The Weak Solutions to Doubly Nonlinear Diffusion Equation with Convection Term

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Abstract: Consider the following convection diffusion equation

$$u_t = div(|\nabla u^m|^{p-2}\nabla u^m) + \sum_{i=1}^{N} \frac{\partial b_i(u^m, x, t)}{\partial x_i}.$$

Supposed that $0 < m < 1$, $p > 1 + \frac{1}{m}$, using Moser iteration technique, we get the local bounded properties of the solution of the regularized problem. By the compactness theorem, the existence of the weak solution of the convection diffusion equation itself is obtained.

Key–Words: Doubly nonlinear diffusion equation, weak solution, convection term

1 Introduction

The objective of the paper is to study the nonnegative weak solution of the doubly nonlinear diffusion equations with a convection term as follows

$$u_t = div(|\nabla u^m|^{p-2}\nabla u^m) + \sum_{i=1}^{N} \frac{\partial b_i(u^m, x, t)}{\partial x_i}, \tag{1}$$

where ∇ is the spatial gradient operator, $(x,t) \in S = \Omega \times (0,\infty)$, $\Omega \subset \mathbb{R}^N$ is a bounded open domain. The initial boundary value conditions are as usual.

$$u(x,0) = u_0(x), \;\; x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \;\; (x,t) \in \partial\Omega \times (0,\infty), \tag{3}$$

where, $p > 1$, $m > 0$, $N \geq 1$, and we assume that

$$0 \leq u_0(x) \in L^{q-1+\frac{1}{m}}(\Omega), \tag{4}$$

where $3 > q > 1$. According to the different exponents of m, p, we have the following classical terminologies about the equation (1).

(i) The case $p = 2, m = 1$, is the ordinary semilinear diffusion equation.

(ii) The case $p = 2, m \neq 1$, is the porous media equation, it is degenerate at $u = 0$ for $m > 1$ and singular at $u = 0$ for $0 < m < 1$.

(iii) The case $p \neq 2, m = 1$, is the p-diffusion equation, it is degenerate at $\nabla u = 0$ for $2 < p < \infty$ and singular at $\nabla u = 0$ for $1 < p < 2$.

(iv) The case $p \neq 2, m \neq 1$, is the doubly nonlinear diffusion equation, singularity and degeneracy at $u = 0$ and $\nabla u = 0$, respectively, occur in arbitrary combinations.

(v) If $m(p-1) > 1 (= 1, < 1)$, then equation (1) is called the slow (normal, fast) diffusion equation respectively.

Equation (1) appears in a number of different physical situations [1].

For example, in the study of water infiltration through porous media, Darcy's linear relation

$$V = -K(\theta)\nabla\phi,$$

satisfactorily describes flow conditions provided the velocities are small. Here V represents the seepage velocity of water, θ is the volumetric moisture content, $K(\theta)$ is the hydraulic conductivity and ϕ is the total potential, which can be expressed as the sum of a hydrostatic potential $\psi(\theta)$ and a gravitational potential z

$$\phi = \psi(\theta) + z. \tag{5}$$

However, (5) fails to describe the flow for large velocities. To get a more accurate description of the flow in this case, several nonlinear versions of (5) have been proposed. One of these versions is

$$V^{\alpha}=-K(\theta)\nabla\phi, \tag{6}$$

where α is a positive constant, cf. [2-4] and their references. If it is assumed that infiltration takes place in a horizontal column of the medium, by the continuity equation

$$\frac{\partial\theta}{\partial t}+\frac{\partial V}{\partial x}=0,$$

(5) and (6) give the equation

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}(D(\theta)^{p}|\theta_x|^{p-1}\theta_x)$$

with $\frac{1}{p}=\alpha$ and $D(\theta)=K(\theta)\psi'(\theta)$. Choosing $D(\theta)=D_0\theta^{m-1}$ (cf. [5-6]), one obtains (1) with $b_i(s,x,t)\equiv 0$, u being the volumetric moisture content.

Another example where equation (1) appears is the one-dimensional turbulent flow of gas in a porous medium (cf. [7]), where u stands for the density, and the pressure is proportional to u^{m-1}; see also [8]. Typical values of p are 1 for laminar (non-turbulent) flow and $\frac{1}{2}$ for completely turbulent flow.

The existence of nonnegative solution of (1) without the convection term $\sum_{i=1}^{N}\frac{\partial b_i(u^m,x,t)}{\partial x_i}$, defined in some weak sense, is well established (see [9], [10] etc.). In 2012, Matas-Merker [11] have supplemented an elementary proof of the existence of weak solutions of (1) by Faedo-Galerkin method, in which some restrictions on the convection term are given, and the initial value $u_0(x)\in L^{m'}(\Omega)$, $m>1$, $m'=\frac{m}{m-1}$ is the conjugate number of m. Recently, the second author of the paper also have studied the existence of nonnegative solution of (1) with the convection term as $\sum_{i=1}^{N}\frac{\partial b_i(u^m)}{\partial x_i}$ in [12]. Other related results had been deeply studied in the tremendous amount of references, for examples, one can refer to [13-19] etc. By the way, the second author has studied the relevant problem for a long time, see [10],[12] and [20-22] please.

2 Some Lemmas and main result

In what follows, we assume that the convection term $\sum_{i=1}^{N}\frac{\partial b_i(u^m,x,t)}{\partial x_i}$ satisfies

(A): For any given $i\in\{1,2,\cdots,N\}$, $b_i(s,x,t)$ is a C^1 function, and there exist constants c such that

$$|b_i(s,x,t)|\le c|s|^{\frac{1}{m}}, \tag{7}$$

$$|b_i'(s,x,t)|=|\frac{\partial b_i(s,x,t)}{\partial s}|\le c|s|^{\frac{1}{m}-1}, \tag{8}$$

and

$$|b_{ix_i}(s,x,t)|=|\frac{\partial b_i(s,x,t)}{\partial x_i}|\le c|s|^{\frac{1}{m}}. \tag{9}$$

As usual, the constants c here and in what follows may be different from one to another. If one compares the condition (A) with the corresponding condition which is posed on the convection term as $\sum_{i=1}^{N}\frac{\partial b_i(u^m)}{\partial x_i}$ in [12], he will find that the exponent $\frac{1}{m}$ in (7)(or $\frac{1}{m}-1$ in (8)) is replaced by a general constant α, which means that $b_i(s)$ in [12] has more general choices compared to $b_i(s,x,t)$ in this paper. In other words, there are essential differences between $\sum_{i=1}^{N}\frac{\partial b_i(u^m)}{\partial x_i}$ and $\sum_{i=1}^{N}\frac{\partial b_i(u^m,x,t)}{\partial x_i}$. By the way, the condition (9) is naturally neglected in [12], the implying condition $m<1$ in (8) is also not needed in [12].

Now we quote the following definition.

Definition 1 *A nonnegative function $u(x,t)$ is called a weak solution of (1)-(3) if u satisfies*

(i)

$$u\in L^{\infty}_{loc}(0,\infty;L^{\infty}(\Omega)), \tag{10}$$

$$u_t\in L^{2}_{loc}(0,\infty;L^{2}(\Omega)), \tag{11}$$

$$u^m\in L^{\infty}_{loc}(0,\infty;W_0^{1,p}(\Omega)), \tag{12}$$

(ii)

$$\int\int_S[u\varphi_t-|\nabla u^m|^{p-2}\nabla u^m\cdot\nabla\varphi$$

$$-\sum_{i=1}^{N}b_i(u^m,x,t)\cdot\varphi_{x_i}]dxdt=0,\ \forall\varphi\in C_0^1(S); \tag{13}$$

(iii)

$$\lim_{t\to 0}\int_{\Omega}|u(x,t)-u_0(x)|dx=0. \tag{14}$$

We need some important lemmas in order to get our results.

Lemma 2 *[23] (Gagliardo-Nirenberg) If $1\le l<N$, $1+\beta\le q$, $1\le r\le q\le(1+\beta)Nl/(N-l)$, suppose that $u^{1+\beta}\in W^{1,l}(\Omega)$, then*

$$\|u\|_q\le c^{1/(1+\beta)}\|u\|_r^{1-\theta}\|u^{1+\beta}\|_{1,l}^{\theta/(1+\beta)},$$

where $\theta=(\beta+1)(r^{-1}-q^{-1})/(N^{-1}-l^{-1}+(\beta+1)r^{-1})$.

Lemma 3 *[24] Let $y(t)$ be a nonnegative function on $(0, T]$. If it satisfies*

$$y'(t)+At^{\lambda\theta-1}y^{1+\theta}(t) \le Bt^{-k}y(t)+Ct^{-\delta}, 0 < t \le T,$$

where $A, \theta > 0$, $\lambda\theta \ge 1$, $B, C \ge 0, k \le 1$, then

$$y(t) \le A^{-\frac{1}{\theta}}(2\lambda + 2BT^{1-k})^{\frac{1}{\theta}}t^{-\lambda}$$

$$+2C(\lambda + BT^{1-k})^{-1}t^{1-\delta}, 0 < t \le T.$$

Lemma 4 *Let $y(\tau)$ be a nonnegative function on $[1, \infty]$. If it satisfies*

$$y'(\tau) + A\tau^{\mu}y^{1+\theta}(\tau) \le B\tau^{-k}, \tau \ge 1,$$

where $A, B, \mu, k \ge 0$, then there are constant $C > 0$ and $\gamma = \min\{(1 + \mu)/\theta, (\mu + k)/(1 + \theta)\}$ such that

$$y(\tau) \le C\tau^{-\gamma}, \tau \ge 1.$$

Lemma 5 *Suppose $L_1 \ge 1$, $r, R, M > 0$, $\lambda_1 > 0$. For $n = 2, 3, \cdots$, let*

$$L_n = RL_{n-1} - M,$$

$$\theta_n = NR(1 - L_{n-1}L_n^{-1})(N(R - 1) + r)^{-1},$$

$$\beta_n = (L_n + M))\theta_n^{-1} - L_n,$$

$$\lambda_n = (1 + \lambda_{n-1}(\beta_n - M))\beta_n^{-1}.$$

Then

$$\lim_{n\to\infty} \lambda_n = \frac{L_1\lambda_1 r + N}{l_1 + MN}.$$

The proof of lemma 4-lemma 5 is easy, one can refer to [25].

In what follows, we assume that

$$p > 1 + \frac{1}{m}, 0 < m < 1,$$

which means that equation (1) is a doubly degenerate parabolic equation. We will consider the regularized problem, use Moser iteration technique, prove the local bounded properties of its solution and obtain the local bounded properties of the L^p-norm of the gradient. By the compactness theorem, we can prove the existence of the solution of the convection diffusion equation itself. At last, the following theorem is obtained.

Theorem 6 *If (A) and $p > 1 + \frac{1}{m}, 0 < m < 1$, $0 \le u_0(x)$ and*

$$u_0(x) \in L^{q-1+\frac{1}{m}}(\Omega), 3 > q > 1,$$

then (1)-(3) has a unique weak solution, which satisfies

$$u^m \in L^{\infty}_{\text{loc}}(0, \infty; L^{q+1-\frac{1}{m}}(\Omega)) \bigcap L^{\infty}_{loc}(0, \infty; W_0^{1,p}(\Omega)), \quad (15)$$

and

$$\|u^m(t)\|_{\infty} \le c(1 + t^{-\lambda})(1 + t)^{-1/(p-1-\frac{1}{m})}, t > 0, \quad (16)$$

where $\lambda = N(pq + (p - 1 - \frac{1}{m})N)^{-1}$. Moreover,

$$\|\nabla u^m\|_p \le c(1 + t^{-\mu})(1 + t)^{-\sigma}, t > 0, \quad (17)$$

where $\mu = 1 + \frac{m-1}{m(p-1)-1}$, $\sigma = \frac{p[m(2\alpha+1)-1]+m}{[m(p-1)-1](p-1)}$.

By the way, we would like to point again that the condition (A) implies that $m < 1$, and so that $p > 2$ by $p > 1 + \frac{1}{m}$. However, in [12], $p > 2$ is an independent condition to assure that (17) is true.

3 The L^{∞} estimation of the solution

Consider the regularized problem

$$u_t = div((|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)$$

$$+\sum_{i=1}^{N}\frac{\partial b_i(u^m, x, t)}{\partial x_i}, \quad (18)$$

$$u(x, 0) = u_{0k}(x) + s, x \in \Omega, \quad (19)$$

$$u(x, t) = s, x \in \partial\Omega, t \ge 0, \quad (20)$$

where $0 \le u_{0k}(x)$ is a suitable smooth function such that

$$\lim_{k\to\infty} \|u_{0k}\|_{q-1+\frac{1}{m}} = \|u_0\|_{q-1+\frac{1}{m}}.$$

By [14], we know that (18)-(20) has a unique nonnegative classical solution u_{ks}. Let $s \to 0$. By a similar way as [9], we are able to prove that

$$u_{ks} \to u_k, \ in \ C(S),$$

$$\nabla u_{ks}^m \rightharpoonup \nabla u_k^m, \ in \ L^p(S),$$

$$u_{kst} \rightharpoonup \nabla u_{kt}, \ in \ L^2(S),$$

$$|\nabla u_{ks}^m|^{p-2}\nabla u_{ksx_i}^m \overset{*}{\rightharpoonup}$$

$$|\nabla u_k^m|^{p-2}\nabla u_{kx_i}^m, weakly\ star\ in\ L^{\infty}_{loc}(0, \infty; L^{\frac{p}{p-1}}(\Omega)),$$

and u_k is the solution of the following problem

$$u_t = div((|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)$$

$$+\sum_{i=1}^{N}\frac{\partial b_i(u^m,x,t)}{\partial x_i}, \quad (21)$$

$$u(x,0)=u_{0k}(x), x\in\Omega, \quad (22)$$

$$u(x,t)=0, x\in\partial\Omega, t\geq 0. \quad (23)$$

In what follows, in the proof of the related lemmas, we only denote u_k as u for simplicity.

Lemma 7 *If u_k is the solution of (21)-(23), then*

$$u_k^m\in L^\infty_{loc}(0,\infty;L^{q-1+\frac{1}{m}}(\Omega))$$

and

$$\|u_k^m\|_{q-1+\frac{1}{m}}\leq c(1+t)^{-\frac{1}{p-1-\frac{1}{m}}}, t\geq 0. \quad (24)$$

Proof: Let $A_n=(q-2)n^{3-q}, B_n=(3-q)n^{2-q}$, and

$$f_n(s)=\begin{cases} s^{q-1}, & \text{if } s\geq\frac{1}{n},\\ A_ns^2+B_ns, & \text{if } 0\leq s<\frac{1}{n}.\end{cases}$$

The assumption $3>q>1$ assures that $f_n\geq 0$ and $f'_n(s)\geq 0$ when $s\geq 0$. Suppose that $n>k$, multiply (21) by $f_n(u^m)$ and integral over Ω. Then

$$\int_\Omega f_n(u^m)div(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)dx$$

$$=-\int_\Omega(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2f'_n(u^m)dx$$

$$\leq-\int_\Omega|\nabla u^m|^pf'_n(u^m)dx$$

$$=-\int_\Omega|\nabla\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p\,dx. \quad (25)$$

Using the second integral mean value theorem, by (9), we have

$$\int_\Omega f_n(u^m)\sum_{i=1}^{N}\frac{\partial b_i(u^m,x,t)}{\partial x_i}dx$$

$$=-\sum_{i=1}^{N}\int_\Omega b_i(u^m,x,t)f'_n(u^m)\frac{\partial u^m}{\partial x_i}dx$$

$$=-\sum_{i=1}^{N}\int_\Omega\frac{\partial}{\partial x_i}\int_0^{u^m}b_i(s)f'_n(s)dsdx$$

$$+\sum_{i=1}^{N}\int_\Omega\int_0^{u^m}b_{ix_i}(s,x,t)f'_n(s)dsdx$$

$$=\sum_{i=1}^{N}\int_\Omega b_{ix_i}(\xi,x,t)\int_0^{u^m}f'_n(s)dsdx$$

$$\leq c\int_\Omega u^{m(q-1+\frac{1}{m})}dx, \quad (26)$$

By(25), (26), we have

$$\int_\Omega f_n(u^m)u_tdx+\int_\Omega|\nabla\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p\,dx$$

$$\leq c\int_\Omega u^{m(q-1+\frac{1}{m})}dx,$$

by Poincare inequality, we have

$$\int_\Omega f_n(u^m)u_tdx+c\int_\Omega|\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p\,dx$$

$$\leq c\int_\Omega\int_\Omega u^{m(q-1+\frac{1}{m})}dx. \quad (27)$$

Let

$$\Omega_{1t}=\Omega\bigcap\{x:|u^m|<\frac{1}{n}\},\Omega_{2t}=\Omega\bigcap\{x:|u^m|\geq\frac{1}{n}\}.$$

Then

$$\int_\Omega|\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p\,dx$$

$$=\int_{\Omega_{1t}\bigcup\Omega_{2t}}|\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p\,dx. \quad (28)$$

On Ω_{1t},

$$|\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p\leq|\int_0^{u^m}|2A_ns+B_n|^{\frac{1}{p}}\,ds|^p$$

$$\leq(2|q-2|+|3-q|)n^{1-q}. \quad (29)$$

On Ω_{2t},

$$|\int_0^{u^m}(f'_n(s))^{\frac{1}{p}}ds|^p=\frac{(q-1)p^p}{(p+q-2)^p}u^{m(p+q-2)}. \quad (30)$$

In addition,

$$\lim_{n\to\infty}\int_\Omega f_n(u^m)u_tdx$$

$$=\frac{1}{m(q-1)+1}\frac{d}{dt}\int_\Omega u^{m(q-1)+1}dx. \quad (31)$$

From (27)-(31), let $n\to\infty$. Then

$$\frac{d}{dt}\int_\Omega u^{m(q-1)+1}dx+c\int_\Omega u^{m[q-1+\frac{1}{m}+p-1-\frac{1}{m}]}dx$$

$$\leq c\int_\Omega u^{m(q-1+\frac{1}{m})}dx. \quad (32)$$

By Jessen inequality, from (32) we get

$$\frac{d}{dt}\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}} + c\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}+p-1-\frac{1}{m}}$$

$$\leq c\int_\Omega u^{m(q-1+\frac{1}{m})}dx,$$

then by Lemma 3, we have

$$\|u^m\|_{q+1-\frac{1}{m}} \leq c(1+t)^{-\frac{1}{p-1-\frac{1}{m}}}.$$

Lemma 8 *If u_k is the solution of (21)-(23), then*

$$\|u_k^m\|_\infty \leq ct^{-\lambda},\ 0<t\leq 1, \tag{33}$$

$$\|u_k^m\|_\infty \leq c(1+t)^{-\frac{1}{p-1-\frac{1}{m}}}, t\geq 1, \tag{34}$$

where $\lambda = \frac{N}{(p-1-\frac{1}{m})N+q}$.

Proof: Multiply (21) by $u^{m(l-1)}$, and integral over Ω, then

$$\int_\Omega u^{m(l-1)}u_t dx$$

$$= \int_\Omega div(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u^{m(l-1)}dx$$

$$+\sum_{i=1}^N\int_\Omega \frac{\partial b_i(u^m,x,t)}{\partial x_i}u^{m(l-1)}dx$$

$$= -(l-1)\int_\Omega(|\nabla u^m|+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 u^{m(l-2)}dx$$

$$-(l-1)\sum_{i=1}^N\int_\Omega\frac{\partial}{\partial x_i}\int_0^{u^m} b_i(s,x,t)s^{(l-2)}dsdx$$

$$+(l-1)\sum_{i=1}^N\int_\Omega\int_0^{u^m} b_{ix_i}(s,x,t)s^{(l-2)}dsdx$$

$$= -(l-1)\int_\Omega(|\nabla u^m|+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 u^{m(l-2)}dx$$

$$+(l-1)\sum_{i=1}^N\int_\Omega b_{ix_i}(\xi,x,t)\int_0^{u^m} s^{(l-2)}dsdx,$$

by (9), we can deduce that

$$\frac{d}{dt}\|u^m\|_{l-1+\frac{1}{m}}^{l-1+\frac{1}{m}}$$

$$+c(l-1+\frac{1}{m})^{2-p}\int_\Omega |\nabla u^{m\frac{p+l-1+\frac{1}{m}-1-\frac{1}{m}}{p}}|^p dx$$

$$\leq \int_\Omega u^{m(l-1+\frac{1}{m})}dx.$$

Set $L = l-1+\frac{1}{m}$. Then

$$\frac{d}{dt}\|u^m\|_L^L + cL^{2-p}\int_\Omega |\nabla u^{m\frac{L+p-1-\frac{1}{m}}{p}}|^p dx \leq \|u^m\|_L^L, \tag{35}$$

where c is a constant independent of l.

Now, if we choose $L_1 = q$, and let

$$L_n = rL_{n-1} - (p-1-\frac{1}{m}),$$

$$\theta_n = rN(1-L_{n-1}L_n^{-1})(p+N(r-1))^{-1},$$

$$\mu_n = (L_n+p-1-\frac{1}{m})\theta_n^{-1} - L_n,$$

$$r > 1+(p-1-\frac{1}{m}),$$

$n = 2, 3, \cdots$. By Lemma 3, we have

$$\|u^m\|_{L_n} \leq c^{p/(L_n+p-1-\frac{1}{m})}$$

$$\cdot\|u^m\|_{L_{n-1}}^{1-\theta_n}\|\nabla u^{m(L_n+p-1-\frac{1}{m})/p}\|_p^{p\theta_n/(p-1-\frac{1}{m}+L_n)}. \tag{36}$$

If we choose $L = L_n$ in (35), by (36), we have

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n} + c^{-p/\theta_n}L_n^{2-p}\|u^m\|_{L_n}^{L_n+\mu_n}\|u^m\|_{L_{n-1}}^{p-1-\frac{1}{m}-\mu_n}$$

$$\leq \|u^m\|_{L_n}^{L_n}.\ 0<t\leq 1. \tag{37}$$

We will prove that there exist two bounded sequences $\{\xi_n\}, \{\lambda_n\}$ such that

$$\|u^m\|_{L_n} \leq \xi_n t^{-\lambda_n},\ 0<t\leq 1. \tag{38}$$

If $n=1$, by Lemma 7, $\lambda_1 = 0$,

$$\xi_1 = \sup_{t\geq 0}\|u^m(t)\|_{q-1+\frac{1}{m}},$$

then (38) is true. If (38) is true for $n-1$, from (37),

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n}$$

$$+c^{-p/\theta_n}L_n^{2-p}\|u^m\|_{L_n}^{L_n+\mu_n}\xi_{n-1}^{p-1-\frac{1}{m}-\mu_n}t^{-(p-1-\frac{1}{m}-\mu_n)\lambda_{n-1}}$$

$$\leq \|u^m\|_{L_n}^{L_n}.\ 0<t\leq 1. \tag{39}$$

we can choose

$$\lambda_n = (\lambda_{n-1}(\mu_n - p + 1 + \frac{1}{m}) + 1)\mu_n^{-1},$$

$$\xi_n = \xi_{n-1}(c^{p/\theta_n}L_n^{p-1}\lambda_n)^{1/\mu_n},\ n = 2, 3, \cdots,$$

by Lemma 3 and (39), we know (38) is also true for n.

Moreover, as $n\to\infty$, $\lambda_n\to\lambda=\frac{N}{(p-1-\frac{1}{m})N+q}$. It is easy to see that $\{\xi_n\}$ is bounded. Thus, by Lemma 5, (33) is true.

To prove (34), we set $\tau=\log(1+t), t\geq 1$, and set

$$w(\tau)=(1+t)^{\frac{1}{p-1-\frac{1}{m}}}u^m(t).$$

By (35), we have

$$\frac{d}{d\tau}\|w(\tau)\|_L^L+cL^{2-p}\|\nabla w^{\frac{L+p-1-\frac{1}{m}}{p}}\|_p^p$$

$$\leq c\|w(\tau)\|_L^L,\ \tau\geq\log 2. \tag{40}$$

By the lemma 3.1 in [26], we have

$$\sup_{t\geq 1}\|u^m(t)(1+t)^{(p-1-\frac{1}{m})^{-1}}\|_\infty=\sup_{\tau\geq\log 2}\|w(\tau)\|$$

$$\leq c\max\left\{1,\sup_{\tau\geq\log 2}\|w(\tau)\|_{q-1+\frac{1}{m}},\sup_{\tau\geq\log 2}\|w(\tau)\|_\infty\right\}$$

$$=c\max\{1,\|u^m(1)\|_\infty,$$

$$\left.\sup_{t\geq 1}\|(1+t)^{\frac{1}{p-1-\frac{1}{m}}}u^m(t)\|_{q-1+\frac{1}{m}}\right\}$$

$$<\infty,$$

which means (34) is true. □

4 The estimate of the gradient

We will get the estimate of the gradient ∇u_k.

Lemma 9 *If u_k is the solution of (21)-(23), then*

$$\|\nabla u_k^m\|_p\leq ct^{-(1+\frac{m-1}{m(p-1)-1})},\ 0<t\leq 1, \tag{41}$$

$$\|\nabla u_k^m\|_p\leq c(1+t)^{-\frac{p-m(p-1)}{(m(p-1)-1)(p-1)}}, t\geq 1. \tag{42}$$

Proof: Multiply (21) by u_t^m, and integral over Ω, then

$$m\int_\Omega u^{m-1}(u_t)^2dx$$

$$=\int_\Omega div((|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u_t^mdx$$

$$+\sum_{i=1}^N\frac{\partial b_i(u^m,x,t)}{\partial x_i}u_t^mdx. \tag{43}$$

$$\int_\Omega div((|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u_t^mdx$$

$$=-\int_\Omega(|\nabla u^m|+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m\nabla u_t^mdx$$

$$=-\frac{1}{2}\int_\Omega(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|_t^2dx,$$

$$=-\frac{1}{2}\int_\Omega\frac{d}{dt}\int_0^{|\nabla u^m|^2}(s+\frac{1}{k})^{\frac{p-2}{2}}dsdx$$

$$=-\frac{1}{2}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2). \tag{44}$$

By the assumption of (A), using Young inequality,

$$\sum_{i=1}^N|\int_\Omega\frac{\partial b_i(u^m,x,t)}{\partial x_i}u_t^mdx|$$

$$\leq\sum_{i=1}^N\int_\Omega|b_i'(u^m)||u_{x_i}^m||u_t^m|dx$$

$$+\sum_{i=1}^N\int_\Omega|b_{ix_i}(u^m,x,t)||u_t^m|dx$$

$$\leq\varepsilon\int_\Omega u^{m-1}(u_t)^2dx+c\int_\Omega|u^m|^{\frac{1}{m}-1}|\nabla u^m|^2dx$$

$$+c\int_\Omega u^2dx. \tag{45}$$

By (43)-(45), we have

$$\int_\Omega u^{m-1}(u_t)^2dx+\frac{1}{m}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2)$$

$$\leq c\int_\Omega|u^m|^{\frac{1}{m}-1}|\nabla u^m|^2dx+c\int_\Omega u^2dx. \tag{46}$$

Multiply (21) by u^m, and integral over Ω, then

$$\frac{1}{m+1}\int_\Omega\frac{d}{dt}u^{m+1}dx=\int_\Omega div(|\nabla u^m|^2$$

$$+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u^mdx+\sum_{i=1}^N\int_\Omega\frac{\partial b_i(u^m,x,t)}{\partial x_i}u^mdx$$

$$=-\int_\Omega(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2dx.$$

and

$$\Gamma_k(|\nabla u^m|^2)\leq\int_\Omega(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2dx$$

$$=-\frac{1}{m+1}\int_\Omega\frac{d}{dt}u^{m+1}dx$$

$$\leq\frac{1}{m+1}\|u^{\frac{m+1}{2}}\|_2\|u^{\frac{m-1}{2}}u_t\|_2,$$

so

$$\frac{1}{m}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2)+(m+1)^2\|u^{\frac{m+1}{2}}\|_2^{-2}\Gamma_k^2(|\nabla u^m|^2)$$

$$\le c\int_\Omega |u^m|^{\frac{1}{m}-1}|\nabla u^m|^2dx+c\int_\Omega u^2dx. \quad (47)$$

By Poincare inequality,

$$\int_\Omega u^2dx\le c\int_\Omega |u^m|^{\frac{1}{m}-1}|\nabla u^m|^2dx.$$

Set $2\gamma=\frac{1}{m}-1$, for $\forall a\in[0,2\gamma]$,

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^2dx\le \|u^m(t)\|_\infty^a$$

$$\cdot\left(\int_\Omega |u^m|^{\frac{(2\gamma-a)p}{p-2}}dx\right)^{\frac{p-2}{p}}\|\nabla u^m\|_p^2. \quad (48)$$

If $2\gamma\ge(p-2)(N+1)/N$, let $a=(2\gamma-(p-2)(1+\frac{q}{N}))^+$. By Lemma 2,

$$\left(\int_\Omega |u^m|^{\frac{(2\gamma-a)p}{p-2}}dx\right)^{\frac{p-2}{p}}$$

$$\le c\|u^m(t)\|_s^{(2\gamma-a)(1-\theta)}\|\nabla u^m\|_p^{p-2}, \quad (49)$$

where

$$\theta=(s^{-1}-(1-\frac{2}{p})(2\gamma-a)^{-1})/(N^{-1}-p^{-1}+s^{-1}),$$

$$s=(2\gamma-p+2-a)N/(p-2)$$

when $2\gamma\ge(p-2)(1+q/N)$, and

$$s=q$$

when

$$(p-2)(1+N^{-1})\le 2\gamma\le(p-2)(1+q/N).$$

By Lemma 7 and Lemma 8, from (48), we have

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^2dx\le ct^{-\lambda a}\|\nabla u^m\|_p^p$$

$$\le ct^{-\lambda a}\Gamma_k(|\nabla u^m|^2).\ 0<t\le 1. \quad (50)$$

If we choose $q=2$ in Lemma 7, we have

$$\|u^m\|_{1+\frac{1}{m}}=\left(\int_\Omega u^{m+1}dx\right)^{\frac{m}{m+1}}\le ct^{-(p-1-\frac{m}{m+1})^{-1}}$$

and

$$\|u^{\frac{m+1}{2}}\|_2^2=\int_\Omega u^{m+1}dx\le ct^{-\frac{m+1}{m(p-1)-1}}. \quad (51)$$

By (47), we have

$$\Gamma_k'(t)+ct^{\frac{m+1}{m(p-1)-1}}\Gamma_k^2(t)$$

$$\le ct^{-\lambda a}\Gamma_k(t),\ 0<t\le 1, \quad (52)$$

If $2\gamma<(p-2)(N+1)/N$ and $p-2\le 2a\le 2\gamma$,

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^2dx\le c\|\nabla u^m\|_1^{2a(1-\theta)}\|\nabla u^m\|_p^{2a\theta+2}$$

$$\le c\|\nabla u^m\|_p^p\le c\Gamma_k(|\nabla u^m|^2).\ 0<t\le 1. \quad (53)$$

If $2\gamma<(p-2)(N+1)/N$ and $p-2\ge 2a\ge 0$, when $0<t\le 1$,

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^2dx$$

$$\le c(1+\|\nabla u^m\|_p^p)\le c(1+\Gamma_k(|\nabla u^m|^2)). \quad (54)$$

(53) and (54) mean that (52) is still true when $2\gamma<(p-2)(N+1)/N$. Using Lemma 4,

$$\Gamma_k(t)\le ct^{-(1+\frac{m-1}{m(p-1)-1})},\ 0<t\le 1.$$

which means (41) is true. Now, we will prove (42). For $t\ge 1$, by (34)

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^2dx\le c\|\nabla u^m\|_p^2\|u^m(t)\|_{2\gamma p/p-2}^{2\gamma}$$

$$\le c(1+t)^{-2\gamma/(p-1-\frac{1}{m})}\|\nabla u^m\|_p^2.\ t\ge 1. \quad (55)$$

$$\Gamma_k(|\nabla u^m|^2)=\int_0^{|\nabla u^m|^2}(s^2+\frac{1}{k})^{\frac{p-2}{2}}ds$$

$$\le c\|\nabla u^m\|_p^p=c(\|\nabla u^m\|_p^2)^{\frac{p}{2}},\ t\ge 1. \quad (56)$$

$$\|u^{\frac{m+1}{2}}\|_2^2=\left(\int_\Omega u^{m+1}dx\right)^2$$

$$\le c(1+t)^{-(p-1-\frac{1}{m})^{-1}},\ t\ge 1. \quad (57)$$

by (47), using (55)-(57)

$$\Gamma_k'(t)+c(1+t)^{-(p-1-\frac{1}{m})^{-1}}\Gamma_k^2(t)$$

$$\le c(1+t)^{2\gamma/(p-1-\frac{1}{m})}(\Gamma_k(t))^{\frac{2}{p}},$$

using Young inequality,

$$\Gamma_k'(t)+c(1+t)^{-(p-1-\frac{1}{m})^{-1}}\Gamma_k^2(t)$$

$$\le c(1+t)^{\frac{-m(2\gamma p+1)}{(m(p-1)-1)(p-1)}}$$

$$=c(1+t)^{-\frac{p-m(p-1)}{(m(p-1)-1)(p-1)}}. \quad (58)$$

By Lemma 4, we know that (42) is true.

Lemma 10 *If u_k is the solution of (21)-(23), then*

$$\int_t^T \int_\Omega u_k^{m-1}(u_{kt})^2 dxds \le ct^{-(1+\frac{m-1}{m(p-1)-1})}$$

$$+ct^{-(\frac{\lambda(1-m)}{2m}+\frac{m-1}{m(p-1)-1})},\ 0 < t \le T. \quad (59)$$

Proof: From (33), (41), (47) and (50), we have

$$\int_t^T \int_\Omega u^{m-1}(u_t)^2 dxds \le \Gamma_k(t)$$

$$+c\int_t^T \int_\Omega |u^m|^{\frac{1}{m}-1}|\nabla u^m|^2 dxds$$

$$\le \Gamma_k(t) + c\int_t^T s^{-\frac{\lambda}{2}(\frac{1}{m}-1)}\Gamma_k(s)ds$$

$$\le ct^{-(1+\frac{m-1}{m(p-1)-1})} + ct^{-(\frac{\lambda(1-m)}{2m}+\frac{m-1}{m(p-1)-1})}. \quad (60)$$

The lemma is proved. □

5 The main result and its proof

Now, we are able to prove the main theorem, which has been quoted before as Theorem 6, and we restate it here as follows.

Theorem 11 *If (A) and $p > 1 + \frac{1}{m}, 0 < m < 1$, $0 \le u_0(x)$ and*

$$u_0(x) \in L^{q-1+\frac{1}{m}}(\Omega), 3 > q > 1,$$

then (1)-(3) has a unique weak solution, which satisfies

$$u^m \in L^\infty_{\mathrm{loc}}(0,\infty; L^{q+1-\frac{1}{m}}(\Omega)) \bigcap L^\infty_{loc}(0,\infty; W_0^{1,p}(\Omega)), \quad (15)$$

and

$$\|u^m(t)\|_\infty \le c(1+t^{-\lambda})(1+t)^{-1/(p-1-\frac{1}{m})}, t > 0, \quad (16)$$

where $\lambda = N(pq + (p-1-\frac{1}{m})N)^{-1}$. Moreover,

$$\|\nabla u^m\|_p \le c(1+t^{-\mu})(1+t)^{-\sigma}, t > 0, \quad (17)$$

where $\mu = 1 + \frac{m-1}{m(p-1)-1}$, $\sigma = \frac{p[m(2\alpha+1)-1]+m}{[m(p-1)-1](p-1)}$.

Proof: From Lemma 7, Lemma 8, Lemma 9 and Lemma 10, using the compactness theory (cf [13]), there is a sequence (still denoted it as $\{u_k\}$) of $\{u_k\}$ such that when $k \to \infty$, we have

$$u_k \overset{*}{\rightharpoonup} u, \mathit{weakly\ star\ in}\ L^\infty_{\mathrm{loc}}(0,\infty; L^{m(q-1)+1}(\Omega)), \quad (61)$$

$$u_{kt} \rightharpoonup u_t,\ \mathit{weakly\ in}\ L^2(0,\infty; L^2(\Omega)), \quad (62)$$

$$\nabla u_k^m \rightharpoonup \nabla u^m,\ \mathit{weakly\ in}\ L^p_{loc}(0,\infty; L^p(\Omega)) \quad (63)$$

$$|\nabla u_k^m|^{p-2}\nabla u_{kx_i}^m \overset{*}{\rightharpoonup} \chi_i,$$

$$\mathit{weakly\ star\ in}\ L^\infty_{loc}(0,\infty; L^{\frac{p}{p-1}}(\Omega)), \quad (64)$$

where $\chi = \{\chi_i : 1 \le i \le N\}$ and every χ_i is a function in $L^\infty_{loc}(0,\infty; L^{\frac{p}{p-1}}(\Omega))$. (61)-(63) are clearly true. In what follows, we only need to prove that

$$\chi = |\nabla u^m|^{p-2}\nabla u^m, \text{ in } L^\infty_{loc}(0,\infty; L^{\frac{p}{p-1}}(\Omega)). \quad (65)$$

By the definition of the weak solution of (21)-(23), let $k \to \infty$. We have

$$\int\int_S (u\varphi_t - \chi\cdot\nabla\varphi$$

$$-\sum_{i=1}^N b_i(u^m, x, t)\cdot\varphi_{x_i})dxdt = 0, \quad (66)$$

for $\forall\varphi \in C_0^\infty(S)$. So, if we are able to prove that

$$\int\int_S |\ \nabla u^m\ |^{p-2}\ \nabla u^m \cdot \nabla\varphi dxdt$$

$$= \int\int_S \chi\cdot\nabla\varphi dxdt, \quad (66)$$

then (65) and (13) are true.

Now, let's prove this fact. For any $\psi \in C_0^\infty(S)$, $0 \le \psi \le 1$; $v^m \in L^p_{loc}(0,T; W_0^{1,p}(\Omega))$, we have

$$\int\int_S \psi(|\ \nabla u_k^m\ |^{p-2}\ \nabla u_k^m$$

$$-\ |\ \nabla v^m\ |^{p-2}\ \nabla v^m)\cdot\nabla(u_k^m - v^m)dxdt \ge 0, \quad (67)$$

If we multiply with $u_k^m\psi$ on two hand sides of (21), then we have

$$\int\int_S \psi\left(|\nabla u_k^m|^2 + \frac{1}{k}\right)^{\frac{p-2}{2}} |\nabla u_k^m|^2 dxdt$$

$$= \frac{1}{m+1}\int\int_S \psi_t u_k^{m+1} dxdt$$

$$-\int\int_S u_k^m \left(|\nabla u_k^m|^2 + \frac{1}{k}\right)^{\frac{p-2}{2}} \nabla u_k^m\cdot\nabla\psi dxdt$$

$$-\sum_{i=1}^N \int\int_S b_i(u_k^m, x, t)(u_{kx_i}^m\psi + u_k^m\psi_{x_i})dxdt. \quad (68)$$

Noticing that $p > 2$, then

$$(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u_k^m|^2 \ge |\nabla u_k^m|^p,$$

$$(|\nabla u_k^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u_k^m|\leq(|\nabla u_k^m|^{p-1}+1),$$

by (67), (68), we have

$$\frac{1}{m+1}\int\int_S\psi_t u_k^{m+1}dxdt$$
$$-\int\int_S u_k^m\left(|\nabla u_k^m|^2+\frac{1}{k}\right)^{\frac{p-2}{2}}\nabla u_k^m\cdot\nabla\psi dxdt$$
$$-\sum_{i=1}^N\int\int_S b_i(u_k^m,x,t)(u_{kx_i}^m\psi+u_k^m\psi_{x_i})dxdt$$
$$+(\frac{1}{k})^{\frac{p-2}{2}}mes\Omega$$
$$-\int\int_S\psi|\nabla u_k^m|^{p-2}\nabla u_k^m\cdot\nabla v^m dxdt$$
$$-\int\int_S\psi|\nabla v^m|^{p-2}\nabla v^m\cdot\nabla(u_k^m-v^m)dxdt\geq 0. \tag{69}$$

Since

$$\left(|\nabla u_k^m|^2+\frac{1}{k}\right)^{\frac{p-2}{2}}\nabla u_k^m$$
$$=|\nabla u_k^m|^{p-2}\nabla u_k^m+\frac{p-2}{2k}\int_0^1(|\nabla u_k^m|^2+\frac{s}{k})^{\frac{p-4}{2}}ds\nabla u_k^m,$$

and

$$\lim_{k\to\infty}\frac{p-2}{k}\int\int_S\int_0^1(|\nabla u_k^m|^2+\frac{s}{k})^{\frac{p-4}{2}}ds$$
$$\cdot\nabla u_k^m\cdot\nabla\psi u_k^m dxdt=0,$$

if we let $k\to\infty$ in (69), we have

$$\frac{1}{m+1}\int\int_S\psi_t u^{m+1}dxdt$$
$$-\sum_{i=1}^N\int\int_S b_i(u^m,x,t)(u_{x_i}^m\psi+u^m\psi_{x_i})dxdt$$
$$-\int\int_S\psi\nabla\chi\cdot\nabla v^m dxdt$$
$$-\int\int_S\psi|\nabla v^m|^{p-2}\nabla v^m\cdot\nabla(u^m-v^m)dxdt\geq 0. \tag{70}$$

Now, we choose $\varphi=\psi u^m$ in (66),

$$\frac{1}{m+1}\int\int_S\psi_t u^{m+1}dxdt$$
$$-\sum_{i=1}^N\int\int_S b_i(u^m,x,t)\cdot(\psi_{x_i}u^m+\psi u_{x_i}^m)dxdt$$
$$-\int\int_S u^m\chi\cdot\nabla\psi dxdt$$
$$-\int\int_S\psi\chi\cdot\nabla u^m dxdt=0.$$

From this formula and (70), we have

$$\int\int_S\psi(\chi-|\nabla v^m|^{p-2}\nabla v^m)\cdot\nabla(u^m-v^m)dxdt$$
$$\geq 0. \tag{71}$$

Let $v^m=u^m-\lambda\varphi,\lambda\geq 0,\varphi\in C_0^\infty(S)$. Then

$$\int\int_S\psi(\chi_i-|\nabla(u^m-\lambda\varphi)|^{p-2}(u^m-\lambda\varphi)_{x_i})\varphi_{x_i}dxdt$$
$$\geq 0.$$

Let $\lambda\to 0$. We obtain

$$\int\int_S\psi(\chi_i-|\nabla u^m|^{p-2}u_{x_i}^m)\varphi_{x_i}dxdt\geq 0. \tag{72}$$

Moreover, if we choose $\lambda\leq 0$, we are able to get

$$\int\int_S\psi(\chi_i-|\nabla u^m|^{p-2}u_{x_i}^m)\varphi_{x_i}dxdt\leq 0. \tag{73}$$

Now, if we choose ψ such that $supp\varphi\subset supp\psi$, and on $supp\varphi$, $\psi=1$, then from (72)-(73), we can get (65), and so (13) is true.

Next, we are to prove (14).

For small $r>0$, denote $\Omega_r=\{x\in\Omega: dist(x,\partial\Omega)\leq r\}$. For any $\eta>0$, let

$$sgn_\eta(s)=\begin{cases}1, & \text{if } s>\eta,\\ \frac{s}{\eta}, & \text{if } |s|\leq\eta,\\ -1, & \text{if } s<-\eta.\end{cases}$$

For any given small $r>0$, large enough k,l, we declare that

$$\int_{\Omega_{2r}}|u_k(x,t)-u_l(x,t)|dx$$
$$\leq\int_{\Omega_r}|u_k(x,0)-u_l(x,0)|dx+c_r(t), \tag{74}$$

where $c_r(t)$ is independent of k,l, and $\lim_{t\to 0}c_r(t)=0$. By (21)

$$\int_0^t\int_{\Omega_r}\varphi(u_{kt}-u_{lt})dxd\tau$$
$$+\int_0^t\int_{\Omega_r}\nabla\varphi[(|\nabla u_k^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m$$

$$-(|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]dxd\tau$$

$$+\sum_{i=1}^N \int_0^t \int_{\Omega_r} [b_i(u_k^m, x, t) - b_i(u_l^m, x, t)]\nabla\varphi dxd\tau$$

$$= 0, \tag{75}$$

for $\forall\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$. Suppose that $\xi(x) \in C_0^1(\Omega_r)$ such that

$$0 \le \xi \le 1;\ \ \xi\mid_{\Omega_{2r}} = 1,$$

and choose $\varphi = \xi sgn_\eta(u_k^m - u_l^m)$ in (75), then

$$\int_0^t \int_{\Omega_r} \xi sgn_\eta(u_k^m - u_l^m)(u_{kt} - u_{lt})dxd\tau$$

$$+\int_0^t \int_{\Omega_r} [(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m$$

$$-(x|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]\nabla\xi sgn_\eta(u_k^m - u_l^m)dxd\tau$$

$$+\int_0^t \int_{\Omega_r} [(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m$$

$$-(x|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]\nabla(u_k^m - u_l^m)$$

$$\cdot\xi sgn'_\eta(u_k^m - u_l^m)dxd\tau$$

$$+\sum_{i=1}^N \int_0^t \int_{\Omega_r} [b_i(u_k^m, x, t) - b_i(u_l^m, x, t)]$$

$$\cdot\nabla\xi sgn_\eta(u_k^m - u_l^m)dxd\tau$$

$$+\sum_{i=1}^N \int_0^t \int_{\Omega_r} [b_i(u_k^m, x, t) - b_i(u_l^m, x, t)]\nabla(u_k^m - u_l^m)$$

$$\cdot\xi sgn'_\eta(u_k^m - u_l^m)dxd\tau = 0. \tag{76}$$

If we notice that the third term and the fifth term in the left hand side on (76) tend to zero when $\eta \to 0$, then we have

$$\lim_{\eta\to 0}\int_0^t \int_{\Omega_r} \xi sgn_\eta(u_k^m - u_l^m)(u_{kt} - u_{lt})dxd\tau$$

$$+\lim_{\eta\to 0}\int_0^t \int_{\Omega_r} [(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m$$

$$-(|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]\nabla\xi sgn_\eta(u_k^m - u_l^m)dxd\tau$$

$$+\sum_{i=1}^N \lim_{\eta\to 0}\int_0^t \int_{\Omega_r} [b_i(u_k^m, x, t) - b_i(u_l^m, x, t)]\nabla$$

$$\cdot\xi sgn_\eta(u_k^m - u_l^m)dxd\tau = 0. \tag{77}$$

At the same time,

$$\lim_{\eta\to 0}\int_0^t \int_{\Omega_r} \xi sgn_\eta(u_k^m - u_l^m)(u_{kt} - u_{lt})dxd\tau$$

$$= \int_0^t \int_{\Omega_r} \xi sgn(u_k^m - u_l^m)(u_{kt} - u_{lt})dxd\tau$$

$$= \int_0^t \int_{\Omega_r} \xi sgn(u_k - u_l)(u_{kt} - u_{lt})dxd\tau$$

$$\lim_{\eta\to 0}\int_0^t \int_{\Omega_r} \xi sgn_\eta(u_k - u_l)(u_{kt} - u_{lt})dxd\tau$$

$$= \lim_{\eta\to 0}\int_0^t \int_{\Omega_r} \xi(\int_0^{u_k-u_l} sgn_\eta(s)ds)_\tau dxd\tau$$

$$= \lim_{\eta\to 0}\int_0^t \int_{\Omega_r} \xi \int_0^{u_k-u_l} sgn_\eta(s)ds\mid_0^t dx$$

$$= \int_{\Omega_r} \xi|u_k - u_l|dx - \int_{\Omega_r} \xi|u_{0k} - u_{0l}|dx. \tag{78}$$

By (77)(78), we have

$$\int_{\Omega_{2r}} \xi|u_k - u_l|dx \le \int_{\Omega_r} |u_{0k} - u_{0l}|dx$$

$$+c\int_0^t \int_{\Omega_r} [(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-1}{2}} + (|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-1}{2}}]dxd\tau$$

$$+c\sum_{i=1}^N \int_0^t \int_{\Omega_r} |b_i(u_k^m, x, t) - b_i(u_l^m, x, t)|dxd\tau,$$

which means (74) is true.

Now, for any given small r, if k, l are large enough, by (74), we have

$$\int_{\Omega_{2r}} |u(x,t) - u_0(x)|dx \le \int_{\Omega_r} |u(x,t) - u_k(x,t)|dx$$

$$+\int_{\Omega_{2r}} |u_{0k}(x) - u_{0l}(x)|dx$$

$$+\int_{\Omega_{2r}} |u_l(x,t) - u_{0l}(x)|dx + \int_{\Omega_{2r}} |u_{0l}(x) - u_0(x)|dx$$

letting $t \to 0$, we get (14).

At last, we are to prove the uniqueness of the solutions. For any positive integer n, let $g_n(s)$ be an odd function and

$$g_n(s) = \begin{cases} 1, & \text{if } s > \frac{1}{n}, \\ n^2s^2e^{1-n^2s^2}, & \text{if } s \le \frac{1}{n}. \end{cases}$$

Let u_1, u_2 be two solutions of (1)-(3) with the initial value $u_{01}(x), u_{02}(x)$ respectively. Multiplying (1) with $g_n(u_1^m - u_2^m)$ and making integral on Ω, we have

$$\int_\Omega g_n(u_1^m - u_2^m)(u_{1t} - u_{2t})dx$$

$$+\int_\Omega[|\nabla|u_1^m|^{p-2}|\nabla u_1^m-|\nabla|u_2^m|^{p-2}|\nabla u_2^m]\times \nabla(u_1^m-u_2^m)g_n'dx$$
$$+\sum_{i=1}^N\int_\Omega[b_i(u_1^m,x,t)-b_i(u_2^m,x,t)]\times (u_1^m-u_2^m)_{x_i}g_n'dx$$
$$=0, \tag{79}$$

$$\lim_{n\to\infty}\int_\Omega g_n(u_1^m-u_2^m)(u_{1t}-u_{2t})dx=\frac{d}{dt}\|u_1-u_2\|_1,$$

$$\int_\Omega[|\nabla|u_1^m|^{p-2}|\nabla u_1^m-|\nabla|u_2^m|^{p-2}|\nabla u_2^m]\times \nabla(u_1^m-u_2^m)g_n'dx \geq 0,$$

$$\sum_{i=1}^N\left|\lim_{n\to\infty}\int_\Omega[b_i(u_1^m,x,t)-b_i(u_2^m,x,t)]\times (u_1^m-u_2^m)_{x_i}g_n'dx\right|$$
$$\leq 6\lim_{n\to\infty}\sum_{i=1}^N \int_{\Omega\cap\{|u_1^m-u_2^m|\leq\frac{1}{n}\}}|b_i'(\xi,x,t)||(u_1^m-u_2^m)_{x_i}|dx$$
$$=0,$$

due to $0\leq g'(s)\leq 6s^{-1}$ when $|s|\leq\frac{1}{n}$, and using the fact of that $b_i'(0,x,t)=0$.

Let $n\to\infty$ in (79). We have

$$\frac{d}{dt}\|u_1-u_2\|_1\leq 0, \tag{80}$$

which implies that

$$\int_\Omega|u_1(x,t)-u_2(x,t)|dx\leq\int_\Omega|u_{01}(x)-u_{02}(x)|dx,$$

is true for $\forall t\geq 0$.

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