dimensional faces. Thus system (8) is bounded.

Lemma 1 Suppose $p + q + b = 0$

1 If $R_0 \leq 1$, $Q_0 = (\frac{A(\mu + \delta_1)}{\mu(\mu + \delta_1 + c)}, 0, 0)$ is the only equilibrium in T and it is globally asymptotically stable. If $R_0 > 1$, Q_0 becomes unstable, whereas $Q_1 = (S_1, E_1, I_1)$ is corresponding to the equilibrium P_1 of system (1), and emerges as a unique

equilibrium in the interior of T , where R_0 is defined by (7)

$$\begin{aligned}
 I_1 &= \frac{1}{\alpha}(A - \mu N_1) \\
 E_1 &= \frac{\delta}{\alpha\varepsilon}(A - \mu N_1) - \frac{pA}{\varepsilon} \\
 S_1 &= \frac{\alpha(1 - p - q - b)A}{\alpha\mu + D(N_1)(A - \mu N_1) + \alpha\delta_1 + \alpha c} \\
 R_1 &= \frac{bA}{\mu + \delta_1} + \frac{\gamma(A - \mu N_1)}{\alpha(\mu + \delta_1)} \\
 &\quad + \frac{\alpha c(1 - p - q - b)A}{(\mu + \delta_1)(\alpha\mu + D(N_1)(A - \mu N_1) + \alpha\delta_1 + \alpha c)}
 \end{aligned}$$

N_1 is the unique root of the equation $G(N) = 0$.

2 When $R_0 > 1$ the solutions of system (7) starting sufficiently close to Q_0 in T move away from Q_0 except that starting on the invariant S -axis approach Q_0 along this axis.

Lemma 2 When $0 < p + q < 1 - b$, the system (7) has the only equilibrium $Q^* = (S^*, E^*, I^*)$, which is corresponding to the equilibrium P^* of system (1), and Q^* is in the interior of the feasible region of T , where

$$\begin{aligned}
 I^* &= \frac{1}{\alpha}(A - \mu N^*)E^* \\
 E^* &= \frac{\delta}{\alpha\varepsilon}(A - \mu N^*) - \frac{pA}{\varepsilon} \\
 S^* &= \frac{\alpha(1 - p - q - b)A}{\alpha\mu + D(N^*)(A - \mu N^*) + \alpha\delta_1 + \alpha c}
 \end{aligned}$$

Lemma 3 The system (8) is uniformly persistent if either $p + q + b = 0$, R_0 between $\frac{\mu}{\mu + \delta_1 + c}$ and $C(A/\mu)/\alpha$ or $0 < p + q < 1 - b$ holds.

Proof. If $0 < p + q < 1 - b$, it is easy to see that the vector field of system (8) is transversal to the boundary of T on all its faces.

If $p + q + b = 0$, the vector field of the system (8) is transversal to boundary of T on all its faces except the S -axis, which is invariant with respect to (8). On the S -axis the equation of $S : \frac{dS}{dt} = A - \mu S - cS$, implies

$$S(t) \rightarrow \frac{A}{\mu + c} \text{ as } t \rightarrow +\infty$$

If $p + q + b = 0$, Q_0 is a only ω -limit point on the boundary of T .

The system is said to be uniformly persistent ([15, 16]) if there exists constancy c with $0 <$

$c < 1$ such that any solution $(S(t), E(t), I(t))$ with initial data $(S(0), E(0), I(0)) \in \overset{\circ}{T}$ satisfies $\liminf_{t \rightarrow \infty} |(S(t), E(t), I(t))| \geq c$.

The conclusion of the lemma follows from Theorem 4.3 in [17], as the maximal invariant set on the boundary ∂T of T is an empty set when $0 < p + q < 1 - b$ and the singleton $\{Q_0\}$ and Q_0 is isolated when $p + q + b = 0$ and R_0 between $\frac{\mu}{\mu + \delta_1 + c}$ and $C(A/\mu)/\alpha$, thus the hypothesis (H) of [17] holds for(8). The conclusion for uniform persistence in Theorem 4.3 of [17] is equivalent to Q_0 being unstable when $p + q + b = 0$ and $R_0 > 1$. Thus, system(8)is uniformly persistent in $\overset{\circ}{T}$.

The main aim of this section is to prove the following Theorem.

Theorem 4 1) When $p + q + b = 0$ and R_0 between $\frac{\mu}{\mu + \delta_1 + c}$ and $C(A/\mu)/\alpha$, the unique endemic equilibrium P_1 of system (1)is globally asymptotically stable in the interior of Γ . Moreover, P_1 attracts all trajectories in Γ except those on the invariant S-axis that converge to P_0 along this axis.

2) When $0 < p + q < 1 - b$, the unique endemic equilibrium P^* of system (1)is globally asymptotically stable in the interior of Γ .

By inspecting the vector field given by (1), we see that if $0 < p + q < 1 - b$, all trajectories starting from the boundary $\partial\Gamma$ of Γ enter the interior $\overset{\circ}{\Gamma}$ of Γ .

When $p + q + b = 0$, they do except those solutions on the S-axis that converge to P_0 , along this invariant axis. Thus we need only to prove that $\overset{\circ}{\Gamma}$ is the attractive region of P^* when $0 < p + q < 1 - b$ and P_1 when $p + q + b = 0$ respectively.

For the results in Theorem 1 and Theorem 3, we utilize a geometric approach to the global stability problem developed in Li and Muldowney [19]and Smith [18]. Here we apply the theory, in particular Theorem A.2 in the appendix, to prove the following theorem. In addition, it is remarked in [18], under the assumptions of theorem A.2 in the appendix, the condition $\bar{q}_2 < 0$ also implies the local stability of the equilibrium \bar{x} , as assuming the contrary, \bar{x} is both the alpha and omega limit of a homo-clinic orbit that is ruled out by the condition $\bar{q}_2 < 0$.

From Lemma 1 we can make the following conclusions.

Lemma 4 1) If $0 < p + q < 1 - b$, Q^* is globally asymptotically stable.

2) When $p + q + b = 0$ and R_0 is between $\frac{\mu}{\mu + \delta_1 + c}$ and $C(A/\mu)/\alpha$, Q_1 is globally asymptotically stable.

Proof. By Lemma 3, there exists a compact set K in the interior of Γ that is absorbing(8). Thus, in the closed set T the system (8) satisfies the assumptions H_3, H_4, H_5 in appendix [17]. Let $x = (S, E, I)$ and $f(X)$ denote the vector field of (8). The Jacobian matrix $J = \frac{\partial f}{\partial x}$

Associated with a general solution of (8) is:

$$J = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & \varepsilon & -\delta \end{bmatrix}$$

Where we have:

$$\begin{aligned} a_{11} &= -D(N)I - ISD'(N) - c - \mu \\ a_{12} &= -SID'(N) \\ a_{13} &= -D(N)S - D'(N)SI \\ a_{21} &= D(N)I + ISD'(N) \\ a_{22} &= D'(N)SI - \mu - \varepsilon \\ a_{23} &= D(N)S + D'(N)SI \end{aligned}$$

The second compound matrix $J^{[2]}$ of J can be calculated as follows:

$$J^{[2]} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \varepsilon & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}$$

where

$$\begin{aligned} b_{11} &= a_{11} + a_{22} = -D(N)I - c - 2\mu - \varepsilon \\ b_{22} &= a_{11} + a_{33} = -\delta - c - \mu - D(N)I - ISD'(N) \\ b_{33} &= a_{22} + a_{33} = -\delta - \mu - \varepsilon + D'(N)SI \\ b_{12} &= a_{23} \quad b_{13} = -a_{13} \quad b_{23} = a_{12} \quad b_{32} = a_{21} \end{aligned}$$

The proof of the theorem consists of choosing a suitable vector norm $|\cdot|$ in R^3 and a 3×3 matrix-valued function $A(x)$, such that the quantity \bar{q}_2 defined by (A.2) in the appendix is negative. We set A as the following diagonal matrix.

$$A(S, E, I) = \text{diag}(1, \frac{E}{I}, \frac{E}{I}) \tag{9}$$

Then A is C^1 and non-singular in the interior of T . Thus

$$\begin{aligned} A_f A^- &= \text{diag}\left(0, \frac{I}{E}\left(\frac{E}{I}\right)_f, \frac{I}{E}\left(\frac{E}{I}\right)_f\right) \\ &= \text{diag}\left(0, \frac{E'}{E} - \frac{I'}{I}, \frac{E'}{E} - \frac{I'}{I}\right) \end{aligned}$$

Therefore the matrix $B = A_f A^{-1} + A J^{[2]} A^{-1}$ can be written in the block form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

With

$$B_{11} = -D(N)I - c - 2\mu - \varepsilon$$

$$B_{21} = \begin{bmatrix} \frac{E\varepsilon}{I} \\ 0 \end{bmatrix}$$

$$B_{12} = \frac{I}{E}(D(N)S + D'(N)SI) \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} c_{11} & -SID'(N) \\ D(N)I + ISD'(N) & c_{22} \end{bmatrix}$$

$$c_{11} = \frac{E'}{E} - \frac{I'}{I} - \mu - \delta - c - D(N)I - ISD'(N)$$

$$c_{22} = \frac{E'}{E} - \frac{I'}{I} - \mu - \delta - \varepsilon + D'(N)IS$$

Choosing the vector norm $|\cdot|$ in $R^{[2]} \cong R^3$ as $|(u, v, w)| = \sup\{|u|, |v| + |w|\}$

The *Lozinskiĭ* measure $\rho(B)$ with respect to $|\cdot|$ can be estimated as follows (see [18]): $\rho(B) \leq \sup\{g_1, g_2\}$, where

$$g_2 = \rho(B_{12}) + |B_{22}| \quad g_1 = \rho_1(B_{11}) + |B_{12}| \quad (10)$$

where $|B_{12}|, B_{21}$ are matrix norms with respect to l_1 vector norm, and ρ_1 denotes the *Lozinskiĭ* measure with respect to l_1 vector norm. More specifically, $|B_{21}| = \frac{E\varepsilon}{I}$ and noting the assumptions (H₁)-(H₂)

$$B_{21} = \frac{IS}{E}(D(N) + |D'(N)I|)$$

Noting that B_{11} is a scalar, its *Lozinskiĭ* measure with respect to any vector norm in ρ_1 is equal to B_{11} . In order to compute $\rho_1(B_{22})$, we add absolute value of off-diagonal elements to the diagonal one in each column of B_{22} , and then take the maximum of two sums. Using $D'(N) \leq 0$ and $\omega = \mu + \varepsilon$, we have

$$\rho_1(B_{22}) = \frac{E'}{E} - \frac{I''}{I} - \delta - c, \quad \text{if } c \leq \varepsilon$$

$$\rho_1(B_{22}) = \frac{E'}{E} - \frac{I''}{I} - \delta - \varepsilon, \quad \text{if } c > \varepsilon$$

Therefore

$$B_{11} = -D(N)I - c - 2\mu - \varepsilon$$

$$B_{12} = \frac{IS}{E}(D(N) + D'(N)I)$$

The reason is that

$$D(N) + D'(N)I \geq D(N) + D'(N)N \\ = (ND(N))' = C'(N) \geq 0$$

and

$$g_1 = -\mu - \omega - c - D(N)I + \frac{IS}{E}(D(N) + D'(N)I) \quad (11)$$

$$g_2 = \frac{E\varepsilon}{I} + \frac{E'}{E} - \frac{I'}{I} - \delta - c \quad \text{if } c \leq \varepsilon$$

$$g_2 = \frac{E\varepsilon}{I} + \frac{E'}{E} - \frac{I'}{I} - \delta - \varepsilon \quad \text{if } c > \varepsilon$$

Rewrite the last two equations of (8) we have

$$\frac{E'}{E} + \omega - \frac{qA}{E} = \frac{D(N)SI}{E} \\ \frac{I'}{I} + \delta - \frac{pA}{I} = \frac{\varepsilon E}{I} \quad (12)$$

Seeing that $D'(N) \leq 0, D(N) \geq 0$ we have if $c \leq \varepsilon$

$$g_1 = -\mu - c - \frac{qA}{E} + \frac{E'}{E} - D(N)I \\ + \frac{IS}{E}D'(N)I \leq \frac{E'}{E} - \mu - c$$

$$g_2 = -c - \frac{pA}{I} + \frac{E'}{E} \quad \text{if } c \leq \varepsilon$$

$$g_2 = -\varepsilon - \frac{pA}{I} + \frac{E'}{E} \quad \text{if } c > \varepsilon$$

$$\rho(B) \leq \frac{E'}{E} - \min\{c, \mu + c, \varepsilon\}$$

Along with the solution $(S(t), E(t), I(t))$ of equation (8) with the initial value $(S(0), E(0), I(0))$ in the compact absorbed set K which is in the internal of T , we have

$$\frac{1}{t} \int_0^t \rho(B) ds \leq \frac{1}{t} \log \frac{E(t)}{E(0)} - M$$

$M = \min\{c, \mu + c, \varepsilon\}$, which implies

$$\bar{q}_2 = \limsup_{t \rightarrow \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t \rho(B(x(s, x_0))) ds \leq -M.$$

3 Simulation and application analysis

As an application, the author discussed the syphilis. Syphilis is an infectious disease which is transmitted by sexually, blood and maternal chronic. In China, the

prevalence of syphilis is almost distributed throughout the country (Figure1). Henan, Yunnan, Guangxi are the most seriously provinces. In order to observe the development trend of syphilis prevalence, the author collected reports of National Notifiable Disease statistics from January 2006 to June 2012, in total of 78 data [20](Figure 2,3)



Figure 1:

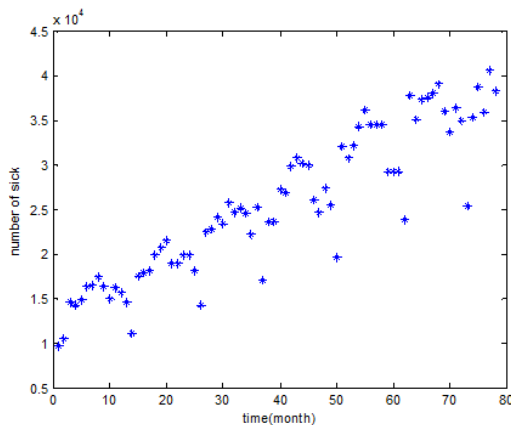


Figure 2: The scatter diagram of Syphilis prevalence

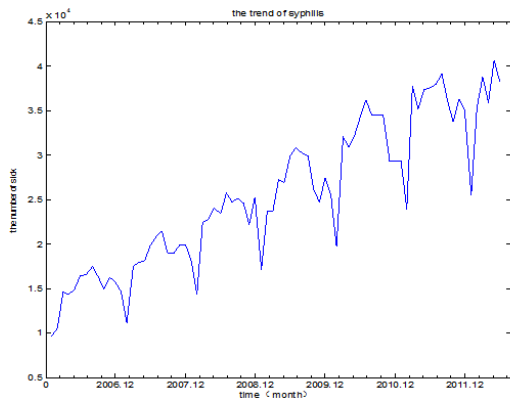


Figure 3: The trend of Syphilis prevalence

We can see that, the number of syphilis fluctuates in periodic, the peak located in 8,9,10 months annual, and 1,2 month are the least months. But the peak of fluctuations is being higher and higher. It is obvious from Figure2 that the trough values are roughly linear.

At present, most of the research is the symptom, pathogenesis and pathological analysis [21, 22]. For the consideration of social and economic impact, human activity change gradually, we weed out trough data, phase prediction(Figure4).

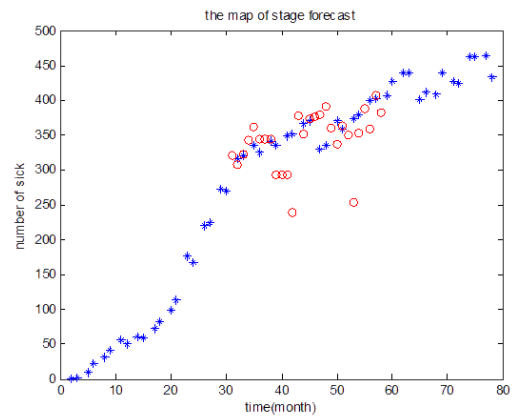


Figure 4: The map of prediction

$$p = q = b = 0.133, \beta = 0.56, \lambda = 0.42, A = 0.12$$

$$\mu = 0.0012, \varepsilon = 0.65, \alpha = 0.21, \gamma = 0.72, \delta_1 = 1$$

The red circle sets are the real data, the star sets are forecast data, and we can see that this method has well done the problem. We can make prediction of the numbers of Syphilis after July 2012 through the prophase data. The author extracts the trough data, and make forecast, using the data fitting (Figure5)

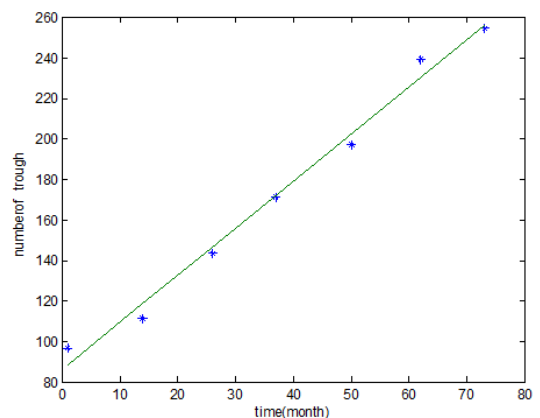


Figure 5: The data fitting for the trough data

Through the above prediction method for syphilis, we can solve the prevalence prediction of this

complex infectious disease. We have note that, this prediction method is only applicable to a short period of time. Thus the application value of the model has been proved.

If we make prediction for a long time using this method, you will be very disappointment(Figure 6).1

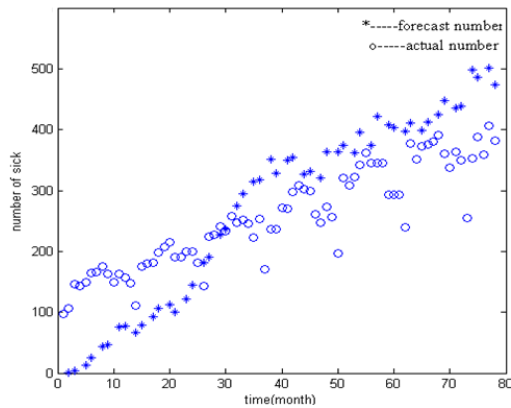


Figure 6: prediction for a long time

In addition, the recruitment rate A is very important, we can see that the number of diseased change with A from figure 7. We also simulate the stability of the endemic equilibrium(Figure 8).

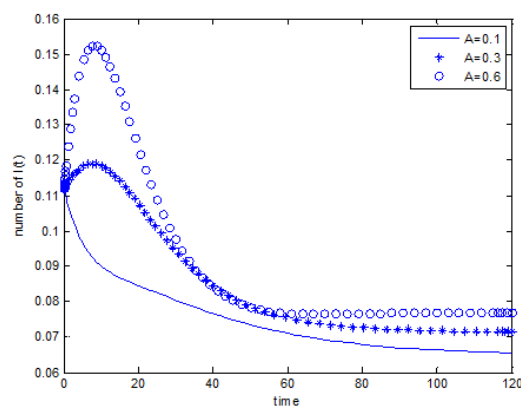


Figure 7: The change of diseased with A

4 Conclusions

This article has been devoted to studying an SEIRS model for disease transmission that include immigration of the infectives, susceptibles, exposed and recovered. For the reason that the immunity of the recovered is temporary, a proportion δ_1 of recovered will come back to susceptibles. We also consider vaccine injection to the susceptibles with a proportion c .

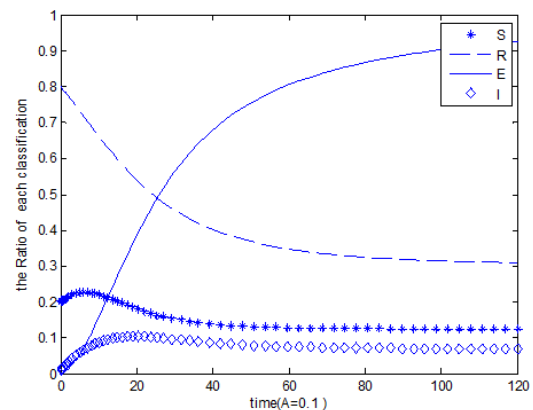


Figure 8: The stability of the endemic equilibrium

The model also incorporates a population size dependent contact rate $C(N)$ and a disease-related death. As the infected fraction cannot be eliminated from the population, this kind of model has unique an endemic equilibrium that is globally asymptotically stable. In order to prove the global asymptotical stability of the endemic equilibrium, we change our system to a three-dimensional asymptotical autonomous system with limit equation. The result is that the global dynamical behaviour of system (1) and the outcome of the disease are completely determined by R_0 . In a special case $p + q + b = 0$, where the new members of immigration are all susceptibles, the model shows a threshold phenomenon. In order to eradicate the disease it would be necessary to isolate the fraction of arriving infected individuals. Our model has some features in common with models that include vertical transmission, but vertical transmission models normally include a flow of new infected proportional to the number of infected already in the population and thus may have a disease-free equilibrium [9, 13].

R_0 is the product of the constant $\frac{\varepsilon}{\delta\omega}$ and the value $C(A/\mu)$ of the adequate contact rate function $C(N)$ at $N = \frac{A}{\mu}$, which is the population size at the disease-free equilibrium. The constant $\frac{1}{\delta} = \frac{1}{\mu + \alpha + \gamma}$ is the average period and $\frac{\varepsilon}{\omega} = \frac{\varepsilon}{\mu + \varepsilon}$ is the fraction of exposed individuals surviving the latent class E . From expression(7) of R_0 , it is easy to see the contribution of the contact rate to the basic reproduction number. If let $C(N)$ be a constancy λ , then R_0 defined by (7) becomes the basis reproduction number $\frac{\lambda\varepsilon}{(\mu + \varepsilon)(\mu + \alpha + \gamma)}$ of the SEIR model with the stan-

dard incidence $\frac{\lambda SI}{N}$ in [8]. If $\delta=0, c=0$, the outcome will be the same to article [10].

Finally, we discussed syphilis incidence and transmission characteristics and predict the development in China. Computer simulation shows that the model can reflect the dynamic and immigration behavioural for disease transmission. So we can take reasonable prevention and intervention methods to provide reference, basing on the theory.

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