Optimal investment problem for an insurer and a reinsurer under the proportional reinsurance model

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Abstract: This paper focuses on the optimal investment problem for an insurer and a reinsurer. The insurer's and reinsurer's surplus processes are both approximated by a Brownian motion with drift and the insurer can purchase proportional reinsurance from the reinsurer. In addition, both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. We first study the optimization problem of minimizing the ruin probability for the insurer. Then according to the optimal reinsurance proportion chosen by the insurer, we study two optimal investment problems for the reinsurer: the problem of maximizing the exponential utility and the problem of minimizing the ruin probability. By solving the corresponding Hamilton-Jacobi-Bellman (HJB) equations, we derive optimal strategies for both the insurer and the reinsurer explicitly. Furthermore, we find that the reinsurer's optimal strategies under the two cases are equivalent for some special parameters. Finally, numerical simulations are presented to illustrate the effects of model parameters on the optimal strategies.

Key–Words: Proportional reinsurance, Optimal investment, For a reinsurer, Hamilton-Jacobi-Bellman (HJB) equation, Exponential utility maximization, Ruin probability minimization

1 Introduction

As investment income has gradually become an important way to increase the profit of insurance company, optimization problems taking both reinsurance and investment into account with different objectives have inspired literally hundreds of researches.

Browne [1] considered a diffusion risk model and obtained the optimal investment strategies of exponential utility maximization and ruin probability minimization. Hipp and Plum [2] used the compound Poisson risk model and studied an insurer's optimal policy to minimize the ruin probability. Schmidli [3] used a Brownian motion with drift to model the claim process and obtained the optimal quota-share reinsurance strategy for an insurer. Later, Yang and Zhang[4] considered the optimal investment problem for an insurer with jump-diffusion risk process. Promislow and Young[5] discussed the problem of minimizing the ruin probability subject to both investment and proportional reinsurance strategies for diffusion risk model. Luo et al. [6] studied a similar optimal reinsurance and investment problem in the case of neither short selling nor borrowing. Bai and Guo [7] investigated the optimal proportional reinsurance and investment problem with multiple risky assets. Cao and Wan [8] considered the optimal reinsurance-investment problem of utility maximization and obtained the explicit solutions for exponential and power utility functions. Gu et al. [9] used the constant elasticity of variance (CEV) model to study the optimal reinsurance and investment problem for an insurer. Zhao et al. [10] discussed the robust portfolio selection problem for an insurer with exponential utility preference. Liang et al. [11, 12] derived the optimal proportional reinsurance and investment strategies for exponential utility maximization under different financial markets. Lin and Li [13] focused on an optimal reinsurance-investment problem for an insurer with jump-diffusion risk model when the stock's price was governed by a CEV model. In Gu et al. [14], the insurer was allowed to purchase excess-of-loss reinsurance and invested in a financial market, and optimal strategies were obtained explicitly. Li et al. [15] investigated the optimal time-consistent reinsurance and investment problem under the mean-variance criterion for an insurer. Li and Li [16] took the state dependent risk aversion into account based on Li et al. [15] with the price process of the risky assets satisfying geometric Brownian motion. Zhao et al. [17] considered the optimal excess-

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of-loss reinsurance and investment problem for an insurer with jump-diffusion risk process under the Heston model. Liang and Bayraktar [18] discussed an optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution were not known but had to be inferred from the observations of claim arrivals. Besides, there are some other interesting topics about the insurance, see Zhang and Ma [19], Xu and Ma [20], Wang et al. [21] and references therein.

However, most of the above researches only consider the investment problem for the insurer and ignore the management of the reinsurer. But the reinsurer also faces ruin and needs to invest in a financial market to manage his/her wealth. Thus we study the investment problem for both the insurer and the reinsurer. From the above literatures, we find that maximizing the expected exponential utility and minimizing the ruin probability are two common investment objectives for the insurer. Moreover, Browne [1], Bai and Guo [7] have shown that maximizing the exponential utility and minimizing the ruin probability produce the same type of investment strategy for zero interest rate. Thus we consider these two objective functions for the reinsurer and examine the equality of the reinsurer's strategy under the two cases.

In this paper, we focus on the optimal investment problem for both the insurer and the reinsurer when the insurer can purchase proportional reinsurance. In our model, the basic claim process is assumed to follow a Brownian motion with drift. The insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. Furthermore, we consider the correlation between the claim process and the risky asset's price. We first derive the insurer's reinsurance-investment strategy for minimizing the ruin probability. Then with the optimal reinsurance proportion, we consider two optimization problems for the reinsurer: the problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the ruin probability. By solving the corresponding Hamilton-Jacobi-Bellman (HJB) equations, we obtain the explicit solutions and value functions for the optimization problems of the reinsurer. Moreover, we illustrate the equality of the reinsurer's optimal investment strategies for the two objective functions. Finally, we provide numerical simulations to show the effects of model parameters on the optimal strategies.

This paper proceeds as follows. In Section 2, we present formulation of the model. Section 3 provides the optimal proportional reinsurance and investment problem for the insurer in the sense of minimizing the ruin probability. In Section 4, with the optimal reinsurance proportion obtained in Section 3, we derive the optimal investment strategies and value functions for the reinsurer's optimization problems. In Section 5, numerical simulations are presented to illustrate our results. Section 6 concludes the paper.

2 Model formulation

Suppose that the claim process $C(t)$ is described by

$$\mathrm{d}C(t) = a\mathrm{d}t - b\mathrm{d}W_0(t), \tag{1}$$

where a and b are positive constants, $W_0(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$. Assume that the premium rate is $c = (1+\theta)a$ with safety loading $\theta > 0$. According to equation (1), the surplus process of the insurer is given by

$$\mathrm{d}R(t) = c\mathrm{d}t - \mathrm{d}C(t) = a\theta\mathrm{d}t + b\mathrm{d}W_0(t).$$

The insurer is allowed to purchase proportional reinsurance to reduce the underlying insurance risk and pays reinsurance premium continuously at rate $(1+\eta)ap(t)$, where $\eta > \theta > 0$ is the safety loading of the reinsurer and $p(t)$ represents the proportion reinsured at time t. Then the surplus process of the insurer and the reinsurer are

$$\mathrm{d}R_1(t) = (\theta - \eta p(t))a\mathrm{d}t + b(1-p(t))\mathrm{d}W_0(t),$$

$$\mathrm{d}R_2(t) = \eta p(t)a\mathrm{d}t + bp(t)\mathrm{d}W_0(t).$$

We assume that both the insurer and the reinsurer can invest their surplus in a financial market consisting of a risk-free asset with price $S_0(t)$ given by

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \quad S_0(0) = 1$$

and a risky asset with price $S(t)$ satisfying

$$\mathrm{d}S(t) = S(t)\left(\mu\mathrm{d}t + \sigma\mathrm{d}W(t)\right),$$

where r is the interest rate, μ, σ denote the appreciation rate and volatility of the risky asset, respectively. $W(t)$ is another standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ and $\mathrm{Cov}[W_0(t), W(t)] = \rho_0 t$. As usual, we assume that $\mu > r$.

Let $\pi_1(t)$ represents the amount invested in the risky asset by the insurer and $\pi_2(t)$ be the amount invested in the risky asset by the reinsurer at time t. For the insurer, a reinsurance-investment strategy $(p(t), \pi_1(t))$ is called admissible if it is $(\mathcal{F}_t)-$progressively measurable, satisfies $0 \leq p(t) \leq 1$ and $\mathrm{E}[\int_0^\infty (\pi_1(t))^2\mathrm{d}t] < \infty$. For the reinsurer, an investment strategy $\pi_2(t)$ is called admissible if it is $(\mathcal{F}_t)-$progressively measurable and satisfies $\mathrm{E}[\int_0^\infty (\pi_2(t))^2\mathrm{d}t] < \infty$.

Corresponding to an admissible strategy $(p(t), \pi_1(t))$, the wealth process $X(t)$ of the insurer follows

$$\mathrm{d}X(t) = [rX(t) + \pi_1(t)(\mu - r) + (\theta - \eta p(t))a]\mathrm{d}t$$
$$+\pi_1(t)\sigma \mathrm{d}W(t) + b(1 - p(t))\mathrm{d}W_0(t). \tag{2}$$

For each $\pi_2(t)$, the wealth process $Y(t)$ of the reinsurer is

$$\mathrm{d}Y(t) = [rY(t) + \pi_2(t)(\mu - r) + \eta p(t)a]\mathrm{d}t$$
$$+\pi_2(t)\sigma \mathrm{d}W(t) + bp(t)\mathrm{d}W_0(t). \tag{3}$$

3 Minimizing the ruin probability for the insurer

In this section, we consider the optimal reinsurance and investment problem for the insurer and he/she aims to minimize the ruin probability. Let $\tau^{p,\pi_1} = \inf\{t : X(t) < 0\}$, for diffusion risk model, it is known that $\tau^{p,\pi_1} = \inf\{t : X(t) = 0\}$ with probability 1. Denote the ruin probability, given the initial reserve x, by

$$\psi^{p,\pi_1}(x) = \mathrm{P}(\tau^{p,\pi_1} < \infty | X(0) = x)$$

and the minimal ruin probability by

$$\psi(x) = \inf_{p,\pi_1} \psi^{p,\pi_1}(x). \tag{4}$$

Our goal is to find the minimal ruin probability $\psi(x)$ and an optimal strategy $(p^*(x), \pi_1^*(x))$ such that $\psi(x) = \psi^{p^*,\pi_1^*}(x)$.

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$\min_{p,\pi_1} \Big\{ [rx + \pi_1(\mu - r) + a(\theta - p\eta)]\, \psi_x + \frac{1}{2}\, [\sigma^2\pi_1^2$$
$$+b^2(1-p)^2 + 2\sigma\pi_1 b(1-p)\rho_0]\, \psi_{xx} \Big\} = 0 \tag{5}$$

with boundary conditions $\psi(0) = 1$ and $\psi(\infty) = 0$.

The following lemma gives a trivial case of this problem.

Lemma 1. *Let ψ be defined by equation (4). Then*

$$\psi(x) = 0, \quad x \geq \frac{a(\eta - \theta)}{r} \tag{6}$$

and the corresponding optimal strategies of the insurer are

$$p^*(x) = 1, \quad \pi_1^*(x) = 0. \tag{7}$$

Proof. From equation (2), we can see that for the initial reserve $x \geq \frac{a(\eta-\theta)}{r}$, if $p(x) = 1, \pi_1(x) = 0$, ruin will not occur, i.e., $\psi^{p,\pi_1}(x) = 0$. By equation (4), the minimal ruin probability is

$$\psi(x) = 0, \quad x \geq \frac{a(\eta - \theta)}{r}$$

and the corresponding optimal reinsurance-investment strategy is

$$p^*(x) = 1, \quad \pi_1^*(x) = 0.$$

□

Lemma 1 implies that we can concentrate on the class of functions $\psi(x)$ such that $\psi(x) = 0$ on $\left[\frac{a(\eta-\theta)}{r}, \infty\right)$.

Differentiating with respect to π_1 in equation (5), we obtain

$$\pi_1^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{\psi_x}{\psi_{xx}} - \frac{b\rho_0(1-p)}{\sigma}. \tag{8}$$

Putting equation (8) into HJB equation (5), after simplification, we have

$$rx\psi_x + a\theta\psi_x - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\psi_x^2}{\psi_{xx}} + \min_p \Big\{ - ap\eta\psi_x$$
$$-\frac{b\rho_0(1-p)(\mu - r)}{\sigma}\psi_x + \frac{b^2(1-p)^2(1-\rho_0^2)}{2}\psi_{xx} \Big\}$$
$$= 0 \tag{9}$$

with $\psi(0) = 1, \psi(x) = 0$ for $x \geq \frac{a(\eta-\theta)}{r}$.

Differentiating with respect to p in equation (9) gives the optimal reinsurance proportion

$$p^0 = 1 + \frac{a\sigma\eta - b\rho_0(\mu - r)}{b^2\sigma(1-\rho_0^2)} \cdot \frac{\psi_x}{\psi_{xx}}. \tag{10}$$

Equation (10) indicates that $p^0(x)$ does not satisfy $0 \leq p^0(x) \leq 1$. If $0 \leq p^0(x) \leq 1$, then $p^*(x)$ coincides with $p^0(x)$. If $p^0(x) \leq 0$, then we set $p^*(x) = 0$. And if $p^0(x) \geq 1$, we simply let $p^*(x)$ be 1.

If $0 < p^0(x) < 1$, then $p^*(x) = p^0(x)$ and equation (9) becomes

$$rx\psi_x + a\theta\psi_x - a\eta\psi_x - \frac{[a\sigma\eta - b\rho_0(\mu - r)]^2}{2b^2\sigma^2(1-\rho_0^2)} \cdot \frac{\psi_x^2}{\psi_{xx}}$$
$$-\frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\psi_x^2}{\psi_{xx}} = 0, \quad 0 < x < \frac{a(\eta - \theta)}{r};$$
$$\psi(0) = 1, \;\; \psi(x) = 0, \quad x \geq \frac{a(\eta - \theta)}{r}. \tag{11}$$

Define

$$M=\frac{(\mu-r)^2}{\sigma^2}+\frac{[a\sigma\eta-b\rho_0(\mu-r)]^2}{b^2\sigma^2(1-\rho_0^2)} \quad (12)$$

for simplicity and we conjecture a solution to equation (11) in the following form

$$\psi(x)=c_1\left\{1-\frac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1},\ 0<x<\frac{a(\eta-\theta)}{r}, \quad (13)$$

where the constant c_1 will be determined later. From equation (10), the corresponding $p^0(x)$ is given by

$$p^0(x)=1-\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}, \quad 0<x<\frac{a(\eta-\theta)}{r}. \quad (14)$$

We next derive the explicit expressions for $\psi(x)$ in the following cases.

Case 1. $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}>0$.

If $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}>0$, equation (14) shows that $p^0(x)\in(0,1)$ is equivalent to

$$x>\beta=\frac{a(\eta-\theta)}{r}-\frac{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}{2ar\sigma^2\eta-2br\rho_0\sigma(\mu-r)}. \quad (15)$$

(1) If $\beta\le 0$, it is easy to show that $c_1=1$ and

$$\psi(x)=\left\{1-\frac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1},\ 0<x<\frac{a(\eta-\theta)}{r}. \quad (16)$$

(2) If $\beta>0$, then $\psi(x)$ becomes

$$\psi(x)=c_2\left\{1-\frac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1},\ \beta<x<\frac{a(\eta-\theta)}{r}. \quad (17)$$

For $0<x\le\beta$, $p^0(x)\le 0$. We turn to the case that $p^*(x)=0$ and equation (9) is transformed into

$$rx\psi_x+a\theta\psi_x-\frac{b\rho_0(\mu-r)}{\sigma}\psi_x+\frac{b^2(1-\rho_0^2)}{2}\psi_{xx}-\frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{\psi_x^2}{\psi_{xx}}=0. \quad (18)$$

To solve equation (18), we denote $f(x)=\frac{\psi_x}{\psi_{xx}}$ and obtain

$$\frac{(\mu-r)^2}{2\sigma^2}(f(x))^2-\left(rx+a\theta-\frac{b\rho_0(\mu-r)}{\sigma}\right)f(x)-\frac{b^2(1-\rho_0^2)}{2}=0. \quad (19)$$

Let

$$N_1=rx+a\theta-\frac{b\rho_0(\mu-r)}{\sigma}.$$

$\psi_x<0$ and $\psi_{xx}>0$ lead to $f(x)<0$, thus

$$f(x)=\frac{\sigma^2N_1-\sigma^2\sqrt{N_1^2+\frac{b^2(\mu-r)^2(1-\rho_0^2)}{\sigma^2}}}{(\mu-r)^2} \quad (20)$$

and

$$\psi(x)=c_3+c_4\int_0^x\exp\left(\int_0^z\frac{1}{f(y)}\mathrm{d}y\right)\mathrm{d}z,\ 0<x\le\beta. \quad (21)$$

In terms of the boundary condition $\psi(0)=1$, we have $c_3=1$. Since $\psi(x)$ is continuously differentiable at $x=\beta$, we derive

$$\begin{aligned}c_2=&\left(1-\frac{r\beta}{a(\eta-\theta)}\right)^{-\frac{M}{2r}}\Bigg[1-\frac{r\beta}{a(\eta-\theta)}\\&+\frac{r}{a(\eta-\theta)}\left(\frac{M}{2r}+1\right)\exp\left(-\int_0^z\frac{1}{f_2(y)}\mathrm{d}y\right)\\&\cdot\int_0^\beta\exp\left(\int_0^z\frac{1}{f_2(y)}\mathrm{d}y\right)\mathrm{d}z\Bigg]^{-1},\\c_4=&-\frac{r}{a(\eta-\theta)}\left(\frac{M}{2r}+1\right)\exp\left(-\int_0^z\frac{1}{f_2(y)}\mathrm{d}y\right)\\&\Bigg[1-\frac{r\beta}{a(\eta-\theta)}+\frac{r}{a(\eta-\theta)}\left(\frac{M}{2r}+1\right)\\&\exp\left(-\int_0^z\frac{1}{f_2(y)}\mathrm{d}y\right)\int_0^\beta\exp\left(\int_0^z\frac{1}{f_2(y)}\mathrm{d}y\right)\mathrm{d}z\Bigg]^{-1}.\end{aligned} \quad (22)$$

Case 2. $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}\le 0$.

If $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}\le 0$, according to equation (14), we find $p^0(x)\ge 1$. Therefore, we let $p^*(x)=1$ and equation (9) reduces to

$$\begin{aligned}&rx\psi_x+a\theta\psi_x-a\eta\psi_x-\frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{\psi_x^2}{\psi_{xx}}=0,\\&0<x<\frac{a(\eta-\theta)}{r};\\&\psi(0)=1,\ \ \psi(x)=0,\quad x\ge\frac{a(\eta-\theta)}{r}.\end{aligned} \quad (23)$$

We conjecture the solution to equation (23) in the following way

$$\psi(x)=c_5\left\{1-\frac{rx}{a(\eta-\theta)}\right\}^{\frac{(\mu-r)^2}{2r\sigma^2}+1},\quad 0<x<\frac{a(\eta-\theta)}{r}. \quad (24)$$

Taking the boundary condition $\psi(0)=1$ into account, we obtain $c_5=1$.

The following theorem gives the optimal strategies for the insurer to minimize his/her ruin probability.

Theorem 2. *For optimization problem (4), the optimal reinsurance, investment strategy and ruin probability function of the insurer are given by*
(1) If $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}>0$ *and* $\beta\le 0$, *the optimal reinsurance and investment strategies are*

$$p^*(x)=\begin{cases}1-\dfrac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}, & 0<x<\dfrac{a(\eta-\theta)}{r},\\ 1, & x\ge\dfrac{a(\eta-\theta)}{r},\end{cases}$$

$$\pi_1^*(x)=\begin{cases}\dfrac{2b^2(\mu-r)-2ab\rho_0\sigma\eta(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}, & 0<x<\dfrac{a(\eta-\theta)}{r},\\ 0, & x\ge\dfrac{a(\eta-\theta)}{r},\end{cases}$$

and the corresponding ruin probability is

$$\psi(x)=\begin{cases}\left\{1-\dfrac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1}, & 0<x<\dfrac{a(\eta-\theta)}{r},\\ 0, & x\ge\dfrac{a(\eta-\theta)}{r}.\end{cases}$$

(2) If $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}>0$ *and* $\beta>0$, *the optimal reinsurance and investment strategies are*

$$p^*(x)=\begin{cases}0, & 0<x\le\beta,\\ 1-\dfrac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}, & \beta<x<\dfrac{a(\eta-\theta)}{r},\\ 1, & x\ge\dfrac{a(\eta-\theta)}{r},\end{cases}$$

$$\pi_1^*(x)=\begin{cases}-\dfrac{\mu-r}{\sigma^2}f(x)-\dfrac{b\rho_0}{\sigma}, & 0<x\le\beta,\\ \dfrac{2b^2(\mu-r)-2ab\rho_0\sigma\eta(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}, & \beta<x<\dfrac{a(\eta-\theta)}{r},\\ 0, & x\ge\dfrac{a(\eta-\theta)}{r},\end{cases}$$

and the ruin probability function is

$$\psi(x)=\begin{cases}1+c_4\displaystyle\int_0^x\exp\left(\int_0^z\frac{1}{f(y)}\mathrm{d}y\right)\mathrm{d}z, & 0<x\le\beta,\\ c_2\left\{1-\dfrac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1}, & \beta<x<\dfrac{a(\eta-\theta)}{r},\\ 0, & x\ge\dfrac{a(\eta-\theta)}{r},\end{cases}$$

where c_2, c_4 *and* f *are defined in equations (22) and (20).*
(3) If $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}\le 0$, *the optimal reinsurance and investment strategies are*

$$p^*(x)=1,\quad x>0,$$

$$\pi_1^*(x)=\begin{cases}\dfrac{2(a\eta-a\theta-rx)}{\mu-r}, & 0<x<\dfrac{a(\eta-\theta)}{r},\\ 0, & x\ge\dfrac{a(\eta-\theta)}{r},\end{cases}$$

and the corresponding ruin probability is

$$\psi(x)=\begin{cases}\left\{1-\dfrac{rx}{a(\eta-\theta)}\right\}^{\frac{(\mu-r)^2}{2r\sigma^2}+1}, & 0<x<\dfrac{a(\eta-\theta)}{r},\\ 0, & x\ge\dfrac{a(\eta-\theta)}{r}.\end{cases}$$

Proof. (1) If $\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}>0$ and $\beta\le 0$, when $0<x<\frac{a(\eta-\theta)}{r}$, we have $0<p^0(x)<1$ and $p^*(x)=p^0(x)$. According to equations (8), (14) and (16), we obtain

$$p^*(x)=1-\frac{2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2},$$

$$\begin{aligned}\pi_1^*(x)&=-\frac{\mu-r}{\sigma^2}\cdot\frac{\psi_x}{\psi_{xx}}-\frac{b\rho_0(1-p)}{\sigma}\\&=\frac{\mu-r}{\sigma^2}\cdot\frac{2a(\eta-\theta)}{M}\left\{1-\frac{rx}{a(\eta-\theta)}\right\}\\&\quad-\frac{b\rho_0}{\sigma}\cdot\frac{[2a\sigma^2\eta-2b\rho_0\sigma(\mu-r)](a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}\\&=\frac{2b^2(\mu-r)-2ab\rho_0\sigma\eta(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2-2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2},\end{aligned}$$

$$\psi(x)=\left\{1-\frac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1}.$$

When $x \geq \frac{a(\eta-\theta)}{r}$, we have

$$p^*(x) = 1, \quad \pi_1^*(x) = 0, \quad \psi(x) = 0$$

from Lemma 1.

(2) If $\frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2} > 0$ and $\beta > 0$, when $0 < x < \beta$, we derive $p^0(x) \leq 0$ and then $p^*(x) = 0$. According to equations (8), (20), (21) and (22), we obtain

$$\begin{aligned}\pi_1^*(x) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\psi_x}{\psi_{xx}} - \frac{b\rho_0(1-p)}{\sigma}\\ &= -\frac{\mu-r}{\sigma^2}f(x) - \frac{b\rho_0}{\sigma},\end{aligned}$$

$$\psi(x) = 1 + c_4\int_0^x \exp\left(\int_0^z \frac{1}{f(y)}\mathrm{d}y\right)\mathrm{d}z.$$

For $\beta < x < \frac{a(\eta-\theta)}{r}$, $p^*(x) = p^0(x)$. According to equations (8), (14), (17) and (22), we have

$$p^*(x) = 1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2},$$

$$\begin{aligned}\pi_1^*(x) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\psi_x}{\psi_{xx}} - \frac{b\rho_0(1-p)}{\sigma}\\ &= \frac{\mu-r}{\sigma^2}\cdot\frac{2a(\eta-\theta)}{M}\left\{1 - \frac{rx}{a(\eta-\theta)}\right\}\\ &-\frac{b\rho_0}{\sigma}\cdot\frac{[2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)](a\eta-a\theta-rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2}\\ &= \frac{2b^2(\mu-r) - 2ab\rho_0\sigma\eta(a\eta-a\theta-rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2},\end{aligned}$$

$$\psi(x) = c_2\left\{1 - \frac{rx}{a(\eta-\theta)}\right\}^{\frac{M}{2r}+1}.$$

When $x \geq \frac{a(\eta-\theta)}{r}$, we have

$$p^*(x) = 1, \quad \pi_1^*(x) = 0, \quad \psi(x) = 0$$

from Lemma 1.

(3) If $\frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r)+(\mu-r)^2b^2} \leq 0$, when $0 < x < \frac{a(\eta-\theta)}{r}$, we see $p^0(x) \geq 1$ and then $p^*(x) = 1$. Equations (8) and (24) lead to

$$\begin{aligned}\pi_1^*(x) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\psi_x}{\psi_{xx}} - \frac{b\rho_0(1-p)}{\sigma}\\ &= \frac{\mu-r}{\sigma^2}\cdot\frac{2\sigma^2a(\eta-\theta)}{(\mu-r)^2}\left\{1 - \frac{rx}{a(\eta-\theta)}\right\}\\ &= \frac{2(a\eta - a\theta - rx)}{\mu-r},\end{aligned}$$

$$\psi(x) = \left\{1 - \frac{rx}{a(\eta-\theta)}\right\}^{\frac{(\mu-r)^2}{2r\sigma^2}+1}.$$

When $x \geq \frac{a(\eta-\theta)}{r}$, we have

$$p^*(x) = 1, \quad \pi_1^*(x) = 0, \quad \psi(x) = 0$$

from Lemma 1. □

4 Optimal investment problems for the reinsurer under two cases

The purpose of this section is to find the reinsurer's optimal investment strategy when the insurer minimizes his/her ruin probability. We consider two optimization problems for the reinsurer: the problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the ruin probability.

4.1 Maximizing the exponential utility of the reinsurer

Suppose that the reinsurer has a utility function U which is strictly concave and continuously differentiable on $(-\infty, \infty)$. For the reinsurance proportion chosen by the insurer, the reinsurer aims to maximize the expected utility of his/her terminal wealth, i.e.

$$\max_{\pi_2} \mathrm{E}[U(Y(T))]. \tag{25}$$

The corresponding Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned}H_t + \sup_{\pi_2}\Big\{&[ry + \pi_2(\mu-r) + ap^*\eta]H_y + \frac{1}{2}\left[\sigma^2\pi_2^2\right.\\ &\left.+b^2(p^*)^2 + 2\sigma\pi_2 bp^*\rho_0\right]H_{yy}\Big\} = 0\end{aligned} \tag{26}$$

with $H(T, y) = U(y)$, where H_t, H_y, H_{yy} denote partial derivatives of first and second orders with respect to t and y.

The first order maximizing condition for the optimal strategy is

$$\pi_2^* = -\frac{\mu-r}{\sigma^2}\cdot\frac{H_y}{H_{yy}} - \frac{bp^*\rho_0}{\sigma}, \tag{27}$$

where p^* is the optimal reinsurance proportion of the insurer. Putting equation (27) into HJB equation (26), after simplification, we have

$$\begin{aligned}&H_t + ryH_y + a\eta p^*H_y - \frac{b\rho_0p^*(\mu-r)}{\sigma}H_y\\ &+\frac{b^2(p^*)^2(1-\rho_0^2)}{2}H_{yy} - \frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}} = 0\end{aligned} \tag{28}$$

with $H(T, y) = U(y)$.

Assume that the reinsurer has an exponential utility function $U(y)$:

$$U(y) = -\frac{1}{m}\mathrm{e}^{-my}, \quad m > 0. \tag{29}$$

According to the utility function described by equation (29), we try to find a solution to equation (28) in the following way

$$H(t, y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}\,k(t) \tag{30}$$

with the boundary condition given by $k(T) = 1$. Thus

$$\begin{aligned}
&H_t = -ry\exp(r(T-t))\exp\{-my\exp(r(T-t))\}\\
&\cdot k(t) - \frac{1}{m}\exp\{-my\exp(r(T-t))\}\,k'(t),\\
&H_y = \exp(r(T-t))\exp\{-my\exp(r(T-t))\}\,k(t),\\
&H_{yy} = -m\exp(2r(T-t))\exp\{-my\exp(r(T-t))\}\\
&\cdot k(t).
\end{aligned} \tag{31}$$

We now derive the reinsurer's investment strategies for different reinsurance proportions.

(1) If $p^*(x) = p^0(x)$, plugging equations (14) and (31) into equation (28) gives

$$\begin{aligned}
&k'(t) - \Bigg\{\Bigg[am\eta - \frac{bm\rho_0(\mu-r)}{\sigma}\\
&-\frac{2(a\sigma\eta - b\rho_0(\mu-r))^2(a\eta - a\theta - rx)m}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\Bigg]\\
&\cdot\exp(r(T-t)) - \Bigg[\frac{b^2m^2(1-\rho_0^2)}{2} - b^2m^2(1-\rho_0^2)\\
&\cdot\frac{(2a\sigma^2\eta - 2b\sigma\rho_0(\mu-r))(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\\
&+\frac{(2a\sigma^2\eta - 2b\sigma\rho_0(\mu-r))^2(a\eta - a\theta - rx)^2}{(a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2)^2}\\
&\cdot\frac{b^2m^2(1-\rho_0^2)}{2}\Bigg]\exp(2r(T-t)) + \frac{(\mu-r)^2}{2\sigma^2}\Bigg\}k(t)\\
&= 0.
\end{aligned} \tag{32}$$

Taking the boundary condition $k(T) = 1$ into consideration, the solution to equation (32) is

$$\begin{aligned}
&k(t) = \exp\Bigg\{\Bigg[\frac{am\eta}{r} - \frac{bm\rho_0(\mu-r)}{r\sigma}\\
&-\frac{2(a\sigma\eta - b\rho_0(\mu-r))^2(a\eta - a\theta - rx)m}{ra^2\sigma^2\eta^2 - 2rab\rho_0\sigma\eta(\mu-r) + r(\mu-r)^2b^2}\Bigg]\\
&\cdot(1 - \exp(r(T-t))) - \Bigg[\frac{b^2m^2(1-\rho_0^2)}{4r} - (1-\rho_0^2)\\
&\cdot\frac{b^2m^2(2a\sigma^2\eta - 2b\sigma\rho_0(\mu-r))(a\eta - a\theta - rx)}{2ra^2\sigma^2\eta^2 - 4rab\rho_0\sigma\eta(\mu-r) + 2r(\mu-r)^2b^2}\\
&+\frac{(2a\sigma^2\eta - 2b\sigma\rho_0(\mu-r))^2(a\eta - a\theta - rx)^2}{(a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2)^2}\\
&\cdot\frac{b^2m^2(1-\rho_0^2)}{4r}\Bigg](1 - \exp(2r(T-t)))\\
&+\frac{(\mu-r)^2}{2\sigma^2}(t-T)\Bigg\}.
\end{aligned} \tag{33}$$

(2) If $p^*(x) = 0$, equation (28) becomes

$$H_t + ryH_y - \frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}} = 0 \tag{34}$$

with $H(T, y) = U(y)$. Introducing equation (31) into equation (34), we obtain

$$k'(t) - \frac{(\mu-r)^2}{2\sigma^2}k(t) = 0. \tag{35}$$

In terms of the boundary condition $k(T) = 1$, the solution to equation (35) is

$$k(t) = \exp\left\{\frac{(\mu-r)^2}{2\sigma^2}(t-T)\right\}. \tag{36}$$

(3) If $p^*(x) = 1$, equation (28) is

$$\begin{aligned}
&H_t + ryH_y + a\eta H_y - \frac{b\rho_0(\mu-r)}{\sigma}H_y\\
&+\frac{b^2(1-\rho_0^2)}{2}H_{yy} - \frac{(\mu-r)^2}{2\sigma^2}\cdot\frac{H_y^2}{H_{yy}} = 0
\end{aligned} \tag{37}$$

with $H(T, y) = U(y)$. Putting equation (31) into equation (37) leads to

$$\begin{aligned}
&k'(t) - \Bigg\{\Bigg[am\eta - \frac{bm\rho_0(\mu-r)}{\sigma}\Bigg]\exp(r(T-t))\\
&-\frac{b^2m^2(1-\rho_0^2)}{2}\exp(2r(T-t)) + \frac{(\mu-r)^2}{2\sigma^2}\Bigg\}k(t)\\
&= 0.
\end{aligned} \tag{38}$$

Taking the boundary condition $k(T) = 1$ into account, we obtain the solution to equation (38) is

$$k(t) = \exp\left\{\left[\frac{am\eta}{r} - \frac{bm\rho_0(\mu-r)}{r\sigma}\right] \cdot(1-\exp(r(T-t))) - \frac{b^2m^2(1-\rho_0^2)}{4r} \cdot(1-\exp(2r(T-t))) + \frac{(\mu-r)^2}{2\sigma^2}(t-T)\right\}. \quad (39)$$

Finally, the following theorem summarizes the above analysis.

Theorem 3. *When minimizing the ruin probability of the insurer, the optimal investment strategy and value function of the reinsurer in the sense of maximizing the exponential utility are given by*
(1) If the optimal reinsurance proportion of the insurer is $p^*(x) = p^0(x)$*, then*

$$\pi_2^*(t) = \frac{\mu-r}{m\sigma^2}\exp(-r(T-t)) - \frac{b\rho_0}{\sigma} \cdot\left[1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\right]$$

and the value function is

$$H(t,y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}k(t),$$

where $k(t)$ *is given by equation (33).*
(2) If $p^*(x) = 0$*, we have*

$$\pi_2^*(t) = \frac{\mu-r}{\sigma^2 m}\exp(-r(T-t))$$

and the value function is

$$H(t,y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}k(t),$$

where $k(t)$ *is given by equation (36).*
(3) If $p^*(x) = 1$*, the reinsurer's optimal investment strategy is*

$$\pi_2^*(t) = \frac{\mu-r}{\sigma^2 m}\exp(-r(T-t)) - \frac{b\rho_0}{\sigma}$$

and the value function is

$$H(t,y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}k(t),$$

where $k(t)$ *is given by equation (39).*

Proof. (1) If $p^*(x) = p^0(x)$, according to equations (27), (30) and (33), we get

$$\begin{aligned}\pi_2^*(t) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{H_x}{H_{xx}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\exp(-r(T-t))}{-m} - \frac{b\rho_0}{\sigma} \\ &\cdot\left[1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\right] \\ &= \frac{\mu-r}{m\sigma^2}\exp(-r(T-t)) - \frac{b\rho_0}{\sigma} \\ &\cdot\left[1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\right],\end{aligned}$$

$$H(t,y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}k(t),$$

where $k(t)$ is defined in equation (33).

(2) For $p^*(x) = 0$, from equations (27), (30) and (36), we derive

$$\begin{aligned}\pi_2^*(t) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{H_x}{H_{xx}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\exp(-r(T-t))}{-m} \\ &= \frac{\mu-r}{\sigma^2 m}\exp(-r(T-t)),\end{aligned}$$

$$H(t,y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}k(t)$$

for $k(t)$ given by equation (36).

(3) For $p^*(x) = 1$, we have

$$\begin{aligned}\pi_2^*(t) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{H_x}{H_{xx}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\exp(-r(T-t))}{-m} - \frac{b\rho_0}{\sigma} \\ &= \frac{\mu-r}{\sigma^2 m}\exp(-r(T-t)) - \frac{b\rho_0}{\sigma},\end{aligned}$$

$$H(t,y) = -\frac{1}{m}\exp\{-my\exp(r(T-t))\}k(t)$$

from equations (27), (30) and $k(t)$ is given in equation (39). □

4.2 Minimizing the ruin probability of the reinsurer

In this subsection, we consider the optimal investment problem in the sense of minimizing the ruin probability for the reinsurer. Let $\tau^{\pi_2} = \inf\{t : Y(t) < 0\}$ be the first time when the surplus of the reinsurance company becomes negative. Since the risk model of the reinsurer is diffusion risk process, we know that $\tau^{\pi_2} = \inf\{t : Y(t) = 0\}$ with probability 1. Denote the ruin probability, given the initial reserve of the reinsurer y, by

$$\phi^{\pi_2}(y) = \mathrm{P}(\tau^{\pi_2} < \infty | Y_0 = y)$$

and the minimal probability of ruin by

$$\phi(y) = \inf \phi^{\pi_2}(y). \tag{40}$$

Our goal is to find the minimal ruin probability $\phi(y)$ and an optimal strategy $\pi_2^*(y)$ such that $\phi(y) = \phi^{\pi_2^*}(y)$.

To solve the above problem, we consider the following HJB equation

$$\min_{\pi_2} \left\{ [ry + \pi_2(\mu - r) + ap\eta]\phi_y + \frac{1}{2}\left[\sigma^2\pi_2^2 + b^2p^2 + 2\sigma\pi_2 bp\rho_0\right]\phi_{yy} \right\} = 0 \tag{41}$$

with boundary conditions $\phi(0) = 1$ and $\phi(\infty) = 0$.

Differentiating with respect to π_2 in equation (41), we obtain

$$\pi_2^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{\phi_y}{\phi_{yy}} - \frac{b\rho_0 p^*}{\sigma}. \tag{42}$$

Putting equation (42) into HJB equation (41), after simplification, we have

$$ry\phi_y + ap^*\eta\phi_y - \frac{b\rho_0 p^*(\mu - r)}{\sigma}\phi_y + \frac{b^2(p^*)^2(1 - \rho_0^2)}{2}\phi_{yy} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\phi_y^2}{\phi_{yy}} = 0 \tag{43}$$

with $\phi(0) = 1, \phi(\infty) = 0$.

From equation (3), we find that when $p^*(x) = 0$, $\pi_2(y) = 0$ and $y \geq 0$, the ruin of the reinsurer will not occur. Thus we next derive the explicit expressions for $\phi(y)$ in the cases that $p^*(x) = p^0(x)$ and $p^*(x) = 1$.

If $p^*(x) = p^0(x)$, equation (43) is transformed into

$$ry\phi_y + a\eta\phi_y - \frac{b\rho_0(\mu - r)}{\sigma}\phi_y - \frac{2(a\sigma\eta - b\rho_0(\mu - r))^2(a\eta - a\theta - rx)\phi_y}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu - r) + (\mu - r)^2b^2} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\phi_y^2}{\phi_{yy}} + \frac{b^2(1 - \rho_0^2)\phi_{yy}}{2}\left[1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu - r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu - r) + (\mu - r)^2b^2}\right]^2 = 0$$

with boundary conditions $\phi(0) = 1$ and $\phi(\infty) = 0$.

Setting $h_1(y) = \frac{\phi_y}{\phi_{yy}}$, we obtain

$$\frac{(\mu - r)^2}{2\sigma^2}(h_1(y))^2 - \left\{ry + a\eta - \frac{b\rho_0(\mu - r)}{\sigma} - \frac{2(a\sigma\eta - b\rho_0(\mu - r))^2(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu - r) + (\mu - r)^2b^2}\right\}h_1(y) - \frac{b^2(1 - \rho_0^2)}{2}\left\{1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu - r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu - r) + (\mu - r)^2b^2}\right\}^2 = 0.$$

Since $\phi_y < 0$ and $\phi_{yy} > 0$, we have $h_1(y) < 0$. Let

$$N_2(y) = ry + a\eta - \frac{b\rho_0(\mu - r)}{\sigma} - \frac{2(a\sigma\eta - b\rho_0(\mu - r))^2(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu - r) + (\mu - r)^2b^2},$$

we derive

$$h_1(y) = \frac{\sigma^2 N_2(y) - \sigma^2\sqrt{(N_2(y))^2 + \frac{b^2(\mu - r)^2(1 - \rho_0^2)}{\sigma^2}(p^0)^2}}{(\mu - r)^2} \tag{44}$$

and

$$\phi(y) = c_6 + c_7\int_0^y \exp\left(\int_0^z \frac{1}{h_1(s)}\mathrm{d}s\right)\mathrm{d}z. \tag{45}$$

In terms of the boundary conditions $\phi(0) = 1$, $\phi(\infty) = 0$, we have $c_6 = 1$ and

$$c_7 = -\frac{1}{\int_0^\infty \exp\left(\int_0^z \frac{1}{h_1(s)}\mathrm{d}s\right)\mathrm{d}z}. \tag{46}$$

If $p^*(x) = 1$, equation (43) becomes

$$ry\phi_y + a\eta\phi_y - \frac{b\rho_0(\mu - r)}{\sigma}\phi_y + \frac{b^2(1 - \rho_0^2)}{2}\phi_{yy} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\phi_y^2}{\phi_{yy}} = 0$$

with $\phi(0) = 1$ and $\phi(\infty) = 0$. Similarly, by setting $h_2(y) = \frac{\phi_y}{\phi_{yy}}$, we get

$$\frac{(\mu - r)^2}{2\sigma^2}(h_2(y))^2 - (ry + a\eta - \frac{b\rho_0(\mu - r)}{\sigma})h_2(y) - \frac{b^2(1 - \rho_0^2)}{2} = 0.$$

Let

$$N_3(y) = ry + a\eta - \frac{b\rho_0(\mu - r)}{\sigma}.$$

Noting that $h_2(y) < 0$, we have

$$h_2(y) = \frac{\sigma^2 N_3(y) - \sigma^2 \sqrt{(N_3(y))^2 + \frac{b^2(\mu-r)^2(1-\rho_0^2)}{\sigma^2}}}{(\mu-r)^2} \tag{47}$$

and

$$\phi(y) = 1 + c_8 \int_0^y \exp\left(\int_0^z \frac{1}{h_2(s)}\mathrm{d}s\right)\mathrm{d}z. \tag{48}$$

Considering the boundary condition $\phi(\infty) = 0$, we have

$$c_8 = -\frac{1}{\int_0^\infty \exp\left(\int_0^z \frac{1}{h_2(s)}\mathrm{d}s\right)\mathrm{d}z}. \tag{49}$$

Concluding the above analysis, we propose the optimal investment strategy for the reinsurer who aims to minimize the ruin probability in the following theorem.

Theorem 4. *Suppose both the objectives of the insurer and reinsurer are minimizing the ruin probability. According to different reinsurance proportions, the optimal investment strategies and ruin probability functions of the reinsurer are*
(1) If $p^*(x) = p^0(x)$, the reinsurer's optimal strategy is

$$\pi_2^*(y) = -\frac{\mu-r}{\sigma^2}h_1(y) - \frac{b\rho_0}{\sigma} \cdot \left[1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\right],$$

and the ruin probability function is

$$\phi(y) = 1 + c_7 \int_0^y \exp\left(\int_0^z \frac{1}{h_1(s)}\mathrm{d}s\right)\mathrm{d}z,$$

where h_1 and c_7 are given by equations (44) and (46).
(2) If $p^*(x) = 0$, then

$$\pi_2^*(y) = 0, \quad \phi(y) = 0.$$

(3) If $p^*(x) = 1$, we obtain

$$\pi_2^*(y) = -\frac{\mu-r}{\sigma^2}h_2(y) - \frac{b\rho_0}{\sigma},$$

and the ruin probability function is

$$\phi(y) = 1 + c_8 \int_0^y \exp\left(\int_0^z \frac{1}{h_2(s)}\mathrm{d}s\right)\mathrm{d}z,$$

where h_2 and c_8 are given in equations (47) and (49).

Proof. (1) For $p^*(x) = p^0(x)$, we know that

$$\phi(y) = 1 + c_7 \int_0^y \exp\left(\int_0^z \frac{1}{h_1(s)}\mathrm{d}s\right)\mathrm{d}z,$$

from the above analysis, where h_1 and c_7 are given by equations (44) and (46). According to equations (42), (44), (45) and (46), we derive

$$\begin{aligned}\pi_2^*(y) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\phi_y}{\phi_{yy}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu-r}{\sigma^2}h_1(y) - \frac{b\rho_0}{\sigma} \\ &\cdot \left[1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma(\mu-r)(a\eta - a\theta - rx)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta(\mu-r) + (\mu-r)^2b^2}\right].\end{aligned}$$

(2) From equation (3), we can see that when $p^*(x) = 0$, the ruin of the reinsurer will not occur if $\pi_2(y) = 0$ and $y \geq 0$. Hence the minimal probability of ruin is $\phi(y) = 0$.
(3) If $p^*(x) = 1$, $\phi(y)$ is derived in the former part. From equations (42), (47), (48) and (49), we obtain

$$\begin{aligned}\pi_2^*(y) &= -\frac{\mu-r}{\sigma^2}\cdot\frac{\phi_y}{\phi_{yy}} - \frac{bp^*\rho_0}{\sigma} \\ &= -\frac{\mu-r}{\sigma^2}h_2(y) - \frac{b\rho_0}{\sigma},\end{aligned}$$

$$\phi(y) = 1 + c_8 \int_0^y \exp\left(\int_0^z \frac{1}{h_2(s)}\mathrm{d}s\right)\mathrm{d}z.$$

□

4.3 The equality of the reinsurer's optimal strategies under the two cases

Browne [1], Bai and Guo [7] both find that with zero interest rate, the insurer's optimal investment strategy that maximizes the expected exponential utility also minimizes the probability of ruin. We now examine the equality of the reinsurer's optimal investment strategies when there is no risk-free asset.

If $r = 0$, for $p^*(x) = p^0(x)$, the optimal investment strategy of exponential utility maximization is

$$\pi_2^* = \frac{\mu}{\sigma^2 m} - \frac{b\rho_0 p^0}{\sigma} \tag{50}$$

and that of ruin probability minimization is

$$\pi_2^* = -\frac{\mu}{\sigma^2}h_1 - \frac{b\rho_0 p^0}{\sigma}, \tag{51}$$

where

$$p^0 = 1 - \frac{2a\sigma^2\eta - 2b\rho_0\sigma\mu}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta\mu + \mu^2b^2}(a\eta - a\theta),$$

$$h_1 = \frac{\sigma^2 N_2 - \sigma^2 \sqrt{N_2^2 + \frac{b^2\mu^2(1-\rho_0^2)}{\sigma^2}(p^0)^2}}{\mu^2},$$

$$N_2 = a\eta - \frac{b\rho_0\mu}{\sigma} - \frac{2(a\sigma\eta - b\rho_0\mu)^2(a\eta - a\theta)}{a^2\sigma^2\eta^2 - 2ab\rho_0\sigma\eta\mu + \mu^2 b^2}.$$

Let $m = -\frac{1}{h_1}$, we have that equations (50) and (51) are equivalent.

For $p^*(x) = 1$,

$$\pi_2^* = \frac{\mu}{\sigma^2 m} - \frac{b\rho_0}{\sigma}, \tag{52}$$

$$\pi_2^* = -\frac{\mu}{\sigma^2} h_2 - \frac{b\rho_0}{\sigma}, \tag{53}$$

where

$$h_2 = \frac{\sigma^2 N_3 - \sigma^2 \sqrt{N_3^2 + \frac{b^2\mu^2(1-\rho_0^2)}{\sigma^2}}}{\mu^2},$$

$$N_3 = a\eta - \frac{b\rho_0\mu}{\sigma}.$$

Let $m = -\frac{1}{h_2}$, equations (52) and (53) are also equivalent.

For $r \neq 0$, the optimal strategy which considering maximizing the exponential utility is related to time t, while that of minimizing the probability of ruin is related to wealth of the reinsurer. Thus, the two strategies are not equivalent for a positive interest rate.

5 Numerical simulations

In this section, we provide some numerical simulations to illustrate the sensitivities of the optimal strategies with respect to the model parameters. Throughout numerical simulations, unless otherwise stated, the basic parameters are given by: $a = 1.5$, $b = 1$, $r = 0.3$, $\mu = 0.5$, $\sigma = 0.3$, $\rho_0 = -0.5$, $\theta = 0.8$, $\eta = 2$, $x = 20$, $y = 20$, $m = 1$, $T = 10$, $t = 5$.

5.1 Numerical simulations of the optimal reinsurance proportion

Figures 1 and 2 show the effects of the insurer's and the reinsurer's safety loading θ and η on the optimal reinsurance proportion p^*. We see that a greater θ yields a greater p^* and the effect of η is opposite. This is because that as θ increases, the insurer will earn more profit. So he/she would like to purchase more reinsurance. However, with the increase of η, the cost of reinsurance will become higher and the insurer will prefer to maintain a stable revenue by purchasing less reinsurance.

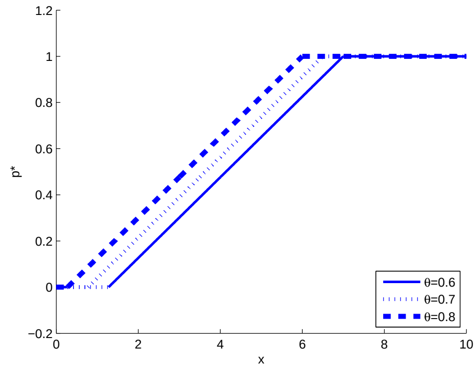

Figure 1: Sensitivity of optimal reinsurance proportion p^* w.r.t. θ.

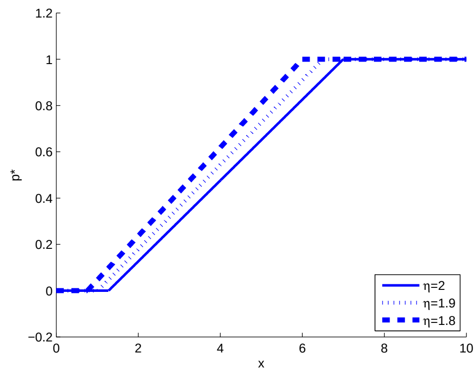

Figure 2: Sensitivity of optimal reinsurance proportion p^* w.r.t. η.

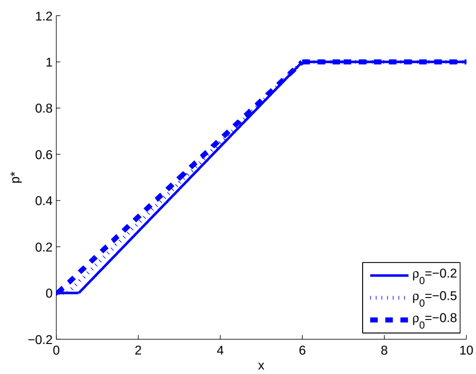

Figure 3: Sensitivity of optimal reinsurance proportion p^* w.r.t. ρ_0 ($\rho_0 < 0$).

Figures 3 and 4 illustrate the effect of the correlation coefficient between risk process and risky asset's price ρ_0 on the optimal reinsurance proportion p^*. We find that no matter ρ_0 is positive or negative, the higher ρ_0 is, the smaller p^* is. This is because that the influence of financial market on the risk process rises when ρ_0 increases. Thus the insurer should purchase more reinsurance to hedge the risk.

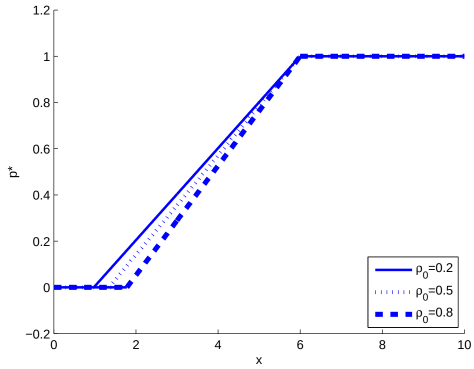

Figure 4: Sensitivity of optimal reinsurance proportion p^* w.r.t. ρ_0 ($\rho_0 > 0$).

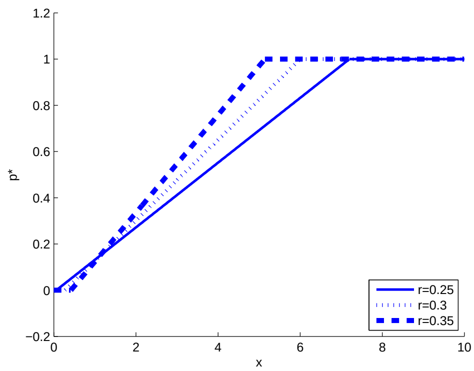

Figure 5: Sensitivity of optimal reinsurance proportion p^* w.r.t. r.

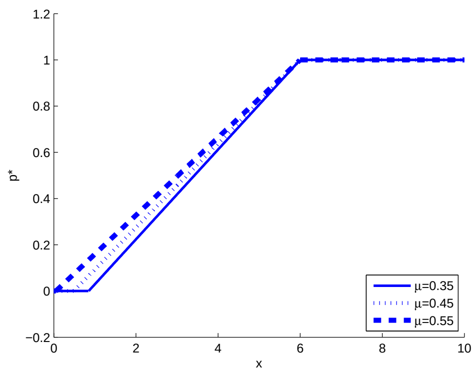

Figure 6: Sensitivity of optimal reinsurance proportion p^* w.r.t. μ.

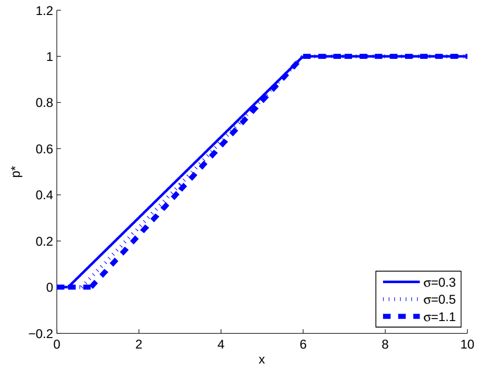

Figure 7: Sensitivity of optimal reinsurance proportion p^* w.r.t. σ.

As shown in Figure 5, the interest rate r exerts a positive effect on the optimal reinsurance proportion p^*. As r increases, the insurer will obtain more profit from investment in the risk-free asset. Therefore, the insurer has more money to purchase the reinsurance and then p^* increases with r. Figure 6 shows that the optimal reinsurance strategy increases with respect to the risky asset's appreciation rate μ. A larger μ implies more return from the investment in the risky asset. Thus the insurer would like to purchase more reinsurance. In Figure 7, we see that the optimal reinsurance proportion is a decreasing function of the risky asset's volatility σ. As σ increases, the risky asset's price is more volatile and then the insurer's revenue from investment is uncertain. Thus he/she will be conservative and reduce the reinsurance proportion.

5.2 Numerical simulations of the insurer's optimal investment strategy

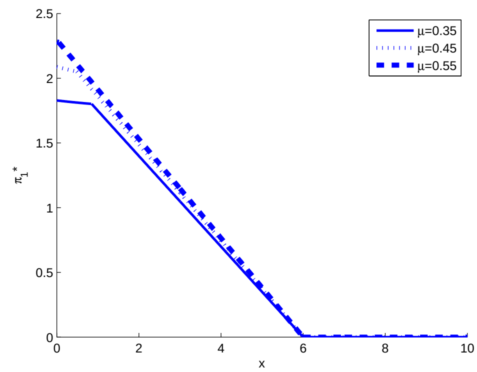

Figure 8: Sensitivity of the insurer's optimal investment strategy π_1^* w.r.t. μ.

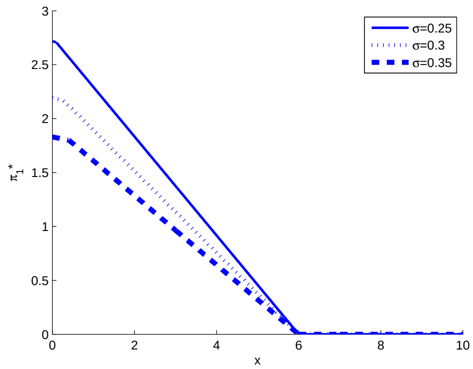

Figure 9: Sensitivity of the insurer's optimal investment strategy π_1^* w.r.t. σ.

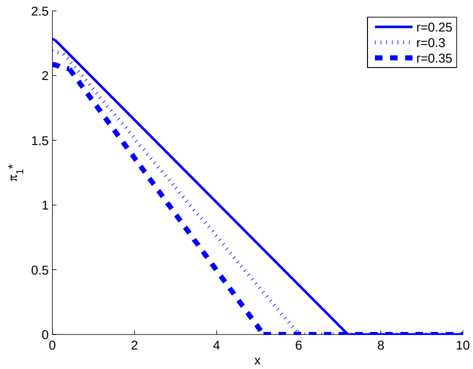

Figure 10: Sensitivity of the insurer's optimal investment strategy π_1^* w.r.t. r.

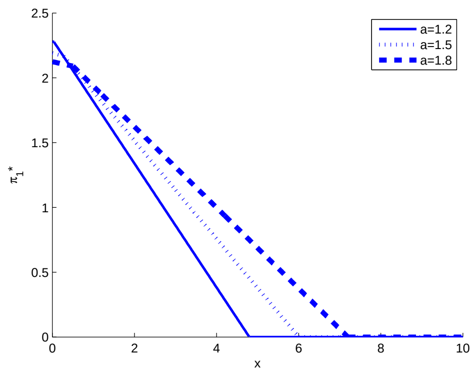

Figure 11: Sensitivity of the insurer's optimal investment strategy π_1^* w.r.t. a.

In Figure 8, we find that the risky asset's appreciation rate μ exerts a positive effect on the insurer's optimal investment strategy π_1^*. The reason is that the risky asset with a larger μ will be more attractive. Therefore, the optimal amount invested in the risky asset raises as μ increases. Figure 9 illustrates the effect of the risky asset's volatility σ on π_1^*. There is a negative relationship between π_1^* and σ. This can be attributed to that a larger σ means more uncertainty of the risky asset's return. Thus the insurer will invest less in the risky asset. As shown by Figure 10, the insurer's optimal investment strategy π_1^* is a decreasing function of the interest rate r. When the interest rate r increases, the risk-free asset is more attractive. Then the insurer will invest more in the risk-free asset and reduce the investment in the risky asset. From Figure 11, we find that there is a positive relationship between the insurer's optimal investment strategy π_1^* and the expectation of claim amount a. This can be explained by the positive relationship between the premium rate and a. As a increases, the premium rate $(1+\theta)a$ increases and the insurer has enough money to invest in the risky asset, thus π_1^* increases.

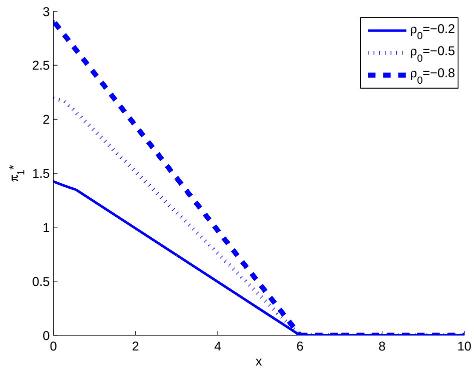

Figure 12: Sensitivity of the insurer's optimal investment strategy π_1^* w.r.t. ρ_0 ($\rho_0 < 0$).

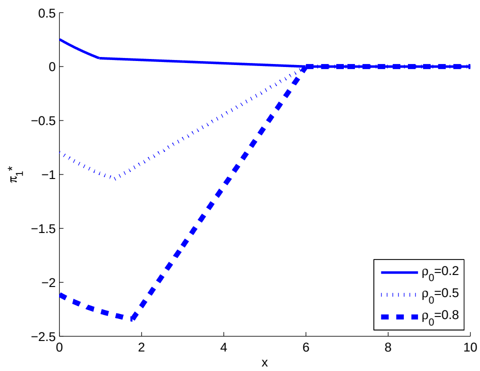

Figure 13: Sensitivity of the insurer's optimal investment strategy π_1^* w.r.t. ρ_0 ($\rho_0 > 0$).

Figures 12 and 13 show the effect of the correlation coefficient ρ_0 on π_1^*. Whether ρ_0 is positive or not, π_1^* decreases with respect to ρ_0. A larger ρ_0 means a stronger correlation between the risk process and the risky asset's price. Thus as ρ_0 increases, the insurer would like to reduce investment in the risky asset to hedge the risk.

5.3 Numerical simulations of the reinsurer's optimal investment strategies under the two cases

In this section, we consider the numerical simulations of the reinsurer's optimal investment strategies under the two cases: maximizing the exponential utility and minimizing ruin probability.

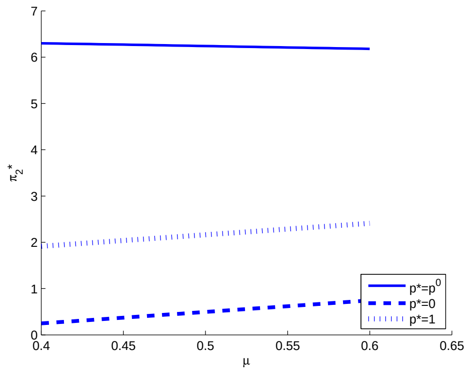

Figure 14: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. μ in the case of maximizing the exponential utility.

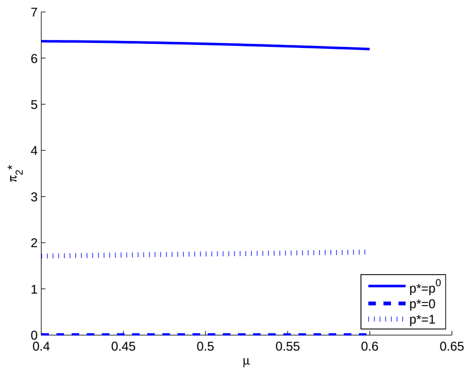

Figure 15: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. μ in the case of minimizing the ruin probability.

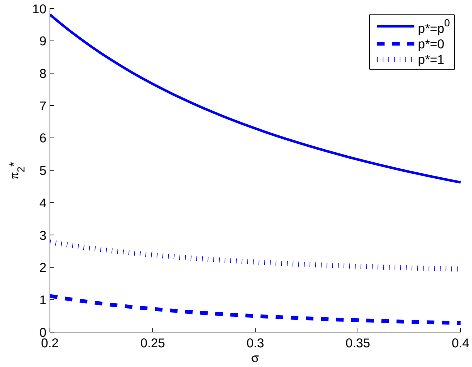

Figure 16: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. σ in the case of maximizing the exponential utility.

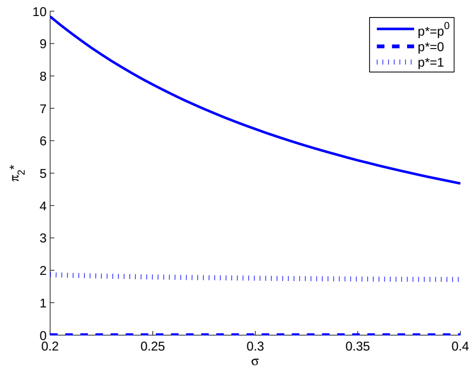

Figure 17: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. σ in the case of minimizing the ruin probability.

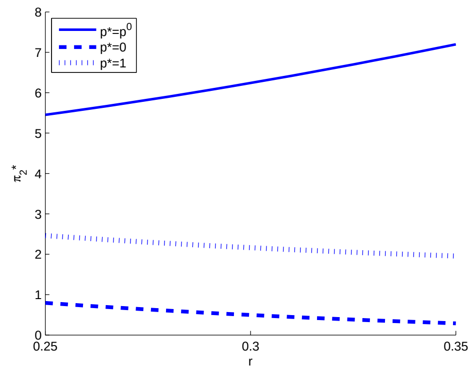

Figure 18: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. r in the case of maximizing the exponential utility.

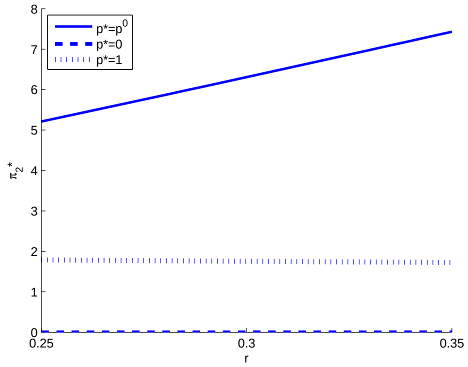

Figure 19: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. r in the case of minimizing the ruin probability.

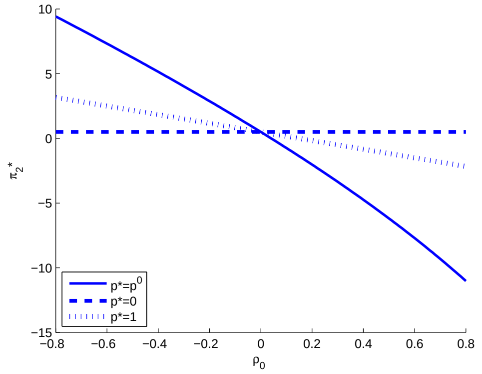

Figure 20: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. ρ_0 in the case of maximizing the exponential utility.

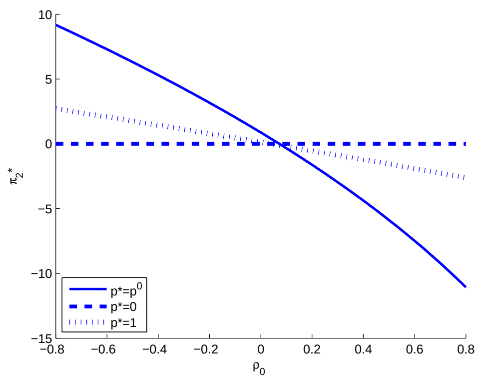

Figure 21: Sensitivity of the reinsurer's optimal investment strategy π_2^* w.r.t. ρ_0 in the case of minimizing the ruin probability.

From Figures 14-21, we find that the effects of μ, σ, r and ρ_0 under the two cases are similar, apart from the case that $p^* = 0$. For the reason that in the case of minimizing ruin probability, if $p^* = 0$, the reinsurer's optimal investment strategy $\pi_2^* = 0$. However, in the case of maximizing the exponential utility, we have $\pi_2^* > 0$ when $p^* = 0$.

From Figures 14 and 15, we see that if $p^* = 1$ or $p^* = 0$, the risky asset's appreciation rate μ exerts a positive effect on the reinsurer's optimal investment strategy π_2^*. In comparison with the case $p^* = 1$ and $p^* = 0$, π_2^* stays stable as μ changes for $p^* = p^0$.

Figures 16 and 17 show that there is a negative relationship between the reinsurer's optimal investment strategy π_2^* and the risky asset's volatility σ, which is consistent with intuition.

In Figures 18 and 19, we find that the reinsurer's optimal investment strategy π_2^* is a decreasing function of r when $p^* = 1$ or $p^* = 0$ while π_2^* increases with r when $p^* = p^0$. This illustrates the intuitive observation that if $p^* = p^0$, p^0 increases with respect to r. Thus, the reinsurer will earn more money from the insurer and prefer to invest more in the risky asset.

Figures 20 and 21 indicate that the higher the correlation coefficient ρ_0, the smaller π_2^* is. According to Figures 9, 12 and 13, both the effects of σ and ρ_0 on the reinsurer's optimal investment strategy are the same as those on the insurer's optimal investment strategy.

6 Conclusion

In this paper, we consider the optimal investment problem for both an insurer and a reinsurer. In our model, the basic claim process is assumed to follow a Brownian motion with drift and the reinsurer can purchase proportional reinsurance from the reinsurer. Both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. The insurer aims to minimize his/her ruin probability and we consider two investment objectives for the reinsurer: maximizing the expected exponential utility of terminal wealth and minimizing the ruin probability. Explicit optimal strategies are obtained via solving the corresponding HJB equation. In particular, we find that optimal investment strategies for the reinsurer under the two cases are equivalent with zero interest rate. Finally, numerical simulations are presented to show the effects of model parameters on the optimal strategies.

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References:

[1] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research,* Vol.20, No.4, 1995, pp. 937–958.

[2] C. Hipp, M. Plum, Optimal investment for insurers, *Insurance: Mathematics and Economics,* Vol.27, No.2, 2000, pp. 215–228.

[3] H. Schmidli, On minimizing the ruin probability by investment and reinsurance, *The Annals of Applied Probability,* Vol.12, No.3, 2002, pp. 797–1137.

[4] H. Yang, L. Zhang, Optimal investment for insurer with jump-diffusion risk process, *Insurance: Mathematics and Economics,* Vol.37, No.3, 2005, pp. 615–634.

[5] D. S. Promislow, V. R. Young, Minimizing the probability of ruin when claims follow Brownian motion with drift, *North American Actuarial Journal,* Vol.9, No.3, 2005, pp. 109–128.

[6] S. Luo, M. Taksar, A. Tsoi, On reinsurance and investment for large insurance portfolios, *Insurance: Mathematics and Economics,* Vol.42, No.1, 2008, pp. 434–444.

[7] L. Bai, J. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, *Insurance: Mathematics and Economics,* Vol.42, No.3, 2008, pp. 968–975.

[8] Y. Cao, N. Wan, Optimal proportional reinsurance and investment based on Hamilton-Jacobi-Bellman equation, *Insurance: Mathematics and Economics,* Vol.45, No.2, 2009, pp. 157–162.

[9] M. Gu, Y. Yang, S. Li, J. Zhang, Constant elasticity of variance model for proportional reinsurance and investment strategies, *Insurance: Mathematics and Economics,* Vol.46, No.3, 2010, pp. 580–587.

[10] H. Zhao, X. Rong, J. Cao, Robust portfolio selection problem for an insurer with exponential utility preference, *WSEAS Transactions on Mathematics,* Vol.10, No.9, 2011, pp. 321–331.

[11] Z. Liang, K. C. Yuen, J. Guo, Optimal proportional reinsurance and investment in a stock market with Ornstein-Uhlenbeck process, *Insurance: Mathematics and Economics,* Vol.49, No.2, 2011, pp. 207–215.

[12] Z. Liang, K. C. Yuen, K. C. Cheung, Optimal reinsurance-investment problem in a constant elasticity of variance stock market for jump-diffusion risk model, *Insurance: Applied Stochastic Models in Business and Industry,* Vol.28, No.6, 2012, pp. 585–597.

[13] X. Lin, Y. Li, Optimal reinsurance and investment for a jump diffusion risk process under the CEV model, *North American Actuarial Journal,* Vol.15, No.3, 2012, pp. 417–431.

[14] A. Gu, X. Guo, Z. Li, Optimal control of excess-of-loss reinsurance and investment for insurers under a CEV model, *Insurance: Mathematics and Economics,* Vol.51, No.3, 2012, pp. 674–684.

[15] Z. Li, Y. Zeng, Y. Lai, Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model, *Insurance: Mathematics and Economics,* Vol.51, No.1, 2012, pp. 191–203.

[16] Y. Li, Z. Li, Optimal time-consistent investment and reinsurance strategies for mean-variance insurers with state dependent risk aversion, *Insurance: Mathematics and Economics,* Vol.53, No.1, 2013, pp. 86–97.

[17] H. Zhao, X. Rong, Y. Zhao, Optimal excess-of-loss reinsurance and investment problem for an insurer with jump-diffusion risk process under the Heston model, *Insurance: Mathematics and Economics,* Vol.53, No.3, 2013, pp. 504–514.

[18] Z. Liang, E. Bayraktar, Optimal reinsurance and investment with unobservable claim size and intensity, *Insurance: Mathematics and Economics,* Vol.55, 2014, pp. 156–166.

[19] J. Zhang, J. Ma, Research on Delayed Complexity Based on Nonlinear Price Game of Insurance Market, *WSEAS Transactions on Mathematics,* Vol.10, No.10, 2011, pp. 368–376.

[20] W. Xu, J. Ma, Study on the dynamic model of a duopoly game with delay in insurance market, *WSEAS Transactions on Mathematics,* Vol.11, No.7, 2012, pp. 599–608.

[21] W. Wang, L. Qian, W. Wang, Hedging strategy for unit-linked life insurance contracts in stochastic volatility models, *WSEAS Transactions on Mathematics,* Vol.12, No.4, 2013, pp. 363–373.