Positive solutions for nonlocal boundary value problems of fractional differential equation

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Abstract: In this paper, we investigate the existence, uniqueness and multiplicity of positive solutions for nonlocal boundary value problems of fractional differential equation. Firstly, we reduce the problem considered to the equivalent integral equation. Secondly, the existence and uniqueness of positive solution is obtained by the use of contraction map principle and some Lipschitz-type conditions. Thirdly, by means of some fixed point theorems, some results on the multiplicity of positive solutions are obtained.

Key–Words: Riemann-Liouville fractional derivative; Nonlocal boundary value problem; Fixed point theorem; successive iteration; Positive solution.

1 Introduction

In this paper, we investigate the positive solutions for nonlocal boundary value problems for fractional differential equations of the form

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0,\ 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = 0, \quad u(1) = \beta u(\xi), \tag{2}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $2 < \alpha \leq 3$ and β, ξ, f satisfying

$$(H_1):\ \beta > 0,\ 0 < \xi < 1,\ 1 - \beta\xi^{\alpha-1} > 0;$$

$$(H_2):\ f \in C\left([0, 1] \times [0, +\infty) \to [0, +\infty)\right).$$

Fractional-order models are found to be more adequate than integer -order models in some real world problems. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, see [1-14] and the reference therein.

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [15] and take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors, for example, see [16-25] and the references therein. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located at intermediate points.

In recent years, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in various fields of science and engineering such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. In consequence, fractional differential equations have been of great interest. For details, see [26-29] and the references therein. Recently, two-point boundary value problems of fractional differential equation have been extensively studied in the literature.

By the use of some fixed point theorems on cones, Zhang [31] obtain the existence of positive solutions for the equation

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0,\ 0 < t < 1,$$

with the boundary condition

$$u(0) + u'(0) = u(1) + u'(1) = 0.$$

Using the adomian decomposition method, Jafari [30] discuss the following fractional boundary value problem

$$D^{\alpha}_{0+}u(x)+\mu f(x, u(x)) = 0,\ 0 < x < 1,\ 1 < \alpha \le 2,$$

$$u(0) = 0, \quad u(1) = c.$$

Recently, in [34], using a fixed point theorem on cone, Bai established the eigenvalue intervals of the following problem

$${}^{C}D^{\alpha}_{0+}u(t) + \lambda h(t)f(u(t)) = 0,\ 0 < t < 1,$$

$$u(0) = u'(1) = u''(0) = 0,$$

where $2 < \alpha \le 3$, ${}^{C}D^{\alpha}_{0+}$ is the standard Caputo differentiation.

Sun etc [33] studied the existence of solutions for the boundary value problem of fractional hybrid differential equations

$$D^{\alpha}_{0+}\left[\frac{x(t)}{f(t, x(t))}\right] + g(t, x(t)) = 0,\ 0 < t < 1,$$

$$x(0) = x(1) = 0,$$

where $1 < \alpha \le 2$, D^{α}_{0+} is the Riemann-Liouville fractional derivative.

Also, the existence and multiplicity of solutions for three-point boundary value problems of fractional differential equations have been studied by some authors.

In [32], Bai investigate the existence and uniqueness of positive solutions for a nonlocal boundary value problem of fractional differential equation

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0,\ 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \beta u(\eta),$$

where

$$1 < \alpha \le 2,\ 0 < \beta\eta^{\alpha-1} < 1,\ 0 < \eta < 1,$$

and D^{α}_{0+} is the standard Riemann-Liouville fractional derivative.

Applying the Schauder fixed point theorem, in [35], an existence result is proved for the following system

$$\begin{aligned} &D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)) = 0,\ 0 < t < 1,\\ &D^{\alpha}v(t) = g(t, u(t), D^{q}u(t)) = 0,\ 0 < t < 1,\\ &u(0) = 0, \quad u(1) = \gamma u(\eta),\\ &v(0) = 0, \quad v(1) = \gamma v(\eta), \end{aligned}$$

where

$$\begin{array}{ll} 1 < \alpha, \beta < 2, & p, q, \gamma > 0,\ 0 < \eta < 1,\\ \alpha - q \ge 1, & \beta - p \ge 1,\\ \gamma\eta^{\alpha-1} < 1, & \gamma\eta^{\beta-1} < 1, \end{array}$$

and D^{α} is the standard Riemann-Liouville fractional derivative.

We find that the above papers considered three-point fractional boundary value problem for $1 < \alpha \le 2$. To the best knowledge of the authors, no work has been done to get positive solution of the three-point fractional boundary value problem for $2 < \alpha \le 3$. The aim of this paper is to fill the gap in the relevant literatures. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

2 Preliminaries and Lemmas

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results.

Definition 1 *[32] The Riemann-Liouville fractional integral operator of order $q > 0$ for function $y \in L'(R^{+})$ is defined as*

$$I^{q}_{0+}y(t) = \frac{1}{\Gamma(q)}\int_{0}^{t}(t-s)^{q-1}y(s)ds,\ \ q > 0.$$

Definition 2 *[32] The Riemann-Liouville fractional derivative of order $q > 0$, $n - 1 < \alpha < n$, $n \in N$ for function y is defined as*

$$D^{q}_{0+}y(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-\alpha-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 3 *[32] The equality $D^{q}_{0+}I^{q}_{0+}f(t) = f(t)$, $q > 0$ holds for $f \in L(0, 1)$.*

Lemma 4 *[32] Let $q > 0$, then the fractional differential equation*

$$D_{0+}^{q}u(t) = 0$$

has solutions

$$u(t) = c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+\cdots+c_nt^{\alpha-n},\ c_i \in R,$$

$$i = 1, 2, \cdots, n, n = [q] + 1.$$

Lemma 5 [32] *Let $q > 0$, then*

$$I_{0+}^{q}D_{0+}^{q}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + \cdots + c_nt^{\alpha-n}$$

for some $c_i \in R$, $i = 1, 2, \cdots, n, n = [q] + 1$.

Lemma 6 *Suppose that $(H_1), (H_2)$ hold. Then for $h(t) \in C[0, 1]$, the following boundary value problem*

$$D_{0+}^{\alpha}u(t) + h(t) = 0,\ 2 < \alpha \leq 3,\ 0 < t < 1, \quad (3)$$

$$u(0) = u'(0) = 0, \quad u(1) = \beta u(\xi), \quad (4)$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)h(s)ds + \frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi, s)h(s)ds, \quad (5)$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (6)$$

Proof: We can apply Lemma 5 and Definition 1 to reduce $D_{0+}^{\alpha}u(t) + h(t) = 0$ to an equivalent integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}.$$

By the boundary conditions of (4), there are $c_2 = c_3 = 0$ and

$$C_1 = \frac{1}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\Big[\int_0^1 (1-s)^{\alpha-1}h(s)ds - \beta\int_0^{\xi}(\xi-s)^{\alpha-1}h(s)ds\Big].$$

Therefore, problem (3),(4) has a unique solution

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\Big[\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds \\ &\quad -\beta\int_0^{\xi}(\xi-s)^{\alpha-1}t^{\alpha-1}h(s)ds\Big] \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds \\ &\quad +\frac{\beta\xi^{\alpha-1}}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds \\ &\quad -\frac{\beta}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\int_0^{\xi}(\xi-s)^{\alpha-1}t^{\alpha-1}h(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t [(1-s)^{\alpha-1}t^{\alpha-1} - (t-s)^{\alpha-1}]h(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds \\ &\quad +\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}\xi^{\alpha-1}h(s)ds \\ &\quad -\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\int_0^{\xi}(\xi-s)^{\alpha-1}h(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t [(1-s)^{\alpha-1}t^{\alpha-1} - (t-s)^{\alpha-1}]h(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds \\ &\quad +\frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\Big[\int_0^{\xi}[(1-s)^{\alpha-1}\xi^{\alpha-1} \\ &\quad -(\xi-s)^{\alpha-1}]h(s)ds \\ &\quad +\int_{\xi}^1 (1-s)^{\alpha-1}\xi^{\alpha-1}h(s)ds\Big] \\ &= \int_0^1 G(t, s)h(s)ds \\ &\quad +\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi, s)h(s)ds. \end{aligned}$$

The proof is completed. □

Lemma 7 *The function $G(t, s)$ defined by (6) satisfies $G(t, s) > 0$ for $t, s \in (0, 1)$.*

Proof: We divide the proof into two cases.

Case 1: when $0 \leq s \leq t \leq 1$,

$$t^{\alpha-1}(1-s)^{\alpha-1} = (t-ts)^{\alpha-1} > (t-s)^{\alpha-1},$$

so, $t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} > 0$.

Case 2: when $0 \le t \le s \le 1$,

$$t^{\alpha-1}(1-s)^{\alpha-1} > 0.$$

The proof is completed. □

Theorem 8 *Let P be a cone in a real Banach space E, $P_c = \{x \in P : \|x\| < c\}$, θ is a nonnegative continuous concave functional on P such that $\theta(x) \le \|x\|$, for all $x \in \overline{P}_c$, and $P(\theta, b, d) = \{x \in P : b \le \theta(x), \|x\| \le d\}$. Suppose that $T : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exist positive constants $0 < a < b < d \le c$ such that*
(C_1) : $\{x \in P(\theta, b, d) : \theta(x) > b\} \ne \emptyset$ and $\theta(Tx) > b$ for $x \in P(\theta, b, d)$;
(C_2) : $\|Tx\| < a$ for $x \in \overline{P}_a$;
(C_3) : $\theta(Tx) > b$ for $x \in P(\theta, b, c)$ with $\|Tx\| > d$.
Then T has at least three fixed points x_1, x_2 and x_3 with

$$\|x_1\| < a, \quad b < \theta(x_2), \quad a < \|x_3\| \quad \textit{with} \quad \theta(x_3) < b.$$

Remark 9 *If $d = c$, then condition (C_1) implies condition (C_3).*

3 Existence and uniqueness solution of the problem (1), (2)

Let $E = C[0,1]$ be endowed with the ordering $u \le v$ if $u(t) \le v(t)$ for all $t \in [0,1]$ and the maximum norm $\|u\| = \max\limits_{0 \le t \le 1} |u(t)|$. Define the cone $P \subset E$ by

$$P = \{u \in E | u(t) \ge 0 \text{ for } t \in [0,1]\}.$$

Lemma 10 *Let $T : P \to E$ be the operator defined by*

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s)) ds + \frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s) f(s,u(s)) ds. \quad (7)$$

Then $T : P \to P$ is completely continuous.

Proof: The operator $T : P \to P$ is continuous in view of nonnegativeness and continuity of $G(t,s)$ and $f(t,u)$.

Let $\Omega \subset P$ be bounded. i.e., there exists a positive constant $M > 0$ such that $\|u\| \le M$, for all $u \in \Omega$. Let $L = \max\limits_{0 \le t \le 1,\ 0 \le u \le M} |f(t,u)| + 1$, then for $u \in \Omega$, we have

$$\begin{aligned} |Tu(t)| &\le |\int_0^1 G(t,s) f(s,u(s)) ds \\ &\quad + \frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s) f(s,u(s)) ds| \\ &\le \left(1 + \frac{\beta}{1-\beta\xi^{\alpha-1}}\right) L \int_0^1 \max_{0 \le t \le 1} G(t,s) ds. \end{aligned}$$

Hence, $T(\Omega)$ is bounded.

On the other hand, given $\varepsilon > 0$, setting

$$\delta = \min\left\{1, \left(\frac{\Gamma(\alpha)\varepsilon\alpha(1-\beta\xi^{\alpha-1})}{2\beta L(\xi^{\alpha-1}-\xi^{\alpha})} 2^{1-\alpha}\right)^{\frac{1}{\alpha-1}}\right\},$$

then for each $u \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and $t_2 - t_1 < \delta$, one has $|Tu(t_2) - Tu(t_1)| < \varepsilon$. That is to say, $T(\Omega)$ is equicontinuity.

In fact,

$$\begin{aligned} & |Tu(t_2) - Tu(t_1)| \\ = & \Big| \int_0^1 [G(t_2,s) - G(t_1,s)] f(s,u(s)) ds \\ & + \frac{\beta t_2^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s) f(s,u(s)) ds \\ & - \frac{\beta t_1^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s) f(s,u(s)) ds \Big| \\ \le & \Big| \int_0^1 [G(t_2,s) - G(t_1,s)] f(s,u(s)) ds \Big| \\ & + \Big| \frac{\beta(t_2^{\alpha-1} - t_1^{\alpha-1})}{1-\beta\xi^{\alpha-1}} \Big| \Big| \int_0^1 G(\xi,s) f(s,u(s)) ds \Big| \\ & \le L \int_0^1 |(G(t_2,s) - G(t_1,s))| ds \\ & + \Big| \frac{\beta L}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s) ds (t_2^{\alpha-1} - t_1^{\alpha-1}) \Big| \\ \le & L(\int_0^1 |G(t_2,s) - G(t_1,s)| ds) \\ & + \frac{\beta L}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})} \left(\frac{\xi^{\alpha-1} - \xi^{\alpha}}{\alpha}\right) |t_2^{\alpha-1} - t_1^{\alpha-1}|. \end{aligned}$$

In the following, we divide the proof into two cases.
Case 1: $\delta \le t_1 < t_2 < 1$, we have

$$t_2^{\alpha-1} - t_1^{\alpha-1} \le \frac{\alpha-1}{\delta^{2-\alpha}}(t_2 - t_1) \le (\alpha-1)\delta^{\alpha-1}.$$

Case 2: $0 \le t_1 < \delta$, $t_2 < 2\delta$, we have

$$t_2^{\alpha-1} - t_1^{\alpha-1} \le t_2^{\alpha-1} < (2\delta)^{\alpha-1}.$$

Consequently, we have

$$t_2^{\alpha-1} - t_1^{\alpha-1} \le 2^{\alpha-1}\delta^{\alpha-1}.$$

On the one hand, $G(t,s)$ is uniformly continuous in $[0,1] \times [0,1]$ because of the continuity of $G(t,s)$. So, for the given $\varepsilon > 0$, whenever $|t_2 - t_1| < \delta$, we have

$$|G(t_2,s) - G(t_1,s)| < \frac{\varepsilon}{2L}.$$

On the other hand,

$$\begin{aligned} & \frac{\beta L}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\left(\frac{\xi^{\alpha-1}-\xi^{\alpha}}{\alpha}\right)|t_2^{\alpha-1}-t_1^{\alpha-1}| \\ < & \frac{\beta L}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\left(\frac{\xi^{\alpha-1}-\xi^{\alpha}}{\alpha}\right)2^{\alpha-1}\delta^{\alpha-1} \\ < & \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, $|Tu(t_2)-Tu(t_1)|<\varepsilon$, which implies that $\{Tu:\ u\in\Omega\}$ is equicontinuous. By means of the Arzela-Ascoli theorem, we have that $T:\ P\to P$ is completely continuous. The proof is completed. □

Lemma 11 *Suppose that* $(H_1),(H_2)$ *hold. Then* u *is a solution of problem (1), (2) if and only if* u *is a fixed point of* T.

Theorem 12 *Assume that* $f(t,u)$ *satisfies*

$$\begin{gathered} |f(t,u)-f(t,v)|<\lambda(t)|u-v|, \\ t\in[0,1],\ u,v\in[0,+\infty). \end{gathered} \tag{8}$$

Then problem (1), (2) has a unique positive solution if

$$\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds<\Gamma(\alpha)(1-\beta\xi^{\alpha-1}). \tag{9}$$

Proof: In the following, we will prove that T^n is a contraction operator for n sufficiently large.

Indeed, from (6), (7), for $u,v\in P$, we have the estimate,

$$\begin{aligned} & |(Tu-Tv)(t)| \\ \le & \int_0^1 G(t,s)|f(s,u(s))-f(s,v(s))|ds \\ & +\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\times \\ & \int_0^1 G(\xi,s)|f(s,u(s))-f(s,v(s))|ds \\ < & \frac{\|u-v\|}{\Gamma(\alpha)}\times \\ & \left(\int_0^t[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]\lambda(s)ds\right. \\ & \left.+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds\right) \\ & +\frac{\beta t^{\alpha-1}\|u-v\|}{\Gamma(\alpha)(1-\beta\xi^{\alpha-1})}\times \\ & \left(\int_0^\xi[\xi^{\alpha-1}(1-s)^{\alpha-1}-(\xi-s)^{\alpha-1}]\lambda(s)ds\right. \\ & \left.+\int_\xi^1 \xi^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds\right) \\ < & \frac{\|u-v\|t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\lambda(s)ds \\ & +\frac{\xi^{\alpha-1}\beta\|u-v\|t^{\alpha-1}}{(1-\beta\xi^{\alpha-1})\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\lambda(s)ds. \end{aligned}$$

Let $L=\int_0^1(1-s)^{\alpha-1}\lambda(s)ds$, we have

$$|(Tu-Tv)(t)|<\frac{L\|u-v\|t^{\alpha-1}}{(1-\beta\xi^{\alpha-1})\Gamma(\alpha)}.$$

Consequently,

$$\begin{aligned} & |(T^2u-T^2v)(t)| \\ \le & \int_0^1 G(t,s)|f(s,(Tu)(s))-f(s,(Tv)(s))|ds \\ & +\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi,s) \\ & |f(s,(Tu)(s))-f(s,(Tv)(s))|ds \\ < & \frac{L\|u-v\|}{\Gamma^2(\alpha)(1-\beta\xi^{\alpha-1})}\left(\int_0^t[t^{\alpha-1}(1-s)^{\alpha-1}\right. \\ & -(t-s)^{\alpha-1}]s^{\alpha-1}\lambda(s)ds \\ & \left.+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)s^{\alpha-1}ds\right) \\ & +\frac{L\beta t^{\alpha-1}\|u-v\|}{\Gamma^2(\alpha)(1-\beta\xi^{\alpha-1})^2}\left(\int_0^\xi[\xi^{\alpha-1}(1-s)^{\alpha-1}\right. \\ & -(\xi-s)^{\alpha-1}]s^{\alpha-1}\lambda(s)ds \\ & \left.+\int_\xi^1 \xi^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)s^{\alpha-1}ds\right) \\ < & \frac{L\|u-v\|t^{\alpha-1}}{\Gamma^2(\alpha)(1-\beta\xi^{\alpha-1})}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds \\ & +\frac{L\xi^{\alpha-1}\beta\|u-v\|t^{\alpha-1}}{(1-\beta\xi^{\alpha-1})^2\Gamma^2(\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds \\ = & \frac{LM\|u-v\|t^{\alpha-1}}{\Gamma^2(\alpha)(1-\beta\xi^{\alpha-1})} \\ & +\frac{LM\xi^{\alpha-1}\beta\|u-v\|t^{\alpha-1}}{(1-\beta\xi^{\alpha-1})^2\Gamma^2(\alpha)} \\ = & \frac{LM\|u-v\|t^{\alpha-1}}{\Gamma^2(\alpha)(1-\beta\xi^{\alpha-1})^2}. \end{aligned}$$

where $M=\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds$. By induction, we have

$$|(T^nu-T^nv)(t)|\le\frac{LM^{n-1}\|u-v\|t^{\alpha-1}}{[\Gamma(\alpha)(1-\beta\xi^{\alpha-1})]^n}.$$

Taking into account that $\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\lambda(s)ds<\Gamma(\alpha)(1-\beta\xi^{\alpha-1})$, we have

$$\frac{LM^{n-1}}{[\Gamma(\alpha)(1-\beta\xi^{\alpha-1})]^n}<\frac{1}{2}$$

for sufficiently large n, and therefore,

$$\|(T^n u - T^n v)\| \leq \frac{1}{2}\|u - v\|,$$

which gives the proof.

In the following, we will give the existence of solution of problem (1), (2).

We set

$$\lambda = \overline{\lim_{\|u\|\to\infty}} \left(\max_{t\in[0,1]} \frac{\|f(t,u)\|}{\|u\|} \right), \tag{10}$$

then $0 < \lambda \leq +\infty$.

$$\eta = \frac{\lambda(1-\beta\xi^{\alpha})}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}, \tag{11}$$

$$\eta' = \frac{\lambda'(1-\beta\xi^{\alpha})}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}. \tag{12}$$

Theorem 13 *Suppose*

$$\eta < 1, \tag{13}$$

then problem (1), (2) has one positive solution.

Proof: According to Lemma 11, we just need to verify that the operator T has a fixed point in E. From 13, choose $\lambda' > \lambda$, such that

$$\eta' < 1. \tag{14}$$

In view of the definition of λ, there exists $N > 0$ such that

$$\|f(t,u)\| < \lambda'\|u\|,\ \forall t \in [0,1],\ \|u\| \geq N. \tag{15}$$

So we have

$$\|f(t,u)\| \leq \lambda'\|u\| + R,\ \forall t \in [0,1],\ u \in [0,+\infty), \tag{16}$$

where

$$R = \max_{t\in[0,1],\ \|u\|\leq N} \|f(t,u)\| < +\infty.$$

From (7), (13)-(16), we have

$$\begin{aligned}
&\|(Tu)(t)\| \\
\leq\ & \max_{t\in[0,1]} \Big| \int_0^1 G(t,s)f(s,u(s))ds \\
&+ \frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s)f(s,u(s))ds \Big| \\
\leq\ & \max_{t\in[0,1]} \int_0^1 |G(t,s)||f(s,u(s))|ds \\
&+ \frac{\beta}{1-\beta\xi^{\alpha-1}} \int_0^1 |G(\xi,s)||f(s,u(s))|ds \\
\leq\ & (\lambda'\|u\| + R)\Big(\int_0^1 |G(t,s)|ds \\
&+ \frac{\beta}{1-\beta\xi^{\alpha-1}} \int_0^1 |G(\xi,s)|ds \Big) \\
\leq\ & (\lambda'\|u\| + R)\frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1}}{\alpha} + \frac{\beta(\xi^{\alpha-1}-\xi^{\alpha})}{\alpha(1-\beta\xi^{\alpha-1})} \right) \\
\leq\ & (\lambda'\|u\| + R)\frac{1}{\Gamma(\alpha)} \left(\frac{1}{\alpha} + \frac{\beta(\xi^{\alpha-1}-\xi^{\alpha})}{\alpha(1-\beta\xi^{\alpha-1})} \right) \\
=\ & (\lambda'\|u\| + R)\frac{1-\beta\xi^{\alpha}}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})} \\
=\ & \lambda' \frac{1-\beta\xi^{\alpha}}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}\|u\| + R^{(1)} \\
=\ & \eta'\|u\| + R^{(1)},
\end{aligned} \tag{17}$$

in which η' is given by (12), $R^{(1)}$ is a positive constant. (17) imply that

$$\|Tu\| \leq \eta'\|u\| + R^{(1)}, \quad \forall u \in E,$$

where $\eta' < 1$. So, we can choose $r > 0$ such that $T(P_r) \subseteq P_r$, where $P_r = \{u \in P | \|u\| \leq r\}$. By using Schauder fixed point theorem, we have that the operator T has a fixed point in P_r. Therefore, problem (1), (2) has one positive solution.

Remark 14 *If when* $\|u\| \to \infty$,

$$\frac{\|f(t,u)\|}{\|u\|} \to 0,\ \ \forall t \in [0,1], \tag{18}$$

then condition (13) satisfied. $(\eta = 0)$.

4 Existence of three positive solutions of the problem (1), (2)

We define the nonnegative continuous concave functional

$$\alpha(u) = \min_{t\in[\xi,1]} |u(t)|, \quad \forall u \in P,$$

$$N_1 = \frac{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}{1-\beta\xi^{\alpha}},$$

$$N_2 = \frac{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}{\beta\xi^{\alpha-1}(\xi^{\alpha-1}-\xi^{\alpha})}.$$

Theorem 15 *Assume that* $(H_1), (H_2)$ *hold and there exist constants with* $0 < a < b < c = d$ *such that the following conditions hold*

(A_1) $f(t,u) < N_1 a$, *for* $(t,u) \in [0,1]\times[0,a]$,

(A_2) $f(t,u) \geq N_2 b$, *for* $(t,u) \in [\xi,1]\times[b,c]$,

(A_3) $f(t,u) \leq N_1 c$, *for* $(t,u) \in [0,1]\times[0,c]$.

Then problem (1), (2) has at least three positive solutions u_1, u_2 and u_3 with

$$\max_{t\in[0,1]} |u_1(t)| < a,$$
$$b < \min_{t\in[\xi,1]} |u_2(t)| < \max_{t\in[0,1]} |u_2(t)| \le c,$$
$$a < \max_{t\in[0,1]} |u_3(t)| \le c,$$
$$\min_{t\in[\xi,1]} |u_3(t)| < b.$$

Proof: We first show that (C_2) of Theorem 8 holds.

If $u \in \overline{P}_c$, then $\|u\| \le c$. By (6), (7) and (A_3), we get

$$\begin{aligned}
&\|(Tu)(t)\| \\
\le\ & \max_{t\in[0,1]} \Big| \int_0^1 G(t,s)f(s,u(s))ds \\
&+\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s)f(s,u(s))ds\Big| \\
\le\ & \max_{t\in[0,1]} \int_0^1 |G(t,s)||f(s,u(s))|ds \\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}} \int_0^1 |G(\xi,s)||f(s,u(s))|ds \\
\le\ & N_1 c\Big(\max_{t\in[0,1]} \int_0^1 |G(t,s)|ds \\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}} \int_0^1 |G(\xi,s)|ds\Big) \\
\le\ & N_1 c\frac{1}{\Gamma(\alpha)}\Big(\max_{t\in[0,1]} \frac{t^{\alpha-1}-t^\alpha}{\alpha} \\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}} \frac{\xi^{\alpha-1}-\xi^\alpha}{\alpha}\Big) \\
\le\ & N_1 c\frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha}+\frac{\beta}{1-\beta\xi^{\alpha-1}}\frac{\xi^{\alpha-1}-\xi^\alpha}{\alpha}\right) \\
=\ & N_1 c\frac{1-\beta\xi^\alpha}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})} \\
\le\ & c,
\end{aligned}$$

which implies that $\|Tu\| \le c,\ u \in \overline{P}_c$. Hence, $T : \overline{P}_c \to \overline{P}_c$. In view of Lemma 10, $T : \overline{P}_c \to \overline{P}_c$ is completely continuous.

In the same way, we can show that if (A_1) holds, then $T\overline{P}_a \subset P_a$. Hence condition (C_2) of Theorem 8 is satisfied.

Next we show that (C_1) of Theorem 8 holds. Choose $u_0(t) = \frac{b+c}{2},\ t \in [0,1]$. It is easy to see that $u_0 \in P,\ \|u_0\| = \frac{b+c}{2} \le c,\ \alpha(u_0) = \frac{b+c}{2} > b$. That is $u_0 \in \{u \in P(\alpha,b,d) :\ \alpha(u) > b\} \ne \emptyset$.

Moreover, if $u \in P(\alpha,b,d)$, we have $b \le u(t) \le c$, for $t \in [\xi,1]$. By (A_2) and (7), we have

$$\begin{aligned}
\alpha(Tu) =\ & \min_{t\in[\xi,1]} |(Tu)(t)| \\
\ge\ & \min_{t\in[\xi,1]} \int_\xi^1 G(t,s)(f(s,u(s))ds \\
&+\frac{\beta\xi^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s)f(s,u(s))ds \\
\ge\ & \frac{\beta\xi^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s)f(s,u(s))ds \\
\ge\ & N_2 b\frac{\beta\xi^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s)ds \\
=\ & N_2 b\frac{\beta\xi^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \frac{\xi^{\alpha-1}-\xi^\alpha}{\alpha\Gamma(\alpha)} = b.
\end{aligned}$$

Hence condition (C_1) of Theorem 8 is satisfied. By Remark 9, condition (C_3) of Theorem 8 is satisfied. To sum up,all the hypotheses of Theorem 8 are satisfied. By Theorem 8, problem (1), (2) has at least three positive solutions u_1, u_2 and u_3 with $\max_{t\in[0,1]} |u_1(t)| < a,\ b < \min_{t\in[\xi,1]} |u_2(t)| < \max_{t\in[0,1]} |u_2(t)| \le c,\ a < \max_{t\in[0,1]} |u_3(t)| \le c,\ \min_{t\in[\xi,1]} |u_3(t)| < b$. The proof is completed. □

5 Existence and uniqueness of successive iteration positive solutions of the problem (1), (2)

Theorem 16 *Assume that $(H_1), (H_2)$ hold. If there exists a positive number Λ such that the following conditions hold*
(B_1) *For each* $t \in [0,1]$, $f(t,\cdot)$ *is nondecreasing*;
(B_2) $f(t,\Lambda) \le N_1\Lambda$, *for all* $t \in [0,1]$;
(B_3) $f(t,0)$ *is not identically zero on any compact subinterval of*$[0,1]$.
Then problem (1), (2) has at least one iteration solution $u^* \in \overline{P}_\Lambda$ with $\|u^*\| \le \Lambda$, $\lim_{n\to\infty} T^n\omega_0 = u^*$ or $\lim_{n\to\infty} T^n v_0 = u^*$, where $\omega_0(t) = \Lambda$, $v_0 = 0$,$t \in [0,1]$.

Proof: We now show that $T :\ \overline{P}_\Lambda \to \overline{P}_\Lambda$. If $u \in \overline{P}_\Lambda$, then $\|u\| \le \Lambda$, we have

$$0 \le \max_{t\in[0,1]} u(t) = \|u\| \le \Lambda.$$

By $(B_1), (B_2)$ and (7), we have

$$\begin{aligned}
&\|(Tu)(t)\| \\
\le\ & \max_{t\in[0,1]} \Big| \int_0^1 G(t,s)f(s,u(s))ds \\
&+\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}} \int_0^1 G(\xi,s)f(s,u(s))ds\Big| \\
\le\ & \max_{t\in[0,1]} \int_0^1 |G(t,s)||f(s,u(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta}{1-\beta\xi^{\alpha-1}}\int_0^1 |G(\xi,s)||f(s,u(s))|ds\\
\le\ & N_1\Lambda\Big(\max_{t\in[0,1]}\int_0^1 |G(t,s)|ds\\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}}\int_0^1 |G(\xi,s)|ds\Big)\\
\le\ & N_1\Lambda\frac{1}{\Gamma(\alpha)}\Big(\max_{t\in[0,1]}\frac{t^{\alpha-1}-t^{\alpha}}{\alpha}\\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}}\frac{\xi^{\alpha-1}-\xi^{\alpha}}{\alpha}\Big)\\
\le\ & N_1\Lambda\frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha}+\frac{\beta}{1-\beta\xi^{\alpha-1}}\frac{\xi^{\alpha-1}-\xi^{\alpha}}{\alpha}\right)\\
=\ & N_1\Lambda\frac{1-\beta\xi^{\alpha}}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}\\
\le\ & \Lambda.
\end{aligned}
$$

Thus we assert that $T:\ \overline{P}_\Lambda\to\overline{P}_\Lambda$.

Let $\omega_n=T^n\omega_0,\ \upsilon_n=T^n\upsilon_0$, then $\omega_n,\ \upsilon_n\in\overline{P}_\Lambda,\ n=1,2,\cdots$. Since T is completely continuous, we assert that $\{\omega_n\}_{n=1}^{\infty},\ \{\upsilon_n\}_{n=1}^{\infty}$ are sequentially compact sets. By (B_1), (B_2) and (7), we have

$$
\begin{aligned}
&\omega_1(t)=(T\omega_0)(t)=\int_0^1 G(t,s)f(s,\Lambda)ds\\
&+\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi,s)f(s,\Lambda)ds\\
\le\ & N_1\Lambda\Big(\int_0^1 G(t,s)ds\\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi,s)ds\Big)\\
\le\ & N_1\Lambda\frac{1}{\Gamma(\alpha)}\Big(\frac{t^{\alpha-1}}{\alpha}\\
&+\frac{\beta}{1-\beta\xi^{\alpha-1}}\frac{\xi^{\alpha-1}-\xi^{\alpha}}{\alpha}\Big)\\
\le\ & N_1\Lambda\frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha}+\frac{\beta}{1-\beta\xi^{\alpha-1}}\frac{\xi^{\alpha-1}-\xi^{\alpha}}{\alpha}\right)\\
=\ & N_1\Lambda\frac{1-\beta\xi^{\alpha}}{\Gamma(\alpha)\alpha(1-\beta\xi^{\alpha-1})}\\
=\ & \Lambda=\omega_0(t).
\end{aligned}
$$

Similarly,

$$\omega_2(t)=(T\omega_1)(t)\le(T\omega_0)(t)=\omega_1(t),\quad t\in[0,1].$$

By induction, then

$$
\begin{aligned}
&\omega_{n+1}(t)=(T\omega_n)(t)\le(T\omega_{n-1})(t)\\
&=\omega_n(t),\ t\in[0,1],\ n=1,2,\cdots.
\end{aligned}
$$

Hence, there exists $u^*\in\overline{P}_\Lambda$, such that $u^*=\lim_{n\to\infty}\omega_n$. Applying the continuity of T and $\omega_{n+1}=T\omega_n$, we get $u^*=Tu^*$. By (B_3), we get

$$
\begin{aligned}
&\upsilon_1(t)=(T\upsilon_0)(t)\\
=\ & \int_0^1 G(t,s)f(s,\upsilon_0(s))ds\\
&+\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi,s)f(s,\upsilon_0(s))ds\\
\ge\ & \int_0^1 G(t,s)f(s,0)ds\\
&+\frac{\beta t^{\alpha-1}}{1-\beta\xi^{\alpha-1}}\int_0^1 G(\xi,s)f(s,0)ds\\
>\ & 0=\upsilon_0(t).
\end{aligned}
$$

Similarly,

$$\upsilon_2(t)=(T\upsilon_1)(t)\ge(T\upsilon_0)(t)=\upsilon_1(t),\quad t\in[0,1].$$

By induction, then

$$
\begin{aligned}
&\upsilon_{n+1}(t)=(T\upsilon_n)(t)\ge(T\upsilon_{n-1})(t)\\
=\ & \upsilon_n(t),\ t\in[0,1],\ n=1,2,\cdots.
\end{aligned}
$$

Hence, there exists $u^*\in\overline{P}_\Lambda$, such that $u^*=\lim_{n\to\infty}\upsilon_n$. Applying the continuity of T and $\upsilon_{n+1}=T\upsilon_n$, we get $u^*=Tu^*$. The proof is completed. □

6 Conclusion

This paper is motivated from some recent papers treating the boundary value problems for three-point fractional differential equations. We first give some notations, recall some concepts and preparation results. Second, we establish a general framework to find the existence and uniqueness solution of the problem (1), (2). Third, some sufficient conditions for the existence of three positive solutions of the problem (1), (2) are established by applying fixed point methods. In the last, the existence and uniqueness of successive iteration positive solutions of the problem (1), (2) are obtained. To the best of our knowledge, no work has been done to get positive solution of the three-point fractional boundary value problem for $2<\alpha\le 3$. The aim of this paper is to fill the gap in the relevant literatures. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

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