The influence of conjugacy class sizes on the structure of finite groups

QINGJUN KONG
Tianjin Polytechnic University
Department of Mathematics
No. 399 Bin Shui West Road
Xiqing District, 300387 Tianjin
CHINA
kqj2929@163.com

Abstract: Let G be a finite p-solvable group and let G^* be the set of p'-elements of primary and biprimary orders of G. We show that when the conjugacy class sizes of G^* are $\{1, m, p^a, mp^a\}$ with $(m, p) = 1$, then the p-complements of G are nilpotent and m is a prime power.

Key–Words: Conjugacy class sizes; Nilpotent groups; Solvable groups; Sylow p-subgroup; Finite groups.

1 Introduction

All groups considered in this paper are finite. If G is a group, then x^G denotes the conjugacy class containing x. Following Baer [1], we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G(in some other papers, $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$ is called conjugacy class size or length of x in G, for example, [2],[3]). We say that a group element has primary or biprimary order respectively if its order is divisible by at most one or two primes. The rest of our notation and terminology are standard. The reader may refer to ref.[4].

There is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist many results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes. In [1], R. Baer proves that a group G is solvable if its elements of prime power order have also prime power index. N. Itô shows in [5] that if the sizes of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. Later in [6], Li Shirong proves that if the finite group G has exactly two conjugacy class lengths of elements of prime power order of G, then G is solvable. Recently, A.Beltrán and M.J.Felipe in [7] show that suppose that the class size of every element of prime power order of G is 1 or m. Then G is nilpotent. More precisely, $m = p^n$ for some prime p, and $G = P \times A$ with A abelian and P a p-group. There exist other deeper results. For instance, in [8], Itô shows that if the conjugacy class sizes of G are $\{1, n, m\}$, then G is solvable. Kong in [9] proves that let G be a group. Assume that the set of conjugacy class sizes of all elements of primary and biprimary orders of G is exactly $\{1, p^a, q^b, p^aq^b\}$, where p and q are two distinct primes and a and b are positive integers, then G is nilpotent. Recently, Kong and Guo in [10] prove that let G be a group and assume that the conjugacy classes sizes of primary and biprimary orders of G are exactly $\{1, p^a, n, p^an\}$ with $(p, n) = 1$, where p is a prime and a and n are positive integers. If there is a p-element in G whose index is precisely p^a, then G is nilpotent and $n = q^b$ for some prime $q \neq p$. In [11], Kong further proves that let G be a group and let G^* be the set of elements of primary, biprimary and triprimary orders of G. Suppose that the conjugacy class sizes of elements of G^* are $\{1, p^a, n, p^an\}$ with $(p, n) = 1$ and $a \geq 0$. Then G is nilpotent and $n = q^b$ for some prime q.

In this paper, we go on studying the nilpotency of a group under some arithmetical conditions on its conjugacy class sizes and will replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to determine completely the structure of G. We put our emphasis on conjugacy class sizes of p'-elements of primary or biprimary orders of G to analyze a new case of groups having four conjugacy class sizes of p'-elements of primary and biprimary orders of G and generalize Theorem A in [11] and Theorem A in [12] and obtain arithmetical conditions on the p-regular conjugacy class sizes which guarantee the nilpotency of the p-complements. Our main result is the following: Let G be a finite p-solvable group and let G^* be the set of p'-elements of primary and biprimary orders of G. Suppose that the conjugacy class sizes of G^* are $\{1, m, p^a, mp^a\}$ with

$(m, p) = 1$, then the p-complements of G are nilpotent and $m = q^b$ for some prime q distinct from p.

We believe that this could probably be a starting point for a proof of a more general result, that is, to show the nilpotency of the p-complements of a group when the set of its p-regular conjugacy class sizes is $\{1, m, n, mn\}$ where m and n are arbitrary coprime numbers.

2 Basic definitions and preliminary results

In this section, we state the necessary results for the proof of our main theorem.

Lemma 1 *[13, Theorem 5] Let G be a finite group and p a prime divisor of $|G|$. Then there is in G no p'−element of primary order whose index is divisible by p if and only if $G = P \times H$, where P is a Sylow p-subgroup of G and H has order prime to p.*

Lemma 2 *[9, Lemma 5] Let G be a group. A prime p does not divide any conjugacy class length of any element of prime power order of G if and only if G has a central Sylow p-subgroup.*

We will make use of Thompson′s $A \times B$ Lemma.

Lemma 3 *[14] Let AB be a finite group represented as a group of automorphisms of a p-group G with $[A, B] = 1 = [A, C_G(B)]$, B a p-group and $A = O^p(A)$. Then $[A, G] = 1$.*

Lemma 4 *Let G be a π-separable group, where π is a non-empty subset of $\pi(G)$. Then*

(a) let $x \in G$ such that $|x^G|$ a π-number, then $x \in O_{\pi,\pi'}(G)$.

(b) the conjugacy class size of any π'-element of primary order in G is a π-number if and only if G has Abelian Hall π'-subgroups. Moreover, $l_{\pi'}(G) \leq 1$.

Proof: (a) Part (a) is Theorem C of [15].

(b) In this case the converse direction is easy and so it is sufficient to prove the direct sense.

We argue by induction on $|G|$. First suppose that $O_\pi(G) \neq 1$. Since the hypothesis is inherited by quotient groups, by induction $G/O_\pi(G)$ has an abelian Hall π'-subgroup $H/O_\pi(G)$. Further, the Schur-Zassenhaus theorem gives that $O_\pi(G)$ has a π-complement H_1 in H. Consequently, $H_1 \cong H/O_\pi(G)$ is an abelian π'-subgroup of G, and we are done.

Hence, we may assume that $O_{\pi'}(G) \neq 1$ since G is π-separable. Moreover, for any π'-element $x \in G$ of primary order, there exists a Hall π'-subgroup K_1 of G and an element $g \in G$ such that $x^g \in K_1^g \leq C_G(x)$. Then

$$x \in C_G(K_1^g) \leq C_G(O_{\pi'}(G)) \leq O_{\pi'}(G),$$

which implies that G has an abelian normal Hall π'-subgroup, as required.

The second assertion follows immediately by applying Lemma 4 (a).

Lemma 5 *Let G be a finite p-solvable group and $\pi = \{p, q\}$ with q and p two primes. Suppose that the sizes of the conjugacy classes of G^* are π-numbers. Then G is solvable, it has abelian π-complements and every p-complement of G has a normal Sylow q-subgroup.*

Proof: We argue by induction on $|G|$. We will prove first that G is solvable. Assume that $O^p(G) < G$. As the hypothesis is inherited by normal subgroups, it follows by induction that $O^p(G)$ is solvable and hence G is solvable too. Thus, we may suppose that $O^{p'}(G) < G$ and use bars to work in $\overline{G} = G/O^{p'}(G)$. Notice that for any $\overline{x} \in \overline{G}$, we may assume that x is a p'-element of primary order and as $|\overline{x}^{\overline{G}}|$ divides $|x^G|$ and $|x^G|$ is a π-number, it follows that $|\overline{x}^{\overline{\overline{G}}}|$ is a q-number. Hence, by applying Lemma 2, we obtain that G is nilpotent. As $O^{p'}(G)$ is solvable by induction, we conclude that G is solvable too. We show now that every p-complement of G has a normal Sylow q-subgroup. If y is a π'-element of G, then in particular we can assume that y is a p'-element of primary order so by hypothesis $|y^G|$ is a π-number. By Lemma 4(b), we have that G has abelian Hall π'-subgroups and $l_{\pi'}(G) \leq 1$. Let T be a π-complement of G, so $O_\pi(G)T \trianglelefteq G$. Suppose first that $O_\pi(G)$=1, so $T = O_{\pi'}(G)$. If x is a q-element of G, by hypothesis $|x^G|$ is a π-number, whence $T \subseteq C_G(x)$. Therefore, $x \in C_G(T) \subseteq T$, whence q does not divide $|G|$ and the thesis of the theorem is trivially true. Accordingly, we will assume that $O_\pi(G) > 1$. If H is any p-complement of G, by induction we have that $HO_\pi(G)/O_\pi(G) = H/H \bigcap O_\pi(G)$ has a normal Sylow q-subgroup. As $H \bigcap O_\pi(G)$ is a q-subgroup of H, we conclude that H has a normal Sylow q-subgroup too, as wanted. □

In the next result, we show that the p-complements of G are indeed nilpotent when we add to the hypotheses of the above theorem the existence of some q-element in G whose index is the highest power of q dividing the sizes of classes of G^*.

Lemma 6 *Let G be a finite p-solvable group and $\pi = \{p, q\}$ with q and p two distinct primes. Suppose that the sizes of classes of G^* are π-numbers. Let q^b be the* *highest power of the prime q which divides the sizes of classes of G^*. Suppose that there exists some q-element $x \in G$ such that $|x^G| = q^b$. Then G has nilpotent p-complements.*

Proof: By Lemma 5 we know that G is solvable. Let Q be a Sylow q-subgroup of G with $x \in Q$. Since $G = QC_G(x)$, then $K = \langle x^g | g \in G\rangle = \langle x^g | g \in Q\rangle$ is a normal q-subgroup of G.

Let T be a π-complement of G with $T \subseteq C_G(x)$. For any $y \in T$ of primary order, we have $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ and then the hypotheses imply that $|C_G(x) : C_G(x) \cap C_G(y)|$ is a p-number. Therefore, $C_K(x) = C_K(x) \cap C_K(y) \subseteq C_K(y)$ and by applying Lemma 3, we obtain $C_K(y) = K$. If we write $Z = C_G(K)$, we have just proved that $T \subseteq Z \trianglelefteq G$, with $|G : Z|$ a π-number. Thus, if y is any π'-element of primary order of Z, then $|y^Z|$ is a π-number, so Lemma 4(b) implies that Z has abelian Hall π'-subgroups and $l_{\pi'}(Z) \leq 1$. Thus, we can write $R = O_{\pi,\pi'}(Z) = O_\pi(Z)T$ with T abelian.

Now, let P_1 be a Sylow p-subgroup of $O_\pi(Z)$. By Frattini′s argument, we can write $R = O_\pi(Z)N_R(P_1)$. Moreover, without loss of generality we may suppose that $T \subseteq NR(P_1)$. Now, for any $y \in T$ of primary order, since $T \subseteq Z \subseteq C_G(x)$, we have $C_G(xy) = C_G(x) \cap C_G(y)$ and $|C_G(x) : C_G(xy)|$ is a p-number by hypothesis. As $R \subseteq C_G(x)$ then $|R : C_R(y)|$ is a p-number too, whence there exists some $Q_1 \in \mathrm{Syl}_q(\mathrm{R})$ such that $Q_1 \subseteq C_R(y)$. Hence, for any $y \in T$ of primary order we have $y \in C_T(Q_1)$ for some $Q_1 \subseteq C_R(y)$. Now, if $g \in TP_1$, then $g \in C_{TP_1}(Q_1)P_1$ for some $Q_1 \subseteq C_R(y)$. Since $R = P_1Q_1T$, then

$$\begin{aligned} TP_1 &\subseteq \bigcup_{g\in TP_1} C_{TP_1}(Q_1^g)P_1 \\ &= \bigcup_{g\in TP_1} (C_{TP_1}(Q_1)P_1)^g, \end{aligned}$$

which forces $TP_1 = C_{TP_1}(Q_1)P_1$. As T is a p-complement of TP_1, there exists some $g \in TP_1$ such that $T^g \subseteq C_{TP_1}(Q_1)$. Thus, $T \times Q_1^{g^{-1}}$ is a p-complement of R.

Now, choose a p-complement H of G such that $T \times Q_1^{g^{-1}} \subseteq H$. Since $R \trianglelefteq G$, we have that $H \cap R$ is a p-complement of R, so $H \cap R = T \times Q_1^{g^{-1}} \trianglelefteq H$. Therefore, $T \trianglelefteq H$. By applying Lemma 5, we conclude that H is nilpotent. □

Lemma 7 *Let G be a p-solvable group whose conjugacy class sizes of G^* are $\{1, p^{a_1}, \cdots, p^{a_r}, q^b, p^{c_1}q^b, \cdots, p^{c_s}q^b\}$, where q is prime distinct from p and $c_i > 0, b, a_i \geq 0$ for all i. Then the p-complements of G are nilpotent.*

Proof: By Lemma 5 we know that G is solvable. If $b = 0$, we use Lemma 4(b) with $\pi = \{p\}$ to obtain that G has abelian p-complements, so we may suppose that $b > 0$.

If there exists some q-element of index q^b, then Lemma 6 applies and the p-complements of G are nilpotent, so the Lemma is proved. Suppose now that there exists some q-element $x \in G$ such that $|x^G| = p^{c_i}q^b$ for some i with $c_i \geq 0$ and let T be a $\{p, q\}$-complement of G, which is abelian by Lemma 4(b), and such that $T \subseteq C_G(x)$. Let Q_1T be a p-complement of $C_G(x)$, where $Q_1 \in \mathrm{Syl}_q(C_G(x))$ and $x \in Q_1$. Now, if $y \in T$ of primary order then $C_G(xy) = C_G(x) \cap C_G(y)$ and notice that the hypotheses of the Lemma imply that $|C_G(x) : C_G(xy)|$ must be a p-power. On the other hand, $T \subseteq C_G(xy)$ since T is abelian, and consequently, there exists some $g \in C_G(x)$ such that TQ_1^g is a p-complement of $C_G(xy)$. Now, if we take a p-complement H of G such that $TQ_1^g \subseteq H$, we can certainly write $H = TQ$, for some $Q \in \mathrm{Syl}_q(\mathrm{G})$, with $Q_1^g \subseteq Q$. As $Q_1^g \subseteq C_Q(xy) \subseteq C_Q(x)$ and $Q_1^g \in \mathrm{Syl}_q(CG(x))$, it follows that $C_Q(xy) = C_Q(x)$ and thus $C_Q(x) \subseteq C_Q(y)$. Furthermore, by Lemma 5, we know that $Q \trianglelefteq H$. Notice that $x \in Q$ and then we can apply Lemma 3 to conclude that $C_Q(y) = Q$, for all $y \in T$. Then $H = Q \times T$, and since T is abelian, we deduce that H is nilpotent and this case is finished too.

As a result, we may assume that every q-element of G is central in G or has p-power index. Choose $y \in G^*$ such that $|y^G| = q^b$ and write $y = y_qy_{q'}$, where y_q and $y_{q'}$ are the q-part and q'-part of y respectively, they are elements of primary orders. Since $C_G(y) \subseteq C_G(y_q)$, it follows that y_q must be central in G and thus, by replacing y by $y_{q'}$, we may assume that y is a $\{p, q\}'$-element. Let $H = QT$ be a p-complement of G, where $Q \in \mathrm{Syl}_q(\mathrm{G})$ and T is a $\{p, q\}$-complement of G with $y \in T$. As $G = QC_G(y)$, then

$$L = \langle y^g | g \in G\rangle = \langle y^g | g \in Q\rangle \subseteq H.$$

As L is normal in G, we deduce that $L \subseteq C_G(x)$ for any $x \in Q$. Then $Q \subseteq C_G(L) \subseteq C_G(y)$, and this is a contradiction. □

Lemma 8 *[9, Theorem 16] Let G be a group. Assume that the set of conjugacy class sizes of all elements of primary and biprimary orders of G is exactly $\{1, p^a, q^b, p^aq^b\}$, where p and q are two distinct primes and a and b are positive integers, then G is nilpotent.*

Lemma 9 *Let G be a p-solvable group whose conjugacy class sizes of G^* are $\{1, r^a, q^b, r^aq^b\}$, where q and r are primes (distinct from or equal to p) and a* *and b positive integers. Then the p-complements of G are nilpotent.*

Proof When $r \neq p \neq q$ we apply Lemma 1 and 8 and obtain that any p-complement of G is nilpotent. Otherwise, we apply Lemma 7. □

Lemma 10 *Let G be a finite p-solvable group and let $\pi = \{p, q\}$. Suppose that the size of any conjugacy class of π'-elements of primary orders is a p-number. Then G is solvable, the π-complements of G are abelian and each p-complement of G has a normal (abelian) q-complement.*

Proof: We show first that G is solvable by induction on $|G|$. If $O^p(G) < G$, then as the hypotheses are inherited by normal subgroups, it clearly follows that G is solvable. Therefore, we will assume that $O^p(G) = G$ and hence, $O^{p'}(G) < G$. Notice that if $O^{p'}(G)$=1, then G is a p'-group and the hypotheses imply that every q'-element of primary order of G is central, so G is trivially solvable. We can assume then that $O^{p'}(G) > 1$ and write $\overline{G} = G/O^{p'}(G)$. It is easy to see that the hypotheses are inherited by factor groups and we will prove it for $\overline{G}$. Let $\overline{x} \in \overline{G}$ a π'-element and factor $x = x_\pi x_{\pi'}$ where x_π and $x_{\pi'}$ are the π-part and π'-part of x respectively, they are elements of primary orders. Then $\overline{x} = \overline{x_{\pi'}}$, so x can be assumed without loss to be a π'-element of primary order. As $|\overline{x}^{\overline{G}}|$ divides $|x^G|$, we obtain that the class size of any π'-element of primary order of $\overline{G}$ is also a p-number, as wanted. By applying the inductive hypothesis to $O^{p'}(G)$ and G, we conclude that G is solvable as wanted.

The fact that the π-complements of G are abelian follows just by applying Lemma 4(b).

We prove now by induction on $|G|$ that each p-complement of G, say H, has a normal q-complement. Let $N = O_{q'}(G)$ and suppose first that $N = 1$. Since the index of any π'-element of primary order $y \in G$ is a p-number, then $O_q(G) \subseteq C_G(y)$ and so $y \in C_G(O_q(G)) \subseteq O_q(G)$, which is a contradiction. Therefore, in this case there are no π'-elements in G, that is, G is a $\{p, q\}$-group and the conclusion of the Lemma is trivial. Hence, we will assume that $N > 1$ and apply the inductive hypothesis to G/N so as to obtain that $HN/N \cong H/H \cap N$ has a normal q-complement. As $H \cap N$ is a q'-subgroup, it follows that H also has a normal q-complement. □

Lemma 11 *[7, Corollary B] Let G be a finite group and suppose that the class size of every element of prime power order of G is 1 or m. Then G is nilpotent. More precisely, $m = p^n$ for some prime p, and $G = P \times A$ with A abelian and P a p-group.*

3 Main results

Theorem 12 *Let G be a finite p-solvable group and let G^* be the set of p'-elements of primary and biprimary orders of G. Suppose that the conjugacy class sizes of G^* are $\{1, m, p^a, mp^a\}$ with $(m, p) = 1$, then the p-complements of G are nilpotent and $m = q^b$ for some prime q distinct from p.*

Proof: We will show that m is a power of some prime $q \neq p$ and then the result will be proved by Lemma 9. First, we show that two p'-elements of primary orders of index p^a and m centralize each other.

Step 1. *If w is a p'-element of primary order of index m and y is a p'-element of primary order of index p^a, then $w \in C_G(y)$.*

Let H be a p-complement of G with $w \in H$. Notice that $G = C_G(w)H$ and that there exists some $g \in H$ such that $w^g \in H^g \subseteq C_G(y)$. Also, as w and y have coprime index, we have $G = C_G(w)C_G(y)$, so we can assume that $g \in C_G(w)$. Thus, $w = w^g \in H^g \subseteq C_G(y)$.

There exist p'-elements of index p^a by hypothesis, so by considering the primary decomposition of such elements, there must exist certain q-elements of index p^a for some prime q. For any such a prime q we prove the following properties (Steps 2-4).

Step 2. *For any p-complement H of G, it holds that every q'-element of primary order of H has index 1 or m in H.*

Since the centralizer of any q-element of index p^a contains some p-complement of G, by conjugacy we may certainly choose some q-element, say y, of index p^a such that $H \subseteq C_G(y)$. Now, let z be any q'-element of primary order of H which centralizes y and then $C_G(zy) = C_G(z) \cap C_G(y) \subseteq C_G(y)$. Thus, z has necessarily index 1 or m in $C_G(y)$. As m is a p'-number, it follows that $C_G(y) = H(C_G(z) \cap C_G(y))$ and we conclude that $|H : C_H(z)| = 1$ or m as required.

Step 3. *If z is a q-element of index $p^a m$, then $C_G(z) = Q_z P_z \times T_z$, where Q_z and P_z are q- and p-subgroups respectively and T_z is an abelian $\{p, q\}'$-subgroup. Furthermore, if z lies in some p-complement H of G, then we can assume that T_z is not central in H.*

By the maximality of the index of z we notice that any $\{p, q\}'$-element of primary order $t \in C_G(z)$ satisfies $C_G(zt) = C_G(z)$, so $C_G(z) \subseteq C_G(t)$ and accordingly $C_G(z)$ can be written as described in the statement. We remark that this part of the step is also true without the assumption of existence of q-elements of index p^a that we are doing.

For the second part, we take some p-complement H of G with $z \in H$ and we will prove that if T_z is central in H, then the theorem is proved. Suppose that $T_z \subseteq Z(H)$, and consequently, $T_z = Z(H)_{q'}$ and

$$m_{q'} = |G|_{\{p,q\}'}/|Z(H)|_{q'}.(I)$$

We distinguish three cases. Assume first that H possesses a q'-element of primary order of index m in G, say w, and take a q-element $y \in H$ of index p^a. We know by Step 1 that $y \in C_G(w)$, so $C_G(wy) = C_G(w) \cap C_G(y)$ and certainly wy has index $p^a m$. On the other hand, (I) implies that $Z(H)_{q'}$ is a $\{p,q\}$-complement of $C_G(wy)$. But as $w \in C_G(wy)$, then $w \in Z(H)$ and this contradicts the fact that w has index m. Therefore, this case cannot happen. Assume now that there exists in H a q'-element of primary order of index $p^a m$. Again (I) shows that $Z(H)$ is a $\{p,q\}$-complement of $C_G(w)$, and as in the above paragraph, this leads to a contradiction.

Finally, we can assume that any q'-element of primary order $w \in H$ has index p^a. This means that the class size of any $\{p,q\}'$-element of primary order of G is a p-number, so by applying Lemma 10, we have that G is solvable and the $\{p,q\}$-complements are abelian. Let s be any prime distinct from p and q and let $S \in \mathrm{Syl}_s(\mathrm{G})$ with $S \subseteq H$. If $S \subseteq Z(H)$, it is trivial that s does not divide m. If $S \nsubseteq Z(H)$, then we take $w \in S - Z(H)$ and by Step 2, w has index m in H. As S is abelian, we have $S \subseteq C_H(w)$, so in particular, s does not divide m either. Therefore, m is a q-power and the theorem is proved.

Step 4. *If z is a q-element of index $p^a m$, lying in some p-complement H of G, then $|H : C_H(z)| = m$ and there exists some element $t \in T_z \cap H - Z(H)$, where T_z is the subgroup defined in Step 3.*

We show first that $|H : C_H(z)| = m$. Write $C_G(z) = Q_z P_z \in T_z$ as in Step 3, with T_z non-central in H and choose a non-central $\{p,q\}'$-element $w \in T_z$. This can be assumed of primary order, say for instance an r-element for a prime $r \neq p, q$. We will distinguish three cases depending on the index of w in G.

Assume first that w has index p^a. In this case, Step 2 asserts that any r'-element of H, in particular z, has index m in H.

Suppose now that w has index m. Observe that any r'-element of primary order of $C_G(w)$ has index 1 or p^a in $C_G(w)$, so by Lemma 10, the $\{p,r\}$-complements of $C_G(w)$ are abelian. On the other hand, as T_z is abelian and $w \in T_z$, we have $C_G(z) \subseteq C_G(w)$, so in particular, Q_z is abelian. We have just shown that any p-complement of $C_G(z)$ is abelian. Moreover, as $|C_G(w) : C_G(z)| = p^a$, then the p-complements of $C_G(z)$ are also p-complements of $C_G(w)$. On the other hand, the fact that $HC_G(w) = G$ implies that $C_H(w)$ is an (abelian) p-complement of $C_G(w)$. Then $C_H(w) \subseteq C_H(z)$ and hence, $C_H(w) = C_H(z)$. Since by Step 2, w has index 1 or m in H, we conclude that z has the same index. But we notice that z cannot be central in H since it has index $p^a m$ in G. So this case is finished too.

Finally, assume that w has index $p^a m$. As T_z is abelian, then $C_G(z) \subseteq C_G(w)$ and by orders, $C_G(z) = C_G(w)$. Taking into account the decomposition of $C_G(z)$ and of $C_G(w)$ given in Step 3 (w is an r-element), we get

$$C_G(z) = C_G(w) = Q_z \times P_z \times T_z$$

with the same notation given there. On the other hand, we can take a q-element $y \in H$ of index p^a in G such that $y \in Z(H)$. Since w has index $p^a m$ we have $C_G(wy) = C_G(w) \cap C_G(y) = C_G(w)$, that is, $C_G(w) \subseteq C_G(y)$ and $|C_G(y) : C_G(w)| = m$. This forces $C_G(y) = HC_G(w)$ and accordingly, $|H : C_H(w)| = m$. Therefore, z also has index m in H, as we wanted to prove.

We prove now the second part of the step. By the first part we have $m_{q'} = |H|_{q'}/|C_H(z)|_{q'}$, but if we consider the decomposition of $C_G(z) = Q_z P_z \times T_z$, we also obtain $m_{q'} = |G_{\{p,q\}'}|/|T_z|$. Thus, $|T_z| = |C_H(z)|_{q'}$. If $T_z \cap H \subseteq Z(H)$, then $T_z \cap H \subseteq Z(H)_{q'} \subseteq C_H(z)_{q'}$. But, on the other hand, $C_H(z)_{q'}$ is clearly contained in the Hall $\{p,q\}'$-subgroup of $C_G(z)$, that is, in T_z. We deduce that $C_H(z)_{q'} = T_z$ and this forces T_z to be central in H, which is a contradiction by Step 3.

It is clear that in G there exist elements of prime orders of index m. From now on we will fix one of these elements x, an r-element, with $r \neq p$, and will choose a p-complement of G, say H, such that $x \in H$. Since $G = HC_G(x)$, then $C_H(x)$ is a p-complement of $C_G(x)$ and by applying Lemma 10 to $C_G(x)$, we can write $C_H(x) = T_x R_x$, with R_x an r-subgroup and T_x an abelian $\{p,r\}'$-subgroup which is normal in $C_H(x)$. We know by Step 1 that any p'-element of prime order of index p^a commutes with any p'-element of prime order of index m, so in particular, every p'-element of prime order of index p^a of H belongs to $C_H(x)$. Now, in the two following steps we prove two properties related to $C_H(x)$.

Step 5. *We may assume that T_x is not central in G.*

We assume that $T_x \subseteq Z(G)$ and work to get a contradiction. We know that every r'-element of prime order in H of index 1 or p^a centralizes x, and consequently, lies in T_x. Thus, there are not r'-elements of prime orders in H of index p^a, whence there cannot exist such elements in G. Accordingly, there must exist some r-element y of index p^a, which can be assumed to lie in $Z(H)$ by conjugacy.

By Step 2, any r'-element of prime order of H has index 1 or m in H. Notice that if any r'-element of prime order of H lies in $Z(H)$, then $H = R \times Z(H)_{r'}$, which is nilpotent and m would be a power of r. In this case the theorem is proved, so we can assume the existence of some r'-element of prime order $y \in H - Z(H)$ of index m in H. Then we have $m_{r'} = |H|_{r'}/|C_H(y)|_{r'}$. But moreover, the structure of $C_G(x)$ provides the equality $m_{r'} = |H|_{r'}/|T_x|$. As $T_x = (Z(G) \cap H)_{r'}$, this yields

$$|T_x| = |Z(G) \cap H|_{r'} = |C_H(y)|_{r'}$$

and consequently, $(Z(G) \cap H)_{r'}$ is an r-complement of $C_H(y)$. This contradicts the fact that y is non-central in $C_H(y)$.

Step 6. *If T_x has an element of index m or $p^a m$, then $C_H(x)$ is abelian.*

Suppose first that there is an element $w \in T_x$ of index m. By considering the primary decomposition of w we can assume without loss that w is a q-element for some prime $q \neq p, r$, since $C_G(w)$ must be equal to the centralizer of some primary component of w. Notice that each $\{p, q\}'$-element of prime order of $C_G(w)$ has index 1 or p^a in $C_G(w)$, and that $C_H(w)$ is a p-complement of $C_G(w)$. So by applying Lemma 10, we can write $C_H(w) = Q_w T_w$ where Q_w is a q-subgroup and T_w an abelian $\{p, q\}'$-subgroup which is normal in $C_H(w)$. We also notice that $|C_H(w)| = |C_H(x)|$, as both subgroups have index $p^a m$ in G. We will prove that in fact both centralizers are equal. As $w \in T_x$, we have that $T_x \subseteq C_H(w)$. On the other hand, x lies in some Sylow r-subgroup of $C_H(w)$, say R_w, which is abelian, so $R_w \subseteq C_H(x)$. Therefore, $T_x R_w \subseteq C_H(x)$ and $T_x R_w \subseteq C_H(w)$. By order considerations, we conclude $C_H(w) = C_H(x) = T_x R_w$. But we know that R_w is abelian and normal in $C_H(w)$, so $C_H(x) = T_x \times R_w$, whence $C_H(x)$ is abelian.

Suppose now that there is an element of prime order $w \in T_x$ of index $p^a m$. Notice that any r-element $z \in C_G(w)$ satisfies $C_G(zw) = C_G(w) \cap C_G(z) = C_G(w)$ by the maximality of the index of w. This means that z is central in $C_G(w)$, so we can write $C_G(w) = T_w P_w \times R_w$, with P_w a p-subgroup, T_w a $\{p, r\}'$-subgroup and R_w an abelian r-subgroup. Moreover, as T_x is abelian, then $T_x \subseteq C_G(w)$, so T_x centralizes R_w. On the other hand, $x \in CG(w)$, so $x \in R_w$ and $R_w \subseteq C_H(x)$. By order considerations R_w is a Sylow r-subgroup of $C_H(x)$, whence we conclude that $C_H(x) = R_w \times T_x$ and $C_H(x)$ is abelian too.

For the rest of the proof we are going to define and work with certain subgroup L_q for any prime $q \neq p, r$. When these subgroups are central for all q we will define and work with a subgroup associated to r.

Step 7. *For any prime $q \neq p, r$, let*

$$L_q = \langle y | y \text{ is a } q\text{−element in } H \text{ with } |y^G| = 1 \text{ or } p^a \rangle.$$

Then L_q is an abelian normal q-subgroup of H.

Suppose that $L_q \subseteq Z(G)$ for all $q \neq p, r$. Then we define

$$L_r = \langle y | y \text{ is an } r\text{−element in } H \text{ with } |y^G| = 1 \text{ or } p^a \rangle.$$

and it is a non-central abelian normal r-subgroup of H. Furthermore, in this case $C_H(x)$ is abelian.

For any prime $q \neq p, r$, we know that $C_H(x)$ has an abelian normal Sylow q-subgroup Q. Likewise, we know by Step 1 that

$$\{y | y \text{ is a } q\text{−element in } H \text{ with } |y^G| = 1 \text{ or } p^a\} \subseteq C_H(x).$$

As a consequence, $L_q \subseteq Q$ and thus, L_q is an abelian q-subgroup of H. The fact that $L_q \trianglelefteq H$ is trivial.

Suppose now that $L_q \subseteq Z(G)$ for all $q \neq p, r$. This implies that there are no q-elements of index p^a for all such primes and hence, there must be an r-element in G (and in H) of index p^a. In particular, $L_r \nsubseteq Z(G)$. On the other hand, by applying Step 5, we deduce that in T_x there must be elements of primary orders of index m or $p^a m$, so by Step 6, $C_H(x)$ is abelian. But by Step 1, we have

$$\{y | y \text{ is an } r\text{−element in } H \text{ with } |y^G| = 1 \text{ or } p^a\} \subseteq C_H(x).$$

so L_r is contained in the Sylow r-subgroup of $C_H(x)$. Therefore, L_r is an abelian r-subgroup of H which is trivially normal in H.

Step 8. *Every q-element of H centralizes L_q for any prime $q \neq p, r$. If $L_q \subseteq Z(G)$ for any $q \neq p, r$, then any r-element of H centralizes L_r.*

Let s be any prime distinct from p and let z be an s-element of H. We will prove that $z \in M = C_H(L_s)$ (we remark that when $s = r$ then we are assuming that $L_q \subseteq Z(G)$ for all $q \neq p, r$).

If z has index p^a, then by definition $z \in L_s$, so trivially $z \in M$. If z has index m, we know by Step 1 that z centralizes any element of primary order of index p^a, so z also lies in M.

Thus, we only have to show that if z has index $p^a m$, then it lies in M too. By Step 3, we write $C_G(z) = S_z P_z \times T_z$ with the notation given there and T_z abelian. Also, by Step 4, there exists some element of primary order $t \in T_z \cap H - Z(H)$, so we have $C_G(z) \subseteq C_G(t)$. In particular $C_{L_s}(z) \subseteq C_{L_s}(t)$, and by applying Lemma 3, we conclude that $t \in M$. Now we distinguish three cases for the index of t in G. If t has index $p^a m$, then $C_G(t) = C_G(z)$, so z lies trivially in M. If t has index p^a, as t is non-central in H then by Step 2 (notice that there are s-elements of index p^a), we get $|H : C_H(t)| = m$. On the other hand, by Step 4, we have $|H : C_H(z)| = m$, and since $C_H(z) \subseteq C_H(t)$ we obtain by order considerations that $C_H(z) = C_H(t)$. It follows that $z \in M$. Finally, suppose that t has index m. There is no loss if we assume that t is an l-element, for some prime $l \neq s, p$, since we can replace t by some of its components in the primary decomposition, with the same index m. By applying Lemma 10 to $C_G(t)$, we get that $C_G(t)$ has abelian $\{p, l\}$-complements, so $C_H(t)$, which is a p-complement of $C_G(t)$, has an abelian normal s-complement, say T_t. Then $z \in T_t$ and $T_t \subseteq C_H(z)$. Therefore, $|C_H(t) : C_H(z)|$ is an l-number. As L_s is a normal s-subgroup of $C_H(t)$, we conclude that $L_s \subseteq C_H(z)$.

Step 9. *We can assume that for any prime $q \neq p, r$, we have $L_q \subseteq Z(H)$. If $L_q \subseteq Z(G)$ for all prime $q \neq p, r$, then $L_r \subseteq Z(H)$.*

Let s be a prime distinct from p. Notice that $L_s \subseteq C_H(x)$ by Step 1. We will consider the following cases:

(a) $s = r$. In this case notice that we are assuming by definition of L_r in Step 7 that $L_q \subseteq Z(G)$ for all prime $q \neq p, r$. Also in this case, as T_x is non-central by Step 5 and there are no $\{p, r\}'$-elements of primary orders of index p^a, then T_x has elements of primary orders of index m or $p^a m$ and by Step 6 we have that $C_H(x)$ is abelian.

(b) $s \neq p, r$. We will distinguish two possibilities:

(1) there are no r-elements of index p^a; and

(2) there are r-elements of index p^a.

In cases (a) and (b)(1) we will see that if $w \in H$ of primary order, then $w \in C_H(L_s)$, so $L_s \subseteq Z(H)$. In case (b)(2) we have by Step 2 that every r'-element of primary order of H has index 1 or m in H. We will prove that if $w \in H$ of primary order, then $w \in R_x^g C_H(L_s)$ with $g \in C_H(x)$. Once this is proved, we have

$$H = \bigcup_{g \in H} R_x^g C_H(L_s),$$

which forces that $H = R_x C_H(L_s)$ and $|H : C_H(L_s)|$ is an r-number. Let $y \in L_s - Z(H)$, then $C_H(L_s) \subseteq C_H(y) \subseteq H$ and $|H : C_H(y)| = m$. Thus m is a r-power and the theorem would be proved. Therefore, $L_s \subseteq Z(H)$ as we want to prove.

Now we prove the properties stated in the above paragraph. Let $w \in H$ and consider the $\{s, s'\}$-decomposition of w. By Step 8 we know that the s-part of w lies in $C_H(L_s)$, so we can assume without loss of generality w is an s'-element of primary order. If w has index m, then $w \in C_H(L_s)$ by Step 1. So we will study the cases in which w has index p^a or mp^a.

Suppose first that w is an element of primary order and w has index p^a. Using Step 1 again we get $w \in C_H(x)$. In case (a) we know that $C_H(x)$ is abelian and $L_r \subseteq C_H(x)$, so clearly $w \in C_H(L_r)$. In case (b), we have $s \neq r$ and thus, $L_s \subseteq T_x$. We consider the $\{r, r'\}$-decomposition of $w = w_r w_{r'}$, where w_r and $w_{r'}$ are elements of primary orders, so $w_{r'} \in T_x$ by Step 1. Since T_x is abelian, we obtain $w_{r'} \in C_H(L_s)$. In case (b)(1), w_r is central in G, so $w \in C_H(L_s)$. In case (b)(2), as $w_r \in C_H(x) = T_x R_x$, then $w_r \in R_x^g$, for some $g \in C_H(x)$. So $w \in R_x^g C_H(L_s)$.

Suppose now that w has index $p^a m$ and consider the primary decomposition of w, that is, $w = w_r w_{r'}$ for some prime r'. If $w_{r'}$ has index p^a or m then, by the above paragraphs, we have $w_{r'} \in C_H(L_s)$. On the other hand, if w_r has index m, then $w_r \in C_H(L_s)$ by Step 1 and $w \in C_H(L_s)$. If w_r has index p^a then, again by the above paragraph, we deduce in cases (a) and (b)(1) that $w_r \in C_H(L_s)$ and, in case (b)(2), we obtain $w_r \in R_x^g C_H(L_s)$, so $w \in R_x^g C_H(L_s)$, for some $g \in C_H(L_s)$. Thus, we can assume that either w_r or $w_{r'}$, for some prime r', has index $p^a m$ in G. We will prove that $w \in C_H(L_s)$ in all cases (a), (b)(1) and (b)(2).

Let us consider w_l (with l either equal to r' or r) such that $|w_l^G| = p^a m$. Notice that $l \neq s$ and $C_G(w) = C_G(w_l)$. Suppose that $L_s \nsubseteq Z(H)$ and take $y \in L_s - Z(H)$. As y has index p^a, by conjugacy we can assume without loss that $H \subseteq C_G(y)$. Then w centralizes y and $C_G(wy) = C_G(w) \cap C_G(y) = C_G(w) = C_G(w_l) \subseteq C_G(y)$. Since $C_G(y) = HC_G(w_l)$, it follows that $C_H(w_l)$ is a p-complement of $C_G(w_l)$. On the other hand, arguing in a similar way as in the first part of Step 3, we get $C_G(w_l) = L_{w_l} P_{w_l} \times A_{w_l}$, with L_{w_l} an l-group, P_{w_l} a p-group and A_{w_l} an abelian $\{p, l\}'$-group, whence $y \in A_{w_l}$. As $L_{w_l} \times A_{w_l}$ is also a p-complement of $C_G(w_l)$, then we can assume up to conjugacy that $C_H(w_l) = L_{w_l} \times A_{w_l} \subseteq H$. Thus $m_s = |G|_s / |A_{w_l}|_s \leq |G|_s / |Z(H)|_s$. If $|A_{w_l}|_s = |Z(H)|_s$, then $|G|_s / |Z(H)|_s = m_s$. Moreover, as $L_s \subseteq C_G(x)$, then $|G|_s / |L_s| \geq m_s$. Since $Z(H)_s \subseteq L_s$, we obtain $L_s = Z(H)_s$, which contradicts our assumption. So we can assume that there are s-elements in A_{w_l} which are not central in H. We distinguish the following three cases.

Suppose first that there is an s-element $z \in A_{w_l}$ of index $p^a m$. Then $C_G(z) = C_G(w_l) = C_G(w)$. By Step 7, we have that $z \in C_H(L_s)$ and then $L_s \subseteq C_G(z) = C_G(w)$. In this case, we obtain

$w \in C_G(L_s)$.

Suppose now that there is an s-element $z \in A_{w_l}$ of index m. As $z, y \in A_{w_l}$, then $C_G(w_l) = C_G(w) \subseteq C_G(y) \cap C_G(z) = C_G(yz)$ and thus, $C_G(yz) = C_G(w)$. Again by Step 7, $L_s \subseteq C_H(z) = C_H(w)$, so $w \in C_H(L_s)$.

Finally, assume that every s-element of A_{w_l} has index p^a, and consequently, that all of them belong to L_s. Then $|A_{w_l}|_s \leq |L_s|$, so $m_s \leq |G|_s/|Ls| \leq |G|_s/|A_{w_l}|_s = m_s$. Therefore, $L_s \subseteq A_{w_l}$. As A_{w_l} is abelian, we conclude that $w_l \in A_{w_l} \subseteq C_H(L_s)$, whence $Ls \subseteq C_H(w_l) = C_H(w)$ and $w \in C_G(L_s)$ as we wanted to prove.

We remark that whenever there exists some s-element of index m in G for some prime $s \neq p$, then Step 9 also holds for s, just by arguing with s instead of r as we have made it in Steps 5-9.

Step 10.*(Conclusion).*

We know that there are p'-elements of primary orders of index p^a, so there is some prime $s \neq p$ such that $L_s \nsubseteq Z(G)$. By Step 9, we have $L_s \subseteq Z(H)$. Notice that if $s = r$, then we are also assuming that $L_q \subseteq Z(G)$ for all prime $q \neq p, r$. We claim first that any s-element $w \in H$ has index 1 or m in H. We distinguish three possibilities according to the index of w in G. If w has index p^a, then $w \in Ls \subseteq Z(H)$. If w has index m, then $HC_G(w) = G$ and it clearly follows that w has index m in H. Finally, if w has index $p^a m$ in G, then by Step 4, w has index m in H, and the claim is proved.

On the other hand, by Step 2, we also have that any s'-element of primary order of H has index 1 or m in H. The rest of the proof consists of showing that any element of primary order of H has index 1 or m too. Then, by applying Lemma 11, we get that H is nilpotent and m is a prime power, so the proof of the theorem will be finished.

Let us take any $z \in H$ in G^* and factor $z = z_s z_{s'}$, where z_s and $z_{s'}$ are elements of primary orders. If one of these factors is central in H, then z would have the same index in H as the other factor, and consequently, z would have index 1 or m in H. Therefore, we will assume that both z_s and $z_{s'}$ are not central in H. We distinguish three cases for the index of z in G. If z has index p^a in G, then as $C_G(z) \subseteq C_G(z_s)$ and since z_s cannot be central in G, it follows that $C_G(z) = C_G(z_s)$, whence z has the same index in H as z_s, that is, m. If z has index m in G, since $HC_G(z) = G$, we easily deduce that z has index m in H too. Thus, we will suppose that z has index $p^a m$ in G.

We have the following possibilities for the index of z_s in G. If z_s has index p^a, then it would be central in H by Step 9, but we are assuming that it is not so. If z_s has index $p^a m$ in G, we certainly have that $C_G(z) = C_G(z_s)$ and then z has the same index in H as z_s. We will assume then that z_s has index m in G.

On the other hand, we analyze the index of $z_{s'}$ in G. Suppose first that $z_{s'}$ has index p^a. If we let $z_{s'} = z_{q_i}$ for some prime $q_i \neq r, s$, it is clear that z_{q_i} has index p^a or 1, whence if $q_i \neq r$, then by Step 9, $z_{q_i} \in Z(H)$. Hence, $z_{q_i} = z_{s'}$ is central in H, so all s'-elements are central in H, contradicting our assumption. Therefore, we can assume that there is some i such that $q_i = r$ and that $z_{s'} = z_r y$ with $y \in Z(H)$ and y of primary order. As $r \neq s$ and z_s has index m, by the remark above this step, we know that Step 9 holds for s, that is, any s'-element of index p^a is central in H. In particular, $z_r \in Z(H)$ and consequently, $z_{s'} \in Z(H)$ too, which is a contradiction.

Suppose now that $z_{s'}$ has index $p^a m$. Then $C_G(z_{s'}) = C_G(z)$, so z has index 1 or m in H.

Finally, let us assume that $z_{s'}$ has index m and consider again the primary decomposition, we can assume $z_{s'} = z_l$. It follows that $C_G(z_s) = C_G(z_l)$ for some prime $l \neq p, r$. Then

$$\begin{aligned} C_G(z) &= C_G(z_s) \cap C_G(z_{s'}) \\ &= C_G(z_s) \cap C_G(z_l) = C_G(z_s z_l) \end{aligned}$$

and accordingly we can assume that $z = z_s z_l$, knowing that both factors have index m in G. Now, by applying Lemma 10, we have that $C_G(z_s)$ has abelian $\{p, s\}$-complements, so we can write $C_H(z_s) = T_1 S_1$, where S_1 is an s-subgroup and T_1 an abelian $\{p, s\}'$-subgroup, with $z_l \in T_1$. Notice that $T_1 \subseteq C_H(z_l)$. Arguing in the same way with $C_G(z_l)$, it has abelian $\{p, l\}$-complements, so in particular we may write $C_H(z_l) = T_2 S_2$, where T_2 is a $\{p, s\}$-subgroup and S_2 is an abelian s-subgroup, with $z_s \in S_2$. Notice that $S_2 \subseteq C_H(z_s) = T_2 S_2$. Also, up to conjugacy and by order considerations we can assume that $S_2 = S_1$, so $C_H(z_s) = S_1 T_1 \subseteq C_H(z_l)$. As both subgroups have the same order, we conclude that $C_H(z_s) = C_H(z_l)$. Therefore,

$$C_H(z) = C_H(z_s) \cap C_H(z_l) = C_H(z_l) = C_H(z_s),$$

whence z has index m in H, as we wanted to prove. Now the proof of the theorem is finished.

Remark 13 *In Theorem 12, we only use p'-elements of primary and biprimary orders of G to guarantee the nilpotency of the p-complements, it can be seen a complete extension of Theorem A in [12].*

4 Conclusion and an application

The results explained in the previous sections show that the method that we replace conditions for all conjugacy classes by conditions referring to only some of the classes in order to investigate the structure of a finite group is very useful. Results of this type are interesting since they can be used to simplify the proofs of new or known properties related to conjugacy classes. Recently, Kang in [17] and [18] characterized the structure of a finite group by using this method. In addition, according to the parallel property of conjugacy class sizes and character degrees in [19] and [20], we may consider using the character degrees to characterize the structure of finite groups. As an application, we can investigate the structure of a finite group when its character degrees of G are exactly $\{1, m, p^a, mp^a\}$, where m is an integer not divisible by p.

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