Probability density function of the time to kill for heterogeneous target

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Abstract: In this paper we propose the concepts about homogeneous and heterogeneous target. The homogeneous target is defined as one that hitting the different position of a target causes the same effect, whereas the heterogeneous target is allowed to differ. The large body of work on stochastic duels studied the homogeneous target. But, in many cases, it is unrealistic to assume that a target is homogeneous. We consider the heterogeneous target in this study, and obtain the probability density function of the time to kill. The general solution is obtained in quadrature form and specific solutions are derived for some particular cases. It is illustrated by an example with particular heterogeneous target.

Key–Words: Stochastic duel; Heterogeneous target; Probability density function; Renewal equation; Military operations research; Applied probability

1 Introduction

The stochastic duel problem has been studied extensively in the past. Ancker [1] studied the fundamental one-on-one stochastic duel model, and provided an excellent review of such model. The general two-on-one stochastic duel model was considered by Gafarian and Ancker [2]. They obtained the general solution for the duel state probabilities, and from these derived the two sides winning probabilities. Gafarian and Manion [3] solved the two-on-two stochastic duel model. The state probabilities for three-on-two stochastic duel were derived by Hong [4]. Friedman [5] and Kikuta [6] considered the many-on-one stochastic duel model. They have obtained an optimal firing policy for the single unit side. Subsequently, Kress [7] investigated the general many-on-one stochastic duel conditioned on the order in which targets are attacked. The solution technique for small- to moderate-size firefights has been generalized to solve the general m-on-n stochastic duel model by Parkhideh [8]. Unfortunately, this solution has strong exponential computation time. Yang and Gafarian [9] discussed the computation time of combat statistics in detail.

Markovian property is a common feature of many real world situations. Markovian theory has been used in many field [10, 11, 12]. Adopting a Markovian analytic approach, Koopman [13] addressed the logical basis of combat modeling and provided an example involving a detection and destruction duel. Wand et al. [14] revisited the one-on-one stochastic duel model and took explicit account of detection in their model. In their scenario envisaged, the defender detects the attacker after some random time interval, while the attacker detects the defenders firing signature with some fixed probability after each shot. McNaught [15] considered the Markovian models of three-on-one stochastic duel involving a hidden defender. In his models, the defender detects the exposed attacking group after an exponentially distributed time interval, while each attacker has a fixed probability of detecting the defender via the flash signature produced after each shot fired by him. Liu [16] applied the theory of one-on-one stochastic duel to guerrilla war, and derived the winning probabilities for the both sides.

Although much work has been done for the stochastic duel problem, an important issue was not mentioned in literature and deserves more considerations. Two sides conduct a duel. Each member on one side is referred to as target by anyone else on his opposing side. The targets were supposed to be homogeneous in literature. The homogeneous target is defined as one that hitting the different position of a target causes the same effect, whereas the heterogeneous target is allowed to differ. Hence, for homogeneous target, there is no need to discuss the position being hit. Obviously, in many cases, it is unrealistic to assume that the target is homogeneous. For example, the effect caused by hitting the leg or hitting the heart of a combatant is completely different. Therefore, researching on heterogeneous target is needed. This paper is motivated by this need.

In this paper, heterogeneous target is considered by explicitly decomposing the whole target region into deadly region and non-deadly region. Figure 1 shows a heterogeneous target region R with deadly region R1 and non-deadly region R_2. For the first time we obtain the probability density function of the time to kill for heterogeneous target. The general solution is obtained in quadrature form and specific solutions are derived for some particular cases. An example with particular heterogeneous target is given.

The organization of this paper is as follows. In Section 2 we present the notations and the assumptions. Then, in Section 3, the probability model is developed. We derive the probability density function of the time to kill for heterogeneous target. In Section 4 we present an example. In Section 5, we draw some concluding remarks. The proofs of Lemma 2 and Lemma 3 can be seen in appendix A and appendix B, respectively.

2 Assumptions and Notations

The basic ingredient in the general solution of a combat model is based on each marksman firing at a passive target (one that does not return fire) [17]. Hence, researching on stochastic duel problem with heterogeneous target, we begin with marksman problem.

Two duelists are denoted by A and B. In order to develop an expression for the probability density function of the time for A to kill B, we assume A fires at a passive target B. Furthermore, we consider B to be a heterogeneous target. And we decompose the whole target region R into deadly region R_1 and non-deadly region R_2 as shown in Figure 1. The following relations hold:

$$R_1 \cup R_2 = R, R_1 \cap R_2 = \emptyset.$$

Suppose the duel satisfies all the basic assumptions:

(a) B is a hidden target.

(b) After B being detected, A fires at B until B is killed.

(c) As firing time (that is, the time between rounds) is a random variable (denoted as η) with a known probability density, $f_\eta(t)$. And each firing time is selected from $f_\eta(t)$, independently and at random.

(d) If A hits deadly region R_1, B is killed immediately.

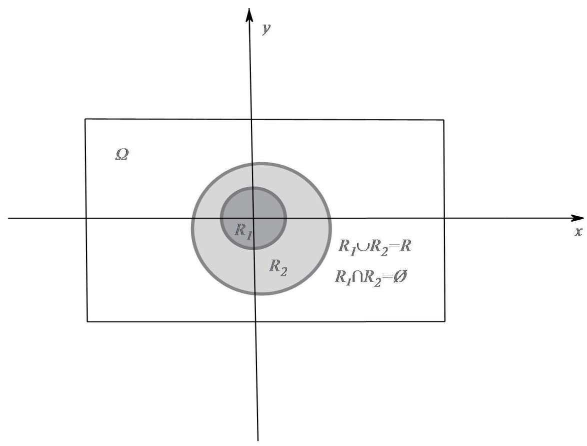

Figure 1: An illustration of deadly region and non-deadly region of the target

(e) If A doesn't hit deadly region R_1, the required number of hitting non-deadly region R_2 to kill B is a discrete random variable (denoted as D) taking possible values $2, 3, 4, \cdots$.

(f) Each time A fires, he has a fixed probability of hitting R_1, and has a fixed probability of hitting R_2. Then each time A fires, he hits B with probability p, where $p = p_1 + p_2$. We denote the probability that B is not hit as q, where $q = 1 - p$.

Other notations which we will use are as follows.

ξ: The time for A to detect B. It is a positive random variable.

$f_\xi(s)$: The probability density function (pdf) of ξ.

η: firing time. The pdf of η is $f_\eta(t)$ (see above (c)). It is a positive random variable.

η_1: The 1st firing time. It is measured from B being detected.

η_i: The ith firing time $(i = 2, 3, \cdots)$. From assumption (c), $\eta_1, \eta_2, \eta_3, \cdots$, are independent and identically distributed with pdf $f_\eta(t)$.

$\Phi_\eta(u)$: The characteristic function of η.

T: The time for A to kill B, measured from the beginning of As searching B.

N: The firing round times of A until killing B. It is a discrete random variable taking possible values $1, 2, 3, \cdots$. And it is realistic to assume that ξ and N are independent.

D: See above (e).

$f_T(t)$: The pdf of T. It is a positive random variable.

$\Phi(u)$: The Fourier transform of $f(x)$. That is,

$$\Phi(u) = \int_{-\infty}^{\infty} f(t) e^{iut} dt.$$

$f^{(n)}(x)$: The n multiple convolution of the positive density function $f(x)$ itself. That is,

$$f^{(n)}(x) = \underbrace{f * f * \cdots * f}_{n}(x),$$

$$f * f(x) = \int_0^x f(x-y)f(y)dy, x \geq 0.$$

3 The Model

Consider the joint probability density function of ξ and T. We have

$$f_{\xi T}(s,t) = f_\xi(s) f_{T|\xi}(t|s), \tag{1}$$

where $f_{T|\xi}(t|s)$ is the conditional density function of T given $\xi = s$ and $f_\xi(s) > 0$.

The probability, $f_{T|\xi}(t|s)dt$, that A takes between time t and $t+dt$ to kill B given $\xi = s$ is

$$\begin{aligned}
& P(t \leq T < t+dt|\xi = s) \\
= \ & P(t \leq T < t+dt, \bigcup_{d=2}^{\infty}(D=d), \\
& \bigcup_{n=1}^{\infty}(N=n)|\xi = s) \\
= \ & \sum_{d=2}^{\infty} P(D=d)P(t \leq T < t+dt, \\
& \bigcup_{n=1}^{\infty}(N=n)|\xi = s, D=d) \\
= \ & \sum_{d=2}^{\infty} P(D=d) \\
& \times \sum_{n=1}^{\infty} P(t \leq T < t+dt, N=n|\xi = s, D=d) \\
= \ & \sum_{d=2}^{\infty} P(D=d) \\
& \times \sum_{n=1}^{\infty} [P(N=n|\xi = s, D=d) \\
& \times P(t \leq T < t+dt|\xi = s, D=d, N=n)].
\end{aligned} \tag{2}$$

Since the density function of As firing time is $f_\eta(t)$, measured from B being detected, the time at which the nth round is fired is the sum of n independent selections from $f_\eta(t)$, so we have

$$\begin{aligned}
& P(t \leq T < t+dt|\xi = s, D=d, N=n) \\
= \ & P(t-s \leq T-s < t-s+dt|\xi = s, \\
& D=d, N=n) \\
= \ & f_\eta^{(n)}(t-s)dt, \qquad t \geq s.
\end{aligned} \tag{3}$$

Also,

$$\begin{aligned}
& P(N=n|\xi = s, D=d) \\
= \ & P(N=n|D=d) \\
= \ & P[(\text{hit } R_1 \text{ onthe } n \text{ thround and} \\
& A \text{ does not hit} R_2), N=n|D=d] \\
& +P[(\text{the number of hitting } R_2 \text{ is } d-1 \\
& \text{on the first } n-1 \text{ rounds and hit } R_2 \text{ on the} \\
& n\text{th round and } A \text{ does not hit } R_1), \\
& N=n|D=d] \\
& + \sum_{j=1}^{d-1} P[(\text{the number of hitting } R_2 \text{ is } j \\
& \text{on the first } n-1 \text{ rounds and hit } R_1 \\
& \text{on the } n \text{ th round}), N=n|D=d],
\end{aligned} \tag{4}$$

where

$$P\left[\left(\begin{array}{c}\text{hit } R_1 \text{ on the } n\text{th round} \\ \text{and } A \text{ does not hit } R_2\end{array}\right), N=n|D=d\right]$$
$$= q^{n-1}p_1, n = 1,2,3,\cdots. \tag{5}$$

$$\begin{aligned}
& P[(\text{the number of hitting } R_2 \text{ is } d-1 \\
& \text{on the first } d-1 \ n-1 \text{ roundsandhit } R_2 \\
& \text{on the } n \text{ th round and } A \text{ does not hit } R_1), \\
& N=n|D=d] \\
= \ & \binom{n-1}{d-1} p_2^{d-1} q^{n-d} p_2, \\
& n = d, d+1, d+2, \cdots.
\end{aligned} \tag{6}$$

$$\begin{aligned}
& P[(\text{the number of hitting } R_2 \text{ is } j \\
& \text{on the first } n-1 \text{ rounds and hit } R_1 \\
& \text{on the } n \text{ th round}), N=n|D=d] \\
= \ & \binom{n-1}{j} p_2^j q^{n-1-j} p_1 \\
& j = 1,2,3,\cdots,d-1, \\
& n = j+1, j+2, j+3, \cdots.
\end{aligned} \tag{7}$$

From (3), (4), (5), (6) and (7) we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} P(t \leq T < t+dt, N=n|\xi = s, D=d) \\
= \ & \sum_{n=1}^{\infty} P(N=n|\xi = s, D=d)P(t \leq \\
& T < t+dt|\xi = s, D=d, N=n) \\
= \ & \sum_{n=1}^{\infty} q^{n-1}p_1 f_\eta^{(n)}(t-s)dt \\
& + \sum_{n=d}^{\infty} \binom{n-1}{d-1} p_2^{d-1} q^{n-d} p_2 f_\eta^{(n)}(t-s)dt \\
& + \sum_{j=1}^{d-1} \sum_{n=j+1}^{\infty} \binom{n-1}{j} p_2^j q^{n-1-j} p_1 f_\eta^{(n)}(t-s)dt \\
& t \geq s.
\end{aligned} \tag{8}$$

Let

$$\phi_1(t-s|d)=\sum_{n=1}^{\infty}q^{n-1}p_1f_\eta^{(n)}(t-s), \quad (9)$$

$$\begin{aligned}&\phi_2(t-s|d)\\ =\ &\sum_{n=d}^{\infty}\begin{pmatrix}n-1\\ d-1\end{pmatrix}p_2^{d-1}q^{n-d}p_2f_\eta^{(n)}(t-s), \quad (10)\end{aligned}$$

$$\begin{aligned}&\phi_{3,j}(t-s|d)\\ =\ &\sum_{n=j+1}^{\infty}\begin{pmatrix}n-1\\ j\end{pmatrix}p_2^{j}q^{n-1-j}p_1f_\eta^{(n)}(t-s),\\ &j=1,2,3,\cdots,d-1. \quad (11)\end{aligned}$$

Then from (8)

$$\begin{aligned}&\sum_{n=1}^{\infty}P(t\le T<t+dt,N=n|\xi=s,D=d)\\ =\ &\phi_1(t-s|d)dt+\phi_2(t-s|d)dt\\ &+\sum_{j=1}^{d-1}\phi_{3,j}(t-s|d)dt.\end{aligned} \quad (12)$$

Thus, from (2) and (12) we have

$$\begin{aligned}&P(t\le T<t+dt|\xi=s)\\ =\ &\sum_{d=2}^{\infty}P(D=d)[\phi_1(t-s|d)dt+\phi_2(t-s|d)dt\\ &+\sum_{j=1}^{d-1}\phi_{3,j}(t-s|d)dt].\end{aligned}$$

That is

$$\begin{aligned}&f_{T|\xi}(t|s)\\ =\ &\sum_{d=2}^{\infty}P(D=d)[\phi_1(t-s|d)+\phi_2(t-s|d)\\ &+\sum_{j=1}^{d-1}\phi_{3,j}(t-s|d)],t\ge s.\end{aligned} \quad (13)$$

In order to obtain an expression of relatively easy computation for the pdf of T, we shall establish several lemmas.

Lemma 1 *Given $\xi=s$ and $D=d$, the Fourier transform of $\phi_1(t-s|d)$ is*

$$\Phi_1(u|d)=\frac{p_1\Phi_\eta(u)}{1-q\Phi_\eta(u)}. \quad (14)$$

Proof: Let $\tau=t-s$, then, from (9)

$$\phi_1(\tau|d)=\sum_{n=1}^{\infty}q^{n-1}p_1f_\eta^{(n)}(\tau). \quad (15)$$

From the convolution property of Fourier transform, (15) may be transformed into

$$\Phi_1(u|d)=\sum_{n=1}^{\infty}q^{n-1}p_1[\Phi_\eta(u)]^n,$$

also, $|q\Phi_\eta(u)|<1$. Hence,

$$\Phi_1(u|d)=\frac{p_1\Phi_\eta(u)}{1-q\Phi_\eta(u)}.$$

□

Lemma 2 *Given $\xi=s$ and $D=d$, the Fourier transform of $\phi_2(t-s|d)$ is*

$$\Phi_2(u|d)=[\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^d. \quad (16)$$

The proof can be seen in Appendix A.

Lemma 3 *Given $\xi=s$ and $D=d$, the Fourier transform of $\phi_{3,j}(t-s|d)$ is*

$$\begin{aligned}&\Phi_{3,j}(u|d)=p_1p_2^j[\tfrac{\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^{j+1}\\ &j=1,2,3,\cdots,d-1.\end{aligned} \quad (17)$$

Theorem 4 *The characteristic function and the density function for the time for A to kill his heterogeneous target B are, respectively,*

$$\begin{aligned}\Phi_T(u)=\ &\Phi_\xi(u)\Big\{\frac{p_1\Phi_\eta(u)}{1-q\Phi_\eta(u)}\\ &+\sum_{d=2}^{+\infty}P(D=d)[\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^d\\ &+\sum_{d=2}^{+\infty}P(D=d)\sum_{j=1}^{d-1}p_1p_2^j[\frac{\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^{j+1}\Big\}, \quad (18)\end{aligned}$$

$$\begin{aligned}f_T(t)=\ &\frac{1}{2\pi}\sum_{d=2}^{+\infty}P(D=d)\int_{-\infty}^{+\infty}e^{-iut}\Phi_\xi(u)\\ &\Big\{\frac{p_1\Phi_\eta(u)}{1-q\Phi_\eta(u)}+[\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^d\\ &+\sum_{j=1}^{d-1}p_1p_2^j[\frac{\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^{j+1}\Big\}du,t\ge 0. \quad (19)\end{aligned}$$

Proof: Since,

$$f_T(t)=\int_{-\infty}^{\infty}f_{\xi T}(s,t)ds=\int_{-\infty}^{\infty}f_\xi(s)f_{T|\xi}(t|s)ds, \quad (20)$$

substituting (13) into (20), we have

$$f_T(t) = \sum_{d=2}^{\infty} P(D=d) \int_{-\infty}^{\infty} f_\xi(s)[\phi_1(t-s|d) + \phi_2(t-s|d) + \sum_{j=1}^{d-1} \phi_{3,j}(t-s|d)]ds. \quad (21)$$

Notice that $\phi_1(t-s|d)$, $\phi_2(t-s|d)$, and $\phi_{3,j}(t-s|d)$ are the function of single variable $t-s$. We have

$$\int_{-\infty}^{\infty} f_\xi(s)\phi_1(t-s|d)ds = f_\xi * \phi_1(t),$$

$$\int_{-\infty}^{\infty} f_\xi(s)\phi_2(t-s|d)ds = f_\xi * \phi_2(t),$$

and

$$\int_{-\infty}^{\infty} f_\xi(s)\phi_{3,j}(t-s|d)ds = f_\xi * \phi_{3,j}(t).$$

Then, taking Fourier transform on both sides of (21), from lemma 1, lemma2, lemma3, and the convolution property of Fourier transform, (18) is followed. Also,

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_T(u)e^{-iut}du, \quad (22)$$

substituting (18) into (22), (19) follows and the theorem is proved.

Corollary 5 *If $p_2 = 0$, i.e. $p_1 + q = 1$, the following results can be derived,*

$$\Phi_T(u) = \frac{p_1\Phi_\xi(u)\Phi_\eta(u)}{1 - q\Phi_\eta(u)}, \quad (23)$$

and

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}\frac{p_1\Phi_\xi(u)\Phi_\eta(u)}{1 - q\Phi_\eta(u)}du, t \geq 0. \quad (24)$$

If $p_2 = 0$, it is implied that the target B can be regarded as homogeneous, and B will be killed immediately as long as he is hit. In this particular case, (23) and (24) are the general solutions for the characteristic function and the density function of the time to kill.

Corollary 6 *If $p_1 = 0$, i.e. $p_2 + q = 1$, the following results can be derived,*

$$\Phi_T(u) = \sum_{d=2}^{+\infty} P(D=d)\Phi_\xi(u)[\frac{p_2\Phi_\eta(u)}{1 - q\Phi_\eta(u)}]^d \quad (25)$$

and

$$f_T(t) = \frac{1}{2\pi}\sum_{d=2}^{+\infty} P(D=d) \int_{-\infty}^{+\infty} e^{-iut}\Phi_\xi(u) [\frac{p_2\Phi_\eta(u)}{1 - q\Phi_\eta(u)}]^d du, t \geq 0. \quad (26)$$

If $p_1 = 0$, it is implied that the target B can be regarded as homogeneous too, but the required times of hitting R_2 to kill B are a discrete random variable taking possible values $2, 3, 4, \cdots$. In this case, the general solutions for the characteristic function and the density function of the time to kill are (25) and (26).

Corollary 7 *If $P(D = 2) = 1$, the following results can be derived,*

$$\Phi_T(u) = \Phi_\xi(u)\Big\{\frac{p_1\Phi_\eta(u)}{1-q\Phi_\eta(u)} + [\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^2 + p_1p_2[\frac{\Phi_\eta(u)}{1 - q\Phi_\eta(u)}]^2\Big\}, \quad (27)$$

and for $t \geq 0$,

$$f_T(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}\Phi_\xi(u)\{\frac{p_1\Phi_\eta(u)}{1-q\Phi_\eta(u)} + [\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^2 + p_1p_2[\frac{\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^2\}du. \quad (28)$$

When $P(D = 2) = 1$, we are told that the target B may be heterogeneous, and that B will be killed as long as A hits R_1, or if A doesn't hit R_1, B will be killed when R_2 is hit twice. (27) and (28) are the solutions for this particular case.

4 Example

Suppose that the random variables ξ and η obey exponential distribution. That is

$$f_\xi(s) = \begin{cases} \lambda e^{-\lambda s} & s \geq 0, \\ 0 & s < 0, \end{cases}$$

$$f_\eta(r) = \begin{cases} \beta e^{-\beta r} & r \geq 0, \\ 0 & r < 0 \end{cases}$$

where λ and β are positive constant. The characteristic functions of ξ and η are, respectively,

$$\Phi_\xi(u) = \frac{\lambda}{\lambda - iu}, \qquad \Phi_\eta(u) = \frac{\beta}{\beta - iu}.$$

Furthermore, assume that $P(D = 2) = 1$. We will calculate the characteristic function and the density function of the random variable T by corollary 7. From (27), we have

$$\Phi_T(u) = \frac{\lambda}{\lambda - iu}\Big\{\frac{p_1\beta}{(1-q)\beta - iu} + [\frac{p_2\beta}{(1-q)\beta - iu}]^2 + p_1p_2[\frac{\beta}{(1-q)\beta - iu}]^2\Big\}$$
$$= \frac{\lambda}{\lambda - iu}\Big[\frac{p_1\beta}{p\beta - iu} + (\frac{p_2\beta}{p\beta - iu})^2$$

$$+p_1p_2(\frac{\beta}{p\beta - iu})^2\Big], \tag{29}$$

Using (28) we have

$$\begin{aligned} f_T(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}\frac{\lambda}{\lambda - iu}\Big[\frac{p_1\beta}{p\beta - iu} \\ &+(\frac{p_2\beta}{p\beta - iu})^2 + p_1p_2(\frac{\beta}{p\beta - iu})^2\Big]du \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}\frac{i\lambda}{u+i\lambda}[\frac{ip_1\beta}{u+ip\beta} \\ &+(\frac{ip_2\beta}{u+ip\beta})^2 + p_1p_2(\frac{i\beta}{u+ip\beta})^2]du. \end{aligned} \tag{30}$$

When $\lambda \neq p\beta$, applying the residue theorem we have

$$\begin{aligned} & f_T(t) \\ = \; & \frac{\lambda p_1\beta e^{-p\beta t}(1-e^{-(\lambda - p\beta)t})}{\lambda - p\beta} \\ & +\frac{\lambda p_2^2\beta^2}{(\lambda - p\beta)^2}\{e^{-\lambda t} + [(\lambda - p\beta)t - 1]e^{-p\beta t}\} \\ & +\frac{\lambda p_1p_2\beta^2}{(\lambda - p\beta)^2}\{e^{-\lambda t} + [(\lambda - p\beta)t - 1]e^{-p\beta t}\} \\ = \; & \frac{p_1\lambda\beta}{\lambda - p\beta}e^{-p\beta t}(1-e^{-(\lambda - p\beta)t}) \\ & +\frac{pp_2\lambda\beta^2}{(\lambda - p\beta)^2}e^{-p\beta t}[e^{-(\lambda - p\beta)t} + (\lambda - p\beta)t - 1]. \end{aligned} \tag{31}$$

When $\lambda = p\beta$, from (30) it follows that

$$\begin{aligned} & f_T(t) \\ = \; & \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}[\frac{-p_1\lambda\beta}{(u+ip\beta)^2} + \frac{-ip_2\lambda\beta}{(u+ip\beta)^3} \\ & +\frac{-ip_1p_2\lambda\beta}{(u+ip\beta)^3}]du. \end{aligned}$$

Applying the residue theorem we have

$$\begin{aligned} & f_T(t) \\ = \; & \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}[\frac{-p_1\lambda\beta}{(u+ip\beta)^2} + \frac{-ip_2\lambda\beta}{(u+ip\beta)^3} \\ & +\frac{-ip_1p_2\lambda\beta}{(u+ip\beta)^3}]du. \end{aligned} \tag{32}$$

From (31) and (32) we thus obtain for $t \geq 0$,

$$f_T(t) = \begin{cases} \frac{p_1\lambda\beta}{\lambda - p\beta}e^{-p\beta t}(1-e^{-(\lambda - p\beta)t}) \\ +\frac{pp_2\lambda\beta^2}{(\lambda - p\beta)^2}e^{-p\beta t} \\ \times[e^{-(\lambda - p\beta)t} + (\lambda - p\beta)t - 1] \\ \quad \text{if} \quad \lambda \neq p\beta \\ p_1\lambda\beta t e^{-p\beta t} + \frac{1}{2}pp_2\lambda\beta^2t^2e^{-p\beta t} \\ \quad \text{if} \quad \lambda = p\beta \end{cases} \tag{33}$$

It is readily demonstrated that $f_T(t)$ has the following properties:

(i): $f_T(t) \geq 0$,

(ii): $\int_{-\infty}^{\infty} f_T(t)dt = 1$.

Proof of property (i)**.** Let $\theta = \lambda - p\beta$. When $\theta = 0$, the result is obvious.
When $\theta \neq 0$, from (33) we have

$$\begin{aligned} f_T(t) = \; & \frac{p_1\lambda\beta}{\theta}e^{-p\beta t}(1-e^{-\theta t}) \\ & +\frac{pp_2\lambda\beta^2}{\theta^2}e^{-p\beta t}(e^{-\theta t} + \theta t - 1). \end{aligned}$$

Let

$$g_1(t) = \frac{1}{\theta}(1-e^{-\theta t}), t \geq 0.$$

When $\theta > 0$ and $1-e^{-\theta t} \geq 0$, then $g_1(t) \geq 0$. When $\theta < 0$, $1-e^{-\theta t} \leq 0$, it also has $g_1(t) \geq 0$. Hence, when $\theta \neq 0$ and $t \geq 0$, we have $g_1(t) \geq 0$. Let $g_2(t) = e^{-\theta t} + \theta t - 1, t \geq 0$. Since

$$\frac{dg_2(t)}{dt} = -\theta e^{-\theta t} + \theta,$$

letting $\frac{dg_2(t)}{dt} = 0$, we have the solution $t = 0$. Also, $\frac{d^2g_2(t)}{dt^2} = \theta^2e^{-\theta t} > 0$. This implies that $g_2(t)$ is a strictly lower convex function and that $t = 0$ is the minimum point. So we have $g_2(t) \geq g_2(0)$, i.e. $g_2(t) \geq 0$. Since $g_1(t) \geq 0$ and $g_2(t) \geq 0$, the result follows immediately.

5 Conclusion

The concepts of homogeneous and heterogeneous target are proposed in this study. In many cases, it is unrealistic to assume that a target is homogeneous. Although much work has been done for the stochastic duel problem, the study on heterogeneous target was not considered in literature. Some important statistics such as the mean time to kill the target, or the winning probabilities for the both sides can be obtained by making use of the pdf of the time to kill. Therefore, this paper shows how to find the pdf of the time to kill for heterogeneous target. This is achieved by using renewal theoretic approach. The general solution is obtained in quadrature form and specific solutions are derived for some particular cases. It is illustrated by an example with particular heterogeneous target.

Appendix A

Proof of Lemma 2

Because $\eta_1, \eta_2, \eta_3, \cdots$, are independent and identically distributed with pdf $f_\eta(t)$, the stochastic process of the number of shots is a renewal process. Using renewal equation we will show that lemma 6 holds.

Given $\xi = s$ and $D = d$, let $W_n|_{\xi=s,D=d}$ be the waiting time until the nth shoot. Obviously,

$$W_n|_{\xi=s,D=d} = s + \sum_{i=1}^{n} \eta_i.$$

Let

$$\begin{aligned} & f_n^0(t|s,d)dt \\ = \; & P\begin{pmatrix} t \le W_n|_{\xi=s,D=d} < t+dt \\ \text{and } A \text{ doesn't hit } R \text{ on the } n \text{ rounds} \end{pmatrix} \\ = \; & q^n f_\eta^{(n)}(t-s)dt, n = 1, 2, \cdots, \end{aligned}$$

and

$$\begin{aligned} & f_n^j(t|s,d)dt \\ = \; & P\begin{pmatrix} t \le W_n|_{\xi=s,D=d} < t+dt \\ \text{and the number of hitting } R_2 \\ \text{is } j \text{ on the } n \text{ rounds and} \\ A\text{doesn't hit } R \text{ on the rest} \\ \text{of rounds} \end{pmatrix} \\ = \; & \begin{pmatrix} n \\ \mathrm{j} \end{pmatrix} p_2^j q^{n-j} f_\eta^{(n)}(t-s)dt \\ & j = 1, 2, \cdots, d-1, n = j, j+1, \cdots, \end{aligned}$$

and

$$\begin{aligned} & f_n^d(t|s,d)dt \\ = \; & P\begin{pmatrix} t \le W_n|_{\xi=s,D=d} < t+dt \text{ and} \\ \text{the number of hitting } R_2 \\ \text{is } d-1 \text{ on the first } n-1 \\ \text{rounds and hit } R_2 \text{ on the} \\ n\text{th round then } B \text{ is killed} \\ \text{and } A \text{ doesn't hit } R \\ \text{on the rest of rounds} \end{pmatrix} \\ = \; & \begin{pmatrix} n-1 \\ d-1 \end{pmatrix} p_2^d q^{n-d} f_\eta^{(n)}(t-s)dt, \\ & n = d, d+1, \cdots. \end{aligned}$$

Then it holds that

$$f_n^0(t|s,d) = q^n f_\eta^{(n)}(t-s), n = 1, 2, \cdots,$$

$$f_n^j(t|s,d) = \begin{pmatrix} n \\ j \end{pmatrix} p_2^j q^{n-j} f_\eta^{(n)}(t-s),$$
$$j = 1, 2, \cdots, d-1, n = j, j+1, \cdots,$$

$$f_n^d(t|s,d) = \begin{pmatrix} n-1 \\ d-1 \end{pmatrix} p_2^d q^{n-d} f_\eta^{(n)}(t-s),$$
$$n = d, d+1, \cdots.$$

Notice that

$$f_n^j(t|s,d), j = 0, 1, 2, \cdots, d,$$

are the function of single variable $t-s$. Denote $f_n^j(t|s,d)$ by $f_n^j(t-s|d)$, i.e.,

$$f_n^j(t-s|d) = f_n^j(t|s,d), j = 0, 1, 2, \cdots, d.$$

These functions

$$f_n^j(t-s|d), j = 0, 1, 2, \cdots, d$$

satisfy the following renewal equations.

1) $f_n^0(t-s|d) = q \int\limits_0^{t-s} f_\eta(y) f_{n-1}^0(t-s-y|d)dy$, with $t-s \ge 0$,$n = 2, 3, 4, \cdots$, where$f_1^0(t-s|d) = qf_\eta(t-s)$.

2) $f_n^j(t-s|d) = p_2 \int\limits_0^{t-s} f_\eta(y) f_{n-1}^{j-1}(t-s-y|d)dy + q \int\limits_0^{t-s} f_\eta(y) f_{n-1}^j(t-s-y|d)dy$, $t-s \ge 0$, $j = 1, 2, \cdots, d-1, n = j, j+1, \cdots$, where $f_1^1(t-s|d) = p_2 f_\eta(t-s)$.

3) $f_n^d(t-s|d) = p_2 \int\limits_0^{t-s} f_\eta(y) f_{n-1}^{d-1}(t-s-y|d)dy$, $n = d, d+1, \cdots$.

Furthermore, let

$$f^0(t-s|d) = \sum_{n=1}^{+\infty} f_n^0(t-s|d),$$

$$f^1(t-s|d) = \sum_{n=1}^{+\infty} f_n^1(t-s|d),$$

$$f^j(t-s|d) = \sum_{n=j}^{+\infty} f_n^j(t-s|d), 2 \le j \le d-1,$$

$$f^d(t-s|d) = \sum_{n=d}^{+\infty} f_n^d(t-s|d).$$

Using the renewal equations we have, for $t \ge s$,

$$\begin{aligned} & f^0(t-s|d) \\ = \; & f_1^0(t-s|d) + \sum_{n=2}^{+\infty} f_n^0(t-s|d) \\ = \; & qf_\eta(t-s) \\ & +q\int_0^{t-s} f_\eta(y)[\sum_{n=2}^{+\infty} f_{n-1}^0(t-s-y|d)]dy \end{aligned}$$

$$\begin{aligned} &= q f_\eta(t-s) \\ &\quad + q\int_0^{t-s} f_\eta(y) f^0(t-s-y|d)dy; \end{aligned} \qquad (34)$$

$$\begin{aligned} & f^1(t-s|d) \\ &= f_1^1(t-s|d) + \sum_{n=2}^{+\infty} f_n^1(t-s|d) \\ &= p_2 f_\eta(t-s) \\ &\quad + p_2\int_0^{t-s} f_\eta(y)\sum_{n=2}^{+\infty} f_{n-1}^0(t-s-y|d)dy \\ &\quad + q\int_0^{t-s} f_\eta(y)\sum_{n=2}^{+\infty} f_{n-1}^1(t-s-y|d)dy \\ &= p_2 f_\eta(t-s) \\ &\quad + p_2\int_0^{t-s} f_\eta(y) f^0(t-s-y|d)dy \\ &\quad + q\int_0^{t-s} f_\eta(y) f^1(t-s-y|d)dy; \end{aligned} \qquad (35)$$

$$\begin{aligned} & f^j(t-s|d) \\ &= f_j^j(t-s|d) + \sum_{n=2}^{+\infty} f_n^j(t-s|d) \\ &= p_2^j f_\eta^{(j)}(t-s) \\ &\quad + \sum_{n=j+1}^{+\infty} [p_2\int_0^{t-s} f_\eta(y) f_{n-1}^{j-1}(t-s-y|d)dy \\ &\quad + q\int_0^{t-s} f_\eta(y) f_{n-1}^{j}(t-s-y|d)dy] \\ &= p_2^j f_\eta^{(j)}(t-s) \\ &\quad + p_2\int_0^{t-s} f_\eta(y)\sum_{n=j+1}^{+\infty} f_{n-1}^{j-1}(t-s-y|d)dy \\ &\quad + q\int_0^{t-s} f_\eta(y)\sum_{n=j+1}^{+\infty} f_{n-1}^{j}(t-s-y|d)dy \\ &= p_2^j f_\eta^{(j)}(t-s) \\ &\quad + p_2\int_0^{t-s} f_\eta(y) f^{j-1}(t-s-y|d)dy \\ &\quad - p_2\int_0^{t-s} f_\eta(y) f_{j-1}^{j-1}(t-s-y|d)dy \\ &\quad + q\int_0^{t-s} f_\eta(y) f^{j}(t-s-y|d)dy \\ &= p_2\int_0^{t-s} f_\eta(y) f^{j-1}(t-s-y|d)dy \\ &\quad + q\int_0^{t-s} f_\eta(y) f^{j}(t-s-y|d)dy, \\ &\quad t \ge s, 2 \le j \le d-1, \end{aligned} \qquad (36)$$

where

$$\begin{aligned} & p_2\int_0^{t-s} f_\eta(y) f_{j-1}^{j-1}(t-s-y|d)dy \\ &= p_2\int_0^{t-s} f_\eta(y) p_2^{j-1} f_\eta^{(j-1)}(t-s-y)dy \\ &= p_2^j f_\eta^{(j)}(t-s), \end{aligned}$$

and

$$\begin{aligned} & f^d(t-s|d) \\ &= p_2\int_0^{t-s} f_\eta(y)\sum_{n=d}^{+\infty} f_{n-1}^{d-1}(t-s-y|d)dy \\ &= p_2\int_0^{t-s} f_\eta(y) f^{d-1}(t-s-y|d)dy. \end{aligned} \qquad (37)$$

We emphasize that $\phi_2(t-s|d) = f^d(t-s|d)$. Taking Fourier transform on both sides of (34), (35), (36), and (37), applying the property of convolution we obtain

$$\Phi^0(u|d) = q\Phi_\eta(u) + q\Phi_\eta(u)\Phi^0(u|d), \qquad (38)$$

$$\begin{aligned} \Phi^1(u|d) &= p_2\Phi_\eta(u) + p_2\Phi_\eta(u)\Phi^0(u|d) \\ &\quad + q\Phi_\eta(u)\Phi^1(u|d), \end{aligned} \qquad (39)$$

$$\begin{aligned} \Phi^j(u|d) &= p_2\Phi_\eta(u)\Phi^{j-1}(u|d) + q\Phi_\eta(u)\Phi^j(u|d), \\ & 2 \le j \le d-1, \end{aligned} \qquad (40)$$

$$\Phi^d(u|d) = p_2\Phi_\eta(u)\Phi^{d-1}(u|d), \qquad (41)$$

where the Fourier transform of $f^j(t-s|d)$, $j = 0, 1, 2, \cdots, d$ is denoted by $\Phi^j(u|d)$.

From (38),(39),and (40) we derive

$$\Phi^0(u|d) = \frac{q\Phi_\eta(u)}{1 - q\Phi_\eta(u)}, \qquad (42)$$

$$\Phi^1(u|d) = \frac{p_2\Phi_\eta(u)[1 + \Phi^0(u|d)]}{1 - q\Phi_\eta(u)}, \qquad (43)$$

$$\Phi^j(u|d) = \frac{p_2\Phi_\eta(u)\Phi^{j-1}(u|d)]}{1 - q\Phi_\eta(u)}, 2 \le j \le d-1 \qquad (44)$$

Substituting (42) into (43) we have

$$\Phi^1(u|d) = \frac{p_2\Phi_\eta(u)}{[1 - q\Phi_\eta(u)]^2}. \qquad (45)$$

Substituting (45) into (44) we have

$$\Phi^2(u|d) = \frac{[p_2\Phi_\eta(u)]^2}{[1 - q\Phi_\eta(u)]^3}.$$

In this way, we obtain

$$\Phi^{d-1}(u|d) = \frac{[p_2\Phi_\eta(u)]^{d-1}}{[1 - q\Phi_\eta(u)]^d}. \qquad (46)$$

Substituting (46) into (41) we derive

$$\Phi^d(u|d) = [\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^d. \tag{47}$$

Since $\phi_2(t-s|d) = f^d(t-s|d)$ and the Fourier transform of $f^d(t-s|d)$ is $\Phi^d(u|d)$, we have

$$\Phi_2(u|d) = [\frac{p_2\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^d,$$

which proves the Lemma 2.

Appendix B

Proof of Lemma 3

The proof of Lemma 3 will be constructed along the same lines as that given above for Lemma 2. In this appendix, there are some notations that have appeared in appendix A, but they may have distinct content.

Given $\xi = s$ and $D = d$, let $W_n|_{\xi=s,D=d}$ be the waiting time until the nth shoot. Obviously,

$$W_n|_{\xi=s,D=d} = s + \sum_{i=1}^{n} \eta_i.$$

Let

$$\begin{aligned} & f_n^0(t|s,d)dt \\ = \ & P\left(\begin{array}{c} t \le W_n|_{\xi=s,D=d} < t+dt \\ \text{and } A \text{ doesn't hit } R \text{ on the } n \text{ rounds} \end{array}\right) \\ = \ & q^n f_\eta^{(n)}(t-s)dt, n = 1,2,\cdots, \end{aligned}$$

$$\begin{aligned} & f_n^k(t|s,d)dt \\ = \ & P\left(\begin{array}{l} t \le W_n|_{\xi=s,D=d} < t+dt \\ \text{and the number of hitting } R_2 \\ \quad \text{is } k \text{ on the } n \text{ rounds and} \\ \quad A \text{doesn't hit } R \text{ on the rest} \\ \quad \text{of rounds} \end{array}\right) \\ = \ & \left(\begin{array}{c} n \\ k \end{array}\right) p_2^k q^{n-k} f_\eta^{(n)}(t-s)dt \\ & k = 1,2,\cdots j, n = k, k+1, k+2\cdots, \end{aligned}$$

$$\begin{aligned} & f_n^{j+1}(t|s,d)dt \\ = \ & P\left(\begin{array}{l} t \le W_n|_{\xi=s,D=d} < t+dt \text{ and} \\ \text{the number of hitting } R_2 \\ \text{is } j \text{ on the first } n-1 \\ \text{rounds and hit } R_2 \text{ on the} \\ n\text{th round then } B \text{ is killed} \\ \quad \text{and } A \text{ doesn't hit } R \\ \quad \text{on the rest of rounds} \end{array}\right) \\ = \ & \left(\begin{array}{c} n-1 \\ j \end{array}\right) p_2^j q^{n-j-1} p_1 f_\eta^{(n)}(t-s)dt, \\ & n = j+1, j+2, \cdots. \end{aligned}$$

Then

$$f_n^0(t|s,d) = q^n f_\eta^{(n)}(t-s), n = 1,2,\cdots,$$

$$f_n^k(t|s,d) = \left(\begin{array}{c} n \\ j \end{array}\right) p_2^k q^{n-k} f_\eta^{(n)}(t-s),$$

$$k = 1,2,\cdots,j, n = k, k+1,\cdots,$$

$$f_n^{j+1}(t|s,d) = \left(\begin{array}{c} n-1 \\ j \end{array}\right) p_2^j q^{n-1-j} p_1 f_\eta^{(n)}(t-s),$$

$$n = j+1, j+2, \cdots.$$

Notice that

$$f_n^k(t|s,d), k = 0,1,2,\cdots,j+1,$$

are the function of single variable $t-s$. Denote $f_n^k(t|s,d)$ by $f_n^k(t-s|d)$, i.e.,

$$f_n^k(t-s|d) = f_n^k(t|s,d), k = 0,1,2,\cdots,j+1.$$

The functions $f_n^k(t-s|d)$, $k = 0,1,2,\cdots,j+1$, satisfy the following renewal equations.

1) $f_n^0(t-s|d) = q\int_0^{t-s} f_\eta(y) f_{n-1}^0(t-s-y|d)dy$, $t-s \ge 0, n = 2,3,4,\cdots$, where $f_1^0(t-s|d) = qf_\eta(t-s)$.

2) $f_n^k(t-s|d) = p_2\int_0^{t-s} f_\eta(y) f_{n-1}^{k-1}(t-s-y|d)dy + q\int_0^{t-s} f_\eta(y) f_{n-1}^k(t-s-y|d)dy$,$t-s \ge 0$, $k = 1,2,\cdots,j, n = k, k+1,\cdots$, where $f_1^1(t-s|d) = p_2 f_\eta(t-s)$.

3) $f_n^{j+1}(t-s|d) = p_1\int_0^{t-s} f_\eta(y) f_{n-1}^j(t-s-y|d)dy$, $n = j+1, j+2,\cdots$.

Furthermore, let

$$f^0(t-s|d) = \sum_{n=1}^{+\infty} f_n^0(t-s|d),$$

$$f^1(t-s|d) = \sum_{n=1}^{+\infty} f_n^1(t-s|d),$$

$$f^k(t-s|d) = \sum_{n=k}^{+\infty} f_n^k(t-s|d), k = 2,3,\cdots,j,$$

$$f^{j+1}(t-s|d) = \sum_{n=j+1}^{+\infty} f_n^{j+1}(t-s|d).$$

Using the renewal equations we have

$$\begin{aligned}
& f^0(t-s|d) \\
= \; & f_1^0(t-s|d) + \sum_{n=2}^{+\infty} f_n^0(t-s|d) \\
= \; & q f_\eta(t-s) + q\int_0^{t-s} f_\eta(y) \\
& [\sum_{n=2}^{+\infty} f_{n-1}^0(t-s-y|d)]dy \\
= \; & q f_\eta(t-s) \\
& + q\int_0^{t-s} f_\eta(y) f^0(t-s-y|d)dy; \quad (48)
\end{aligned}$$

$$\begin{aligned}
& f^1(t-s|d) \\
= \; & f_1^1(t-s|d) + \sum_{n=2}^{+\infty} f_n^1(t-s|d) \\
= \; & p_2 f_\eta(t-s) \\
& + p_2\int_0^{t-s} f_\eta(y) \sum_{n=2}^{+\infty} f_{n-1}^0(t-s-y|d)dy \\
& + q\int_0^{t-s} f_\eta(y) \sum_{n=2}^{+\infty} f_{n-1}^1(t-s-y|d)dy \\
= \; & p_2 f_\eta(t-s) \\
& + p_2\int_0^{t-s} f_\eta(y) f^0(t-s-y|d)dy \\
& + q\int_0^{t-s} f_\eta(y) f^1(t-s-y|d)dy; \quad (49)
\end{aligned}$$

$$\begin{aligned}
& f^k(t-s|d) \\
= \; & f_k^k(t-s|d) + \sum_{n=k+1}^{+\infty} f_n^k(t-s|d) \\
= \; & p_2^k f_\eta^{(k)}(t-s) \\
& + \sum_{n=k+1}^{+\infty} [p_2\int_0^{t-s} f_\eta(y) f_{n-1}^{k-1}(t-s-y|d)dy \\
& + q\int_0^{t-s} f_\eta(y) f_{n-1}^k(t-s-y|d)dy] \\
= \; & p_2^k f_\eta^{(k)}(t-s) \\
& + p_2\int_0^{t-s} f_\eta(y) \sum_{n=k+1}^{+\infty} f_{n-1}^{k-1}(t-s-y|d)dy \\
& + q\int_0^{t-s} f_\eta(y) \sum_{n=k+1}^{+\infty} f_{n-1}^k(t-s-y|d)dy \\
= \; & p_2^k f_\eta^{(k)}(t-s) \\
& + p_2\int_0^{t-s} f_\eta(y) f^{k-1}(t-s-y|d)dy \\
& - p_2\int_0^{t-s} f_\eta(y) f_{k-1}^{k-1}(t-s-y|d)dy \\
& + q\int_0^{t-s} f_\eta(y) f^k(t-s-y|d)dy \\
= \; & p_2\int_0^{t-s} f_\eta(y) f^{k-1}(t-s-y|d)dy \\
& + q\int_0^{t-s} f_\eta(y) f^k(t-s-y|d)dy, \\
& t \geq s, \quad 2 \leq k \leq j, \quad (50)
\end{aligned}$$

where

$$\begin{aligned}
& p_2\int_0^{t-s} f_\eta(y) f_{k-1}^{k-1}(t-s-y|d)dy \\
= \; & p_2\int_0^{t-s} f_\eta(y) p_2^{k-1} f_\eta^{(k-1)}(t-s-y)dy \\
= \; & p_2^k f_\eta^{(k)}(t-s),
\end{aligned}$$

and

$$\begin{aligned}
& f^{j+1}(t-s|d) \\
= \; & p_1\int_0^{t-s} f_\eta(y) \sum_{n=j+1}^{+\infty} f_{n-1}^j(t-s-y|d)dy \\
= \; & p_1\int_0^{t-s} f_\eta(y) f^j(t-s-y|d)dy. \quad (51)
\end{aligned}$$

Notice that

$$\phi_{3,j}(t-s|d) = f^{j+1}(t-s|d).$$

Taking Fourier transform on both sides of (48), (49), (50), and (51), applying the property of convolution we obtain

$$\Phi^0(u|d) = q\Phi_\eta(u) + q\Phi_\eta(u)\Phi^0(u|d), \quad (52)$$

$$\begin{aligned}\Phi^1(u|d) = p_2\Phi_\eta(u) + p_2\Phi_\eta(u)\Phi^0(u|d) \\ + q\Phi_\eta(u)\Phi^1(u|d), \quad (53)\end{aligned}$$

$$\begin{aligned}\Phi^k(u|d) \; = \; & p_2\Phi_\eta(u)\Phi^{k-1}(u|d) + q\Phi_\eta(u)\Phi^k(u|d), \\ & 2 \leq k \leq j, \quad (54)\end{aligned}$$

$$\Phi^{j+1}(u|d) = p_2\Phi_\eta(u)\Phi^j(u|d), \quad (55)$$

where the Fourier transform of $f^k(t-s|d)$,$k = 0, 1, 2, \cdots, j+1$, are denoted by $\Phi^k(u|d)$. From (52), (53), and (54) we derive

$$\Phi^0(u|d) = \frac{q\Phi_\eta(u)}{1 - q\Phi_\eta(u)}, \quad (56)$$

$$\Phi^1(u|d) = \frac{p_2\Phi_\eta(u)[1 + \Phi^0(u|d)]}{1 - q\Phi_\eta(u)}, \quad (57)$$

$$\Phi^k(u|d) = \frac{p_2\Phi_\eta(u)\Phi^{k-1}(u|d)]}{1 - q\Phi_\eta(u)}, 2 \leq k \leq j. \quad (58)$$

Substituting (56) into (57) we have

$$\Phi^1(u|d) = \frac{p_2\Phi_\eta(u)}{[1-q\Phi_\eta(u)]^2}. \tag{59}$$

Substituting (59) into (58) we have

$$\Phi^2(u|d) = \frac{[p_2\Phi_\eta(u)]^2}{[1-q\Phi_\eta(u)]^3}.$$

In this way, we obtain

$$\Phi^j(u|d) = \frac{[p_2\Phi_\eta(u)]^j}{[1-q\Phi_\eta(u)]^{j+1}}. \tag{60}$$

Substituting (60) into (55) we derive

$$\Phi^{j+1}(u|d) = p_1p_2^j[\frac{\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^{j+1}. \tag{61}$$

Since $\phi_{3,j}(t-s|d) = f^{j+1}(t-s|d)$ and the Fourier transform of $f^{j+1}(t-s|d)$ is $\Phi^{j+1}(u|d)$, we have

$$\Phi_{3,j}(u|d) = p_1p_2^j[\frac{\Phi_\eta(u)}{1-q\Phi_\eta(u)}]^{j+1}$$

which is what we want to show.

Acknowledgements: This work was supported by National Statistical Science Research Key Program of China (Grant No. 2013LZ45), the Fundamental Research Funds for the Central Universities (Grant No. NUST2012YBXM015) and National Natural Science Foundation of China (Grant No. 11201229).

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