On the validity of Thompson's conjecture for alternating groups A_{p+4} of degree $p+4$

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Abstract: Let G be a group. Let $\pi(G)$ be the set of prime divisor of $|G|$. Let $GK(G)$ denote the graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We use $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. Let $N(G)$ be the set of nonidentity orders of conjugacy classes of elements in G. Some authors have proved that the groups A_n where $n = p, p+1, p+2$ with $s(G) \geq 2$, are characterized by $N(G)$. Then if $s(G) = 1$, we know that Liu and Yang proved that alternating groups A_{p+3} are characterized by $N(G)$. As the development of this topics, we will prove that if G is a finite group with trivial center and $N(G) = N(A_{p+4})$ with $p+i$ composite and $1 \leq i \leq 4$, then G is isomorphic to A_{p+4}.

Key–Words: Element order, Alternating group, Thompson's conjecture, Conjugacy classes, Simple group.

1 Introduction

All groups under considerations are finite and simple groups mean nonabelian simple groups. Let $N(G) := \{n : G$ has a conjugacy class of size $n\}$. Regarding $N(G)$, J.G.Thompson in 1987 put forward the following well-known conjecture.

Thompson's Conjecture (see [24, Question 12.38]). If L is a finite simple non-Abelian group, G is a finite group with trivial center, and $N(G) = N(L)$, then $G \cong L$.

It is well-known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist many results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes (see [21, 22]).

Let $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $GK(G)$ be the graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We use $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. A classification of all finite simple groups with disconnected prime graph was obtained in [16, 27]. Based on these results, Chen proved the following result (see [6, 7]).

Theorem 1. *Let G be a finite group with $Z(G) = 1$. If M is a nonabelian simple group such that $N(G) = N(M)$ and $s(M) \geq 2$, then G is isomorphic to M.*

Ahanjideh and Xu et al. proved the following result (see [2, 4, 1, 3] and [29]).

Theorem 2. *Let G be one of the following groups: $L_n(q)$, $D_n(q)$, B_n, C_n, ${}^2D_N(q)$ and $E_7(q)$ with trivial center. Then G is characterized by $N(G)$.*

Alavi and Daneshkhah got the following (see [5]).

Theorem 3. *Let G be the alternating groups A_n with $n = p, p+1, p+2$ with trivial center. Then G is characterized by $N(G)$.*

So is there a group with connected prime graph for which Thompson's conjecture would be true? Up to now, we have the following (see [11], [20], [23], [26] and [28])

Theorem 4. *Let G be the alternating groups A_{10}, A_{16}, A_{22}, A_{26} or A_{p+3} with trivial center. Then G are characterized by $N(G)$.*

We know that, in A_{p+4}, $s(A_{p+4}) = 1$ and $2 \sim p, 3 \sim p$. Recently, Yang and Liu prove that Thompson's conjecture is true for A_{27} (see [30]). As the development of this topics, we will prove that Thompson's conjecture is valid for the alternating groups A_{p+4} of degree $p+4$ with $p+i$ composite and $1 \leq i \leq 4$.

We introduce some notations used to the proof of the main theorem. For a group, let $Z(G)$ be its center. For any $1 \neq x \in G$, let x^G denote the conjugacy classes in G containing x and $C_G(x)$ denote the centralizer of x in G. Let G be a group and p a prime. Then denote by G_p the Sylow p-subgroup of G. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ denote the automorphism and outer-automorphism group of G, respectively. Let $\omega(G)$ denote the set of element order of G. The other notations are standard (see [9], for instance).

2 Preliminary Results

Lemma 5. *[26, Lemma 1.2] [4, Lemma 2.3] Let $x, y \in G$, $(|x|, |y|) = 1$, and $xy = yx$. Then*

(1) $C_G(xy) = C_G(x) \cap C_G(y)$;

(2) $|x^G|$ divides $|(xy)^G|$;

(3) If $|x^G| = |(xy)^G|$, then $C_G(x) \leq C_G(y)$.

Lemma 6. *[26, Lemma 3] If P and H are finite groups with trivial centers, and $N(P) = N(H)$, then $\pi(P) = \pi(H)$.*

Lemma 7. *[26, Lemma 4] Suppose that G is a finite group with trivial center and p is a prime from $\pi(G)$ such that p^2 does not divide $|x^G|$ for all x in G. Then a Sylow p-subgroup of G is elementary abelian.*

Lemma 8. *[26, Lemma 5]Let K be a normal subgroup of G, and $\overline{G} = G/K$.*

(1) If $\overline{x}$ is the image of an element x of G in $\overline{G}$. Then $|\overline{x}^{\overline{G}}|$ divides $|x^G|$.

(2) If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.

(3) If $y \in K$, then $|y^K|$ divides $|y^G|$.

Let $\exp(n, r)$ denote the nonnegative integer a such that $r^a \mid n$ but $r^{a+1} \nmid n$.

Lemma 9. *[19] Let A_{p+4} be the alternating group of degree $p + 4$, where p is a prime. Then the following hold.*

(1) $\exp(|A_{p+4}|, 2) = \sum_{i=1}^{\infty}[\frac{p+4}{2^i}] - 1$. In particular, $\exp(|A_{p+4}|, 2) \leq p + 3$.

(2) $\exp(|A_{p+4}|, r) = \sum_{i=1}^{\infty}[\frac{p+4}{r^i}]$ for each $r \in \pi(A_{p+4})\backslash\{2\}$. Furthermore, $\exp(|A_{p+4}|, r) < \frac{p+4}{2}$, where $3 \leq r \in \pi(A_{p+4})$. In particular, if $r > [\frac{p+4}{2}]$, then $\exp(|A_{p+4}|, r) = 1$.

Let S_n be the symmetric group of degree n. Assume that the cycle has c_1 1-cycles, c_2 2-cycles, and so on, up to c_k k-cycles, where $1c_1 + 2c_2 + \cdots + kc_k = n$. Then the number of conjugacy class in S_n is

$$z = n!(\prod_{i=1}^{k} i^{c_i} \prod_{i=1}^{k} c_i!)^{-1}. \quad (1)$$

Let A_n be the alternating group of degree n.

Lemma 10. *[13] Let $x \in A_n$. Then for the size of the conjugacy class x^G of x in A_n, we have:*

(1) If for all even i, $c_i = 0$ and for all odd i, $i \in \{0, 1\}$, then $|x^G| = z/2$.

(2) In all other cases, $|x^G| = z$.

In particular, $|x^G| \geq z/2$.

Lemma 11. *[17, Lemma 1] If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here*

- *$s(n) = 6$ for $n \geq 48$;*
- *$s(n) = 5$ for $42 \leq n \leq 47$;*
- *$s(n) = 4$ for $38 \leq n \leq 41$;*
- *$s(n) = 3$ for $18 \leq n \leq 37$;*
- *$s(n) = 2$ for $14 \leq n \leq 17$;*
- *$s(n) = 1$ for $6 \leq n \leq 13$.*

In particular, for every natural number $n > 6$, there exists a prime p such that $\frac{n+1}{2} < p < n - 1$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

Lemma 12. *[12, Lemma 8] Let $q > 1$ be an integer, m be a nature number, and p be an odd prime. If p divides $q - 1$, then $(q^m - 1)_p = m_p \cdot (q - 1)_p$.*

Lemma 13. *[18] Let G be a finite non-abelian simple group and p is the largest prime divisor of $|G|$ with $p \| |G|$. Then $p \nmid |\mathrm{Out}(G)|$.*

Lemma 14. *[31]Let a, b and n be positive integers such that $(a, b) = 1$. Then there exists a prime p with the following properties:*

- *p divides $a^n - b^n$,*
- *p does not divide $a^k - b^k$ for all $k < n$,*

with the following exceptions: $a = 2, b = 1$; $n = 6$ and $a + b = 2^k$; $n = 2$.

Lemma 15. *[10][14] With the exceptions of the relations* $(239)^2 - 2(13)^4 = -1$ *and* $(3)^5 - 2(11)^2 = 1$ *every solution of the equation*

$$p^m - 2q^n = \pm 1; \qquad p, q \quad prime; \qquad m, n > 1,$$

has exponents $m = n = 2$*; i. e., it comes from a unit* $p - q.2^{\frac{1}{2}}$ *of the quadratic field* $Q(2^{\frac{1}{2}})$ *for which the coefficients* p *and* q *are primes.*

Remark 16. *If* $b = 1$*, the prime* p *is called the Zsigmondy prime. If* p *is a Zsigmnody of* $a^n - 1$*, then Fermat's little theorem shows that* $n \mid p - 1$*. Put* $Z_n(a) = \{p : p$ *is a Zigmondy prime of* $a^n - 1\}$*. If* $r \in Z_n(a)$ *and* $r \mid a^m - 1$*, then* $n \mid m$*.*

Let L be a nonabelian simple group and let O denote the order of the outer-automorphism group of L.

Lemma 17. *[15] Let L be a nonabelian simple group. Then the orders and their outer-automorphism of L are as listed in Tables 1, 2 and 3.*

3 Main theorem and its proof

In this section, we shall give the main theorem and its proof.

Theorem 18. *Let* G *be a finite group with trivial center and* $N(G) = N(A_{p+4})$ *with* $p + i$ *composite and* $1 \le i \le 4$*. Then* G *is isomorphic to* A_{p+4}*.*

Proof: We know that if $k = 3$, the groups A_{p+3} are characterized by $N(G)$ (see [23]). Then in the following we only consider when $p \ge 23$ and let $L = A_{p+4}$.

We divide the proof into the following lemmas.

Lemma 19. *The following hold.*

(1) If $2 \ne r \le [\frac{p+4}{2}]$*, then we can write* $p+4 = kr + m$ *with* $0 \le m < r$ *and conjugacy class sizes of* r*-elements of* L *are* $\frac{(p+4)!}{(p+4-ir)! \cdot r^i \cdot i!}$ *for possible* i *with* $1 \le i \le k = [\frac{p+4}{r}]$*.*

In particular, if r *is an odd prime divisor of* $|G|$*, then conjugacy class sizes of* r*-element of* L *are* $\frac{(p+4)!}{(p+4-r)! \cdot r}$*,* $\frac{(p+4)!}{2 \cdot k! \cdot r^2}$*, where* $p + 4 = 2r + k$ *and* $0 \le k < r$*.*

(2) If $r = 2$*, then we can write* $p + 4 = 2k + m$ *with* $0 \le m \le 1$ *and conjugacy class sizes of* 2*-elements of* L *are* $\frac{(p+4)!}{(p+4-2i)! \cdot 2^{2i} \cdot (2i)!}$ *for possible* i *with* $1 \le i \le k = [\frac{p+4}{2}]$*.*

(3) If $r > [\frac{p+4}{2}]$*, then we can write* $p + 4 = r + m$ *with* $0 \le m < r$ *and conjugacy class size of* r*-elements of* L *is* $\frac{(p+4)!}{(p+4-r)! \cdot r}$*.*

In particular, if $r = p$*, then the conjugacy class size of* p*-elements of* L *is* $\frac{(p+4)!}{4! \cdot p}$*.*

(4) The following numbers from $N(G)$ *are maximality with respect to divisibility.*

(4.1) $\frac{(p+4)!}{2mr^2}$ *if* $2 \cdot r + m = p + 4$ *with* m *odd;*

(4.2) One of the following holds: $\frac{(p+4)!}{2 \cdot 3 \cdot p}$ *if* $p+4 = r+4$*;* $\frac{(p+4)!}{2 \cdot (m-2) \cdot r}$ *if* $p+4 = r+m$ *with* $6 \le m$ *even.*

(5) p'*-numbers in* $N(L) \backslash \{1\}$ *are* $\frac{(p+4)!}{4! \cdot p}$*;* $\frac{(p+4)!}{2 \cdot 3 \cdot p}$*;* $\frac{(p+4)!}{2^2 \cdot 2 \cdot p}$*;* $\frac{(p+1)(p+2)(p+3)(p+4)}{2^2 \cdot 2}$*;* $\frac{(p+2)(p+3)(p+4)}{3}$*.*

Proof: From equation 1 and Lemma 10, we get the desired results. □

Lemma 20. *Let* G *be a finite group with trivial center and* $N(G) = N(L)$*. Then* $|L| \mid |G|$ *and* $\pi(G) = \pi(L)$*.*

Proof: Since $|x^G||C_G(x)| = |G|$, every member form $N(G)$ divides the order of G and $|L| \mid |G|$. So by Lemmas 6, we have that $\pi(G) = \pi(L)$. □

Lemma 21. *Suppose that* G *is a finite group with trivial center and* $N(G) = N(L)$*. Then the following hold.*

(1) There exist different primes r_1, r_2, p *from* $\pi(L)$ *such that* $r_1, r_2, p > [\frac{p+4}{2}]$*. In particular, the Sylow* r*-subgroup* S *of* G *is a cyclic group of order* r *where* $r \in \{r_1, r_2, p\}$*. There does not exist an element of order* $r_1 \cdot r_2$*,* $r_1 \cdot p$ *or* $r_2 \cdot p$*.*

(2) For all $n \in N(G)$*, if* n *is divisible at most by* r^a*, then the Sylow* r*-subgroup* S *of* G *is of order* r^a*.*

Proof: (1) By Lemma 11, there exist different prime numbers r_1, r_2, p from $\pi(G)$ such that $r_1, r_2, p > [\frac{p+4}{2}]$.

By Lemmas 19 and 20, it is easy to see that the primes r_1, r_2, p are prime divisors of $|G|$ and r_1^2, r_2^2, p^2 do not divide $|x^G|$ for all $x \in G$. On the other hand, by Lemma 7 we have that S is elementary abelian.

Let $|S| \ge p^2$. Consider an element y of G with $|y^G| = \frac{(p+4)!}{2m \cdot r^2}$ if $2r + m = p + 4$with $m < r$ by Lemma 19.

We consider two cases: $p \mid |y|$ and $p \nmid |y|$.

Table 1: The simple classical groups

L	Lie; rank **L**	d	O	$\|L\|$
$L_n(q)$	$A_{n-1}(q)$ $n-1$	$(n, q-1)$	$2df$, if $n \geq 3$; df, if $n = 2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1)$
$U_n(q)$	${}^2A_{n-1}(q)$ $[n/2]$	$(n, q+1)$	$2df$, if $n \geq 3$ df, if $n = 2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)$
$PSp_{2m}(q)$	$C_m(q)$ m	$(2, q-1)$	df, $m \geq 3$; $2f$, if $m = 2$	$\frac{1}{d}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$
$\Omega_{2m+1}(q)$ q odd	$B_m(q)$ m	2	$2f$	$\frac{1}{2}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$
$P\Omega_{2m}^{+}(q)$ $m \geq 3$	$D_m(q)$ m	$(4, q^m-1)$	$2df$, if $m \neq 4$ $6df$, if $m = 4$	$\frac{1}{d}q^{m(m-1)(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1)}$
$P\Omega_{2m}^{-}(q)$ $m \geq 2$	${}^2D_m(q)$ $m-1$	$(4, q^m+1)$	$2df$	$\frac{1}{d}q^{m(m-1)(q^m+1)\prod_{i=1}^{m-1}}(q^{2i}-1)$

Table 2: The simple exceptional groups

L	**L**	d	O	$\|L\|$
$G_2(q)$	2	1	f, if $p \neq 3$ $2f$, if $p = 3$	$q^6(q^2-1)(q^6-1)$
$F_4(q)$	4	1	$(2,p)f$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$E_6(q)$	6	$(3, q-1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-1)$
$E_7(q)$	7	$(2, q-1)$	df	$\frac{1}{d}q^{63}\prod_{i\in\{2,6,8,10,12,14,18\}}(q^i-1)$
$E_8(q)$	8	1	f	$q^{120}\prod_{i\in\{2,8,12,14,18,20,24,30\}}(q^i-1)$
${}^2B_2(q), q = 2^{2m+1}$	1	1	f	$q^2(q^2+1)(q-1)$
${}^2G_2(q), q = 3^{2m+1}$	1	1	f	$q^3(q^3+1)(q-1)$
${}^2F_4(q), q = 2^{2m+1}$	2	1	f	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$
${}^3D_4(q)$	2	1	$3f$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
${}^2E_6(q)$	4	$(3, q+1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-(-1)^i)$

Table 3: The simple sporadic groups

L	d	O	$\|L\|$
M_{11}	1	1	$2^4\cdot3^2\cdot5\cdot11$
M_{12}	2	2	$2^6\cdot3^3\cdot5\cdot11$
M_{22}	12	2	$2^7\cdot3^2\cdot5\cdot7\cdot11$
M_{23}	1	1	$2^7\cdot3^2\cdot5\cdot7\cdot11\cdot23$
M_{24}	1	1	$2^{10}\cdot3^3\cdot5\cdot7\cdot11\cdot23$
J_1	1	1	$2^3\cdot3\cdot5\cdot7\cdot11\cdot19$
J_2	2	2	$2^7\cdot3^3\cdot5^2\cdot7$
J_3	3	2	$2^7\cdot3^5\cdot5\cdot17\cdot19$
J_4	1	1	$2^{21}\cdot3^3\cdot5\cdot7\cdot11^3\cdot23\cdot29\cdot31\cdot37\cdot43$
HS	2	2	$2^9\cdot3^2\cdot5^3\cdot7\cdot11$
Suz	6	2	$2^{13}\cdot3^7\cdot5^2\cdot7\cdot11\cdot13$
McL	3	2	$2^7\cdot3^6\cdot5^3\cdot7\cdot11$
Ru	2	1	$2^{14}\cdot3^3\cdot5^3\cdot7\cdot13\cdot29$
$He(F_7)$	1	2	$2^{10}\cdot3^3\cdot5^2\cdot7^3\cdot17$
Ly	1	1	$2^8\cdot3^7\cdot5^6\cdot7\cdot11\cdot31\cdot37\cdot67$
ON	3	2	$2^9\cdot3^4\cdot5\cdot7^3\cdot11\cdot19\cdot31$
Co_1	2	1	$2^{21}\cdot3^9\cdot5^4\cdot7^2\cdot11\cdot13\cdot23$
Co_2	1	1	$2^{18}\cdot3^6\cdot5^3\cdot7\cdot11\cdot23$
Co_3	1	1	$2^{10}\cdot3^7\cdot5^3\cdot7\cdot11\cdot23$
Fi_{22}	6	2	$2^{17}\cdot3^9\cdot5^2\cdot7\cdot11\cdot13$
Fi_{23}	1	1	$2^{18}\cdot3^{13}\cdot5^2\cdot7\cdot11\cdot13\cdot17\cdot23$
Fi'_{24}	3	2	$2^{21}\cdot3^{16}\cdot5^2\cdot7^3\cdot11\cdot13\cdot17\cdot23\cdot29$
$HN(F_5)$	1	2	$2^{14}\cdot3^6\cdot5^6\cdot7\cdot11\cdot19$
$Th(F_3)$	1	1	$2^{15}\cdot3^{10}\cdot5^3\cdot7^2\cdot13\cdot19\cdot31$
$BM(F_2)$	2	1	$2^{41}\cdot3^{13}\cdot5^6\cdot7^2\cdot11\cdot13\cdot17\cdot19\cdot23\cdot31\cdot47$
$M(F_1)$	1	1	$2^{46}\cdot3^{20}\cdot5^9\cdot7^6\cdot11^2\cdot13^3\cdot17\cdot19\cdot23\cdot29\cdot31\cdot41\cdot47\cdot59\cdot71$

Case 1. $p \nmid |y|$.

Let x be an element of $C_G(y)$ having order p. Then $C_G(xy) = C_G(x) \cap C_G(y)$, $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$ by Lemma 5. Since S is abelian, $S \leq C_G(x)$. Hence, $p \nmid |x^G|$. It follows that $|x^G|$ equals to $\frac{(p+4)!}{4!\cdot p}$; $\frac{(p+4)!}{2\cdot 3\cdot p}$; $\frac{(p+4)!}{2^2\cdot 2\cdot p}$; $\frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$; $\frac{(p+2)(p+3)(p+4)}{3}$, by Lemma 19.

If $|x^G| = \frac{(p+4)!}{4!\cdot p}$, $\frac{(p+4)!}{2\cdot 3\cdot p}$ or $\frac{(p+4)!}{2^2\cdot 2\cdot p}$, then obviously, there is no number from $N(G)$ such that $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$.

Therefore $|x^G|$ equals to $\frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$, or $\frac{(p+2)(p+3)(p+4)}{3}$. In the following, we will consider the following two subcases.

Subcase 1: $|x^G| = \frac{(p+2)(p+3)(p+4)}{3}$.

If $m = 3$, then obviously, $r \mid \frac{(p+2)(p+3)(p+4)}{3}$. Therefore $\frac{(p+4)!}{r} \mid |(xy)^G|$, a contradiction since $|x^G| \mid |(xy)^G|$, $|y^G| \mid |(xy)^G|$ and the maximality of $|y^G| = \frac{(p+4)!}{2m\cdot r^2}$.

If $m \geq 5$ is odd or $m = 1$, then $r \nmid \frac{(p+2)(p+3)(p+4)}{3}$. Thus $|(xy)^G| = |y^G|$ since the maximality of $|y^G|$ and so by Lemma 5, $C_G(y) \leq C_G(x)$. On the other hand, $p \nmid |x|$ and $p \| |C_G(x)|$. Since $|S| \geq p^2$, then $p \mid |x^G|$. It follows from Lemma 1.2 of [6], that there is a p-element w such that $1 \neq w \in C_G(x)$, $C_G(wx) < C_G(x)$ and $p \mid \frac{|C_G(x)|}{|C_G(wx)|} = 1$, a contradiction.

Subcase 2: $|x^G| = \frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$.

If $m = 1$ or 3, then $r \mid \frac{(p+1)(p+2)(p+3)}{3}$. Therefore $\frac{(p+4)!}{r} \mid |(xy)^G|$.

If $m \geq 5$ is odd, then $r \nmid \frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$. On the other hand, obviously, $p \nmid |x|$ and $p \| |C_G(x)|$. Since $|S| \geq p^2$, then $p \mid |x^G|$. Thus by Lemma 1.2 of [6], we also get a contradiction as "Subcase 1".

Case 2. $p \mid |y|$.

Let $|y| = p \cdot t$. Since S is elementary abelian, the numbers p and t are coprime. Let $u = y^p, v = y^t$. Then $y = uv$, $C_G(uv) = C_G(u) \cap C_G(v)$. Therefore, $|v^G| \mid |y^G| = \frac{(p+4)!}{2\cdot m\cdot r^2}$ if $2r + m = p + 4$ and $1 \leq m < r$.

On the other hand, the element v of G is of order p. Since the Sylow p-subgroup of G is elementary abelian, then $p \nmid |v^G|$. It follows that $|v^G|$ equals to $\frac{(p+4)!}{4!\cdot p}$; $\frac{(p+4)!}{2\cdot 3\cdot p}$; $\frac{(p+4)!}{2^2\cdot 2\cdot p}$; $\frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$; $\frac{(p+2)(p+3)(p+4)}{3}$; by Lemma 19.

If $|v^G|$ equals to $\frac{(p+4)!}{4!\cdot p}, \frac{(p+4)!}{2\cdot 3\cdot p}$ or $\frac{(p+4)!}{2^2\cdot 2\cdot p}$, then $|v^G| \mid |y^G|$, a contradiction. Hence $|v^G| = \frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$ or $|v^G| = \frac{(p+2)(p+3)(p+4)}{3}$. We consider the following two subcases.

Subcase 1: $|v^G| = \frac{(p+2)(p+3)(p+4)}{3}$.

If $m = 3$, then obviously, $r \mid \frac{(p+2)(p+3)(p+4)}{3}$. But $r \nmid \frac{(p+4)!}{2m\cdot r^2}$, a contradiction since $|x^G| \mid |(xy)^G|$, $|y^G| \mid |(xy)^G|$ and the maximality of $|y^G| = \frac{(p+4)!}{2m\cdot r^2}$.

If $m = 1$ or $m \geq 5$ is odd, then $r \nmid \frac{(p+2)(p+3)(p+4)}{3}$. Obviously $p \| |C_G(v)|$ and $p \nmid |v|$. Since $|S| \geq p^2$, then $p \mid |v^G|$. It follows from Lemma 1.2 of [6], that there is a p-element w such that $1 \neq w \in C_G(v)$, $C_G(wv) < C_G(w)$ and $p \mid \frac{|C_G(w)|}{|C_G(wv)|} = \frac{|(wv)^G|}{w^G} = 1$(since $wv = vw$, then by Lemma 5, $|v^G| \mid |(wv)^G| = |w^G|$).

Subcase 2: $|v^G| = \frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$.

If $m = 1, 3$, then obviously, $r \mid \frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$. But $r \nmid \frac{(p+4)!}{2mr^2}$, a contradiction since $|x^G| \mid |(xy)^G|$, $|y^G| \mid |(xy)^G|$ and the maximality of $|y^G| = \frac{(p+4)!}{2m\cdot r^2}$.

If $m \geq 5$ is odd, then similarly we can rule out as "Subcase 1".

Therefore the Sylow p-subgroup of G is of order p.

Similarly we can prove the other two cases.

There does not exist an element of order $r_1 \cdot r_2$, $r_1 \cdot p$ or $r_2 \cdot p$.

(2) Without loss of generality, we assume that n is divisible at most by r^2.

Assume that $|S| \geq r^3$. Consider an element x of G such that

$$|x^G| = \begin{cases} \frac{(p+4)!}{2\cdot 3\cdot p}, & \text{if}\, r+4 = p+4 \\ \frac{(p+4)!}{2m(m-2)p}, & \text{if}\, r+m = p+4 \\ & \text{with}\, 6 \leq m \text{ even} \end{cases}$$

by Lemma 19.

Let $r \nmid |x|$. Then there is an element y of G of order r. By Lemma 5 we have that $C_G(xy) = C_G(x) \cap C_G(y)$, $|x^G| \mid |(xy)^G|$, and $|y^G| \mid |(xy)^G|$.

If y is an r-central element, then $S \leq C_G(y)$ and $|y^G|$ is an r'-number and hence, $|y^G|$ equals to one of the numbers which lie in between $\frac{(p+4)!}{2m!\cdot r^2}$ to $\frac{(p+4)!}{2mr^2}$ where $2 \cdot r + m = p + 4$ with m odd. It follows that $\frac{(p+4)!}{2} \in N(G)$, a contradiction since there is no number from $N(G)$ such that $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$ and the maximality of $|x^G|$.

If y is a noncentral r-element, then we choose an element z of order p such that $r^2 \| |z^G|$ (we choose a maximal number $|z^G|$ from $N(G)$). By hypothesis, $r \mid |C_G(z)|$ and obviously, $r \nmid |z|$. Then by Lemma 1.2 of [6], there is an r-element w such that $1 \neq w \in C_G(z)$, $C_G(wz) < C_G(w)$, and

$$r \mid \frac{|C_G(w)|}{|C_G(wz)|} = \frac{|(wz)^G|}{|w^G|}.$$

On the other hand, we also have $z \in C_G(w)$. Since $wz = zw$, then $|(wz)^G| = |z^G|$ and so $C_G(z) \le C_G(w)$. Therefore $z \in C_G(z) \le C_G(w)$. It follows that $|C_G(z)| = |C_G(wz)|$ since the maximality of $|z^G|$, and we get $r \mid \frac{|C_G(w)|}{|C_G(wz)|} = 1$, a contradiction.

Let $r \mid |x|$, then we write $|x| = rt$. If S is elementary abelian, then $(r,t) = 1$. Set

$$u = x^r, \quad v = x^t.$$

Then $x = uv$, $C_G(x) = C_G(u) \cap C_G(v)$. Hence $|v^G| \mid x^G$ and $|u^G| \mid |x^G|$. Since S is abelian and v is an element of order r, then $|v^G|$ equals to one of the numbers that lie in between $\frac{(p+4)!}{2m!\cdot r^2}$ to $\frac{(p+4)!}{2m\cdot r^2}$ where $2r+m = p+4$. It follows that $|(uv)^G| = |v^G|$ and so $r^3 \mid |S| \mid |v^G|$ and hence, $r \nmid |C_G(v)| = |C_G(uv)| = |C_G(x)|$ contradicting the fact that x is an r-element.

Therefore S is non-abelian. So we chose an element z of order p such that $r^2 \mid |z^G|$ (we only consider the element $|z^G|$ which is the maximality with respect to divisibility from $N(G)$). By hypothesis, $r \mid |C_G(z)|$ and obviously, $r \nmid |z|$. Then by Lemma 1.2 of [6], there is a r-element w such that $w \in C_G(z)$, $C_G(wz) < C_G(z)$ and we get $r \mid \frac{|C_G(w)|}{|C_G(wz)|} = \frac{|(wz)^G|}{|w^G|} = 1$, a contradiction.

For the other case, we similarly can prove as the case "$|S| = r^2$".

The Lemma is proved. □

Lemma 22. *Suppose that G is a finite group with trivial center and $N(G) = N(L)$. Let $\pi = \{2,3\}$. Then $O_{\pi,\pi'}(G) = O_\pi(G)$. In particular, G is insoluble.*

Proof: Let $K = O_\pi(G)$, $\overline{G} = G/K$ and denote by $\overline{x}$ and by $\overline{H}$ the images of an element x and a subgroup H of G in $\overline{G}$, respectively. Assume that the result is not true, then there is a prime $r \in \pi(L)\backslash\pi$ with $O_r(\overline{G}) \neq 1$.

Let $r \in \{r_1, r_2, p\}$ with $O_r(\overline{G}) \neq 1$. Then $\overline{G}$ contains a Hall $\{r \cdot s\}$-subgroup of order $r \cdot s$ with $s \in \{r_1, r_2, p\} - \{r\}$. However, Hall $\{r,s\}$-subgroup must be cyclic contradicting Lemma 21.

Let P be a Sylow r-subgroup of $\overline{G}$ where $r \in \pi(L)\backslash\{r_1, r_2, p\}$. If $O_r(\overline{G}) \neq 1$, then $A = Z(O_r(\overline{G}))$ is a nontrivial normal subgroup of $\overline{G}$. Let $\overline{x}$ be an element of order p in $\overline{G}$. So we have that $|\overline{x}^{\overline{G}}|$ is a divisor of $\frac{(p+4)!}{4!\cdot p}$, $\frac{(p+4)!}{2\cdot 3\cdot p}$, $\frac{(p+4)!}{2^2\cdot 2\cdot p}$, $\frac{(p+1)(p+2)(p+3)(p+4)}{2^2\cdot 2}$ or $\frac{(p+2)(p+3)(p+4)}{3}$.

By coprime action lemma, $A = C_A(\overline{x}) \times [A, \overline{x}]$. In the following, we consider two cases "$r \le [\frac{p+4}{2}]$ and $r \ge [\frac{p+4}{2}]$".

- $r > [\frac{p+4}{2}]$ and $r \neq r_1, r_2, p$. In this case, by Lemma 21, we have that the Sylow r-subgroup of G is of order r. Hence there is a Hall $\{r,p\}$-subgroup H. Since H must be cyclic, then there is an element of order $r \cdot p$, a contradiction by the proof of Lemma 21.
- $r \le [\frac{p+4}{2}]$ and $r \neq 2, 3$. If $2r + m = p+4$, then the index of $C_A(\overline{x})$ in A is at most r^2. By Lemma 14, there exists a least divisor m of $\phi(p)$ such that p divides $r^m - 1$, and the subgroup $[A, \overline{x}] < \overline{x} >$ must be abelian. It follows that $[A, \overline{x}] = 1$, and $A = C_A(\overline{x})$. Let P be a Sylow p-subgroup of G. If $\overline{y}$ is a nontrivial element of $Z(P) \cap A$, then the order of $C_{\overline{G}}(\overline{y})$ is a multiple of p. By Lemma 8, y lies in the center of a Sylow p-subgroup of G. This contradicts Lemma 19. Thus $O_r(\overline{G}) = 1$.

It follows that $O_r(\overline{G}) = 1$ for $r \in \{5, 7, ..., p\}$.

Therefore $O_{\pi,\pi'}(G) = O_\pi(G)$. In particular, G is insoluble. □

Lemma 23. *There is a normal series $1 \le K \le H \le G$ such that $H/K \cong A_{p+3}$.*

Proof: By Lemmas 20 and 21, $|G| = \frac{(p+4)!}{2}$.

By Lemma 22, we have that $H/K \le \overline{G} \le \mathrm{Aut}(H/K)$, where $M : H/K = S_1 \times S_2 \times \cdots \times S_k$ is a direct product of non-abelian simple groups S_1, S_2, $\cdots$, S_k. Since G cannot contain a Hall $\{r_1, r_2, p\}$-subgroup, numbers r_1, r_2, and p divide the order of exactly one of these groups that is listed as Lemma 17, and so we assume that they divide S_1. Since $S_1 \lhd \overline{G}$, we let G^* and M^* denote the factor groups $\overline{G}/S_1$ and M/S_1, respectively. If $k > 1$. Then a Sylow 2-subgroup of G^* is a non-trivial and its center Z has a nontrivial intersection with M^*. Consider a nontrivial element y of $T = S_2 \times \cdots \times S_k$ such that its image in $\overline{G}$ lies in Z. Since y centralizes S_1, it lies in the center of a Sylow 2-subgroup of $\overline{G}$ and centralizes an element of order p, a contradiction. Thus $M = S_1$, and $\overline{G}$ is almost simple. Therefore

$$H/K \le \overline{G} \le \mathrm{Aut}(H/K).$$

Obviously, $r_1, r_2, p \mid |H/K|$ (in fact, if $r_1, r_2, p \mid |G/H|$, then $r_1, r_2, p \mid |G/H| \mid |\mathrm{Out}(H/K)|$, a contradiction from Lemma 17; if $r_1, r_2, p \mid |K|$, then there is an element of order $r \cdot p$ with $r \in \{r_1, r_2\}$ contradicting Lemma 21). In the following, we always assume that $r \in \pi(G) = \pi(L)$. In the following, we consider S_1 which is listed as in Tables 1, 2 and 3.

Case 1. $H/K \cong A_n$ with $n \ge 6$.

Then $n = p, p+1, \cdots, p+k$ with $p+2, p+4, \cdots$ composite and $p+k+1$ prime. If $k \ge 5$, then $\frac{(p+k)!}{2} \mid (p+3)!$, a contradiction. Therefore H/K is isomorphic to A_p, A_{p+1}, A_{p+2}, A_{p+3} or A_{p+4}.

Let x be an element of order p in H. Then $|x^H|$ is p'-number since $|H|_p = p$. Let $H/K \cong A_p$.

Since $|A_p| \mid (p+4)!$, then $3 \mid |K|$. We have $|x^H| = \frac{(p-1)!}{2}$. On the other hand, $|x^G| = \frac{(p+3)!}{6p}$. It follows that $|x^K| \mid \frac{(p+1)(p+2)(p+3)(p+4)}{4!}$ and so there is an element of $r \cdot p$ or of order $r' \cdot p$ with $5 \le r' < r < p$ and r and r' divide one of the prime divisor of the numbers $p+1$, $p+2$, $p+3$ and $p+4$, which contradicts Lemma 21.

Similarly, we can rule out these cases "$H/K \cong A_{p+1}$, $H/K \cong A_{p+2}$ and $H/K \cong A_{p+3}$".

Therefore $H/K \cong A_{p+4}$.

Case 2. H/K is not isomorphic to a sporadic simple group according to Table 3.

Case 3. H/K is isomorphic to a simple group of Lie type.

Let $q' = r'^{f'}$.

1. $S \cong B_n(q)$ with $n \ge 2$.

   In this situation, by hypothesis, $\pi(G) = \{2, 3, 5, 7, \cdots, p\}$ and so

   $$\frac{1}{(2, q-1)} q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1) \mid (p+4)!.$$

   It follows that $p \mid q$ or $p \mid \prod_{i=1}^{n}(q^{2i}-1)$. If $p \mid q$, then q is a power of p. Since $|G_p| = p$ by hypothesis, this is impossible as $n \ge 2$. Therefore $p \mid \prod_{i=1}^{n}(q^{2i}-1)$. It follows that $p \mid q^{2t}-1$ for some $1 \le t \le n$ as p is prime. If $p \mid q^2-1$, then $p \mid q^4 - 1$ and hence $p \mid q^{2n}-1$. Since $|G_p| = p$, then $p \nmid q^{2n-2}-1$. Then without loss of generality, we assume that $p = q^n - 1$ or $p = q^n + 1$ and hence, $2 \mid q$ by Lemma 14. By Fermat's little theorem, $2n \le p-1$ and so, $n^2 \le 2n+5$. It follows that $n = 2, 3$ and order considerations rule out this case.

2. $S \cong D_n(q)$ with $n \ge 4$.

   Therefore we have that $\frac{1}{(4,q^n-1)} q^{n(n-1)} (q^n - 1) \prod_{i=1}^{n-1}(q^{2i}-1) \mid (p+4)!$. Since the Sylow p-subgroup of G is of order p, $p \nmid q$ as otherwise, $q = p$ and thus $n = 1$, a contradiction. It follows that $p \mid q^n - 1$ or $p \mid q^{2t}-1$ for some integer $1 \le t \le n-1$. If $p \mid q^2-1$, then by Remark 16, $n \mid p-1$ and so $n+5 \le p+4$. By Lemma 9, $\frac{n(n+1)}{2} \le \frac{n+3}{2}$ and hence, $n = 1$, a contradiction. If $p \mid q^{2n-2}-1$, then similarly, $2n-2 \le p-1$ and so $\frac{n(n+1)}{2} \le \frac{2n+3}{2}$. But the equation has no solution in $\mathbb{N}$ since $n \ge 4$.

3. $S \cong {}^2A_n(q)$ with $n \ge 2$.

   In this situation, $\frac{1}{(n+1,q+1)} q^{\frac{1}{2}n(n+1)} \prod_{i=1}^{n}(q^{i+1} - (-1)^{i+1}) \mid (p+4)!$. Since the Sylow p-subgroup of G is of order p and $n \ge 2$, we obtain that $p \mid q^{t+1} - (-1)^{t+1}$ for some integer $1 \le t \le n$.

   Let n be odd. Then $p \mid q^{n+1}+1$. If q is odd, then $2 \| q^{n+1}+1$. If $p = \frac{q^{n+1}+1}{2}$, then by Lemma 15, we have a contradiction. Hence $2n+2 \le p-1$. By Lemma 9, $\frac{n(n+1)}{2} \le \frac{2(n+1)+5}{2}$ and so $n = 3$. Order consideration and Lemma 13 imply that it is impossible. Hence q is even. Similarly we can rule out.

   Let n be even. Then $p \mid q^{n+1}-1$. If q is odd, then by Lemma 12, $n+1 \mid p-1$ and hence, by Lemma 9, $\frac{n(n+1)}{2} \le \frac{n+6}{2}$. So $n = 2$, order consideration rules out. So q is even. Similarly we have $n+4 \le p+2$ and $\frac{n(n+1)}{2} \le \frac{n+6}{2}$. Therefore $n = 2$. Order consideration and Lemma 13 rule out this case.

4. $S \cong E_8(q)$.

   Therefore we have that $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1) \mid (p+4)!$. It follows that $p \mid q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$. Hence $p \mid q^t - 1$, where $t \in \{14, 18, 20, 24, 30\}$.

   Let $t = 14$. If q is odd, then by Lemma 11, there is a prime $r > p$, a contradiction. Hence $p \mid q^{30}-1$ and by Remark 16, $30+5 \le p+4$. It follows from Lemmas 12 and 9, that $2^{14} \cdot (q-1)_2^8 \le 2^{35-1}$ and so $q = 3, 5$, order consideration rules out. If q is even, then similarly we have that $q^{120} \mid 2^{35-1}$, a contradiction. Similarly, we can exclude that $H/K \cong E_6(q), E_7(q)$ and $F_4(q)$.

5. $S \cong G_2(q)$.

   Then we have $q^6(q^6-1)(q^2-1) \mid (p+4)!$. It follows that $p \mid q^6-1$ or $p \mid q^2-1$. If $p \mid q^2-1$, then $p \mid q^6-1$. Hence we only consider $p \mid q^6-1$ and $6 \mid p-1$. If q is odd, then by Lemma 9, $6 \le \frac{6+5}{2}$, a contradiction. Hence q is even. Similarly, we have $6 \le \frac{6+4}{2}$, a contradiction.

6. $S \cong {}^2E_6(q)$.

   It is easy to see that $\frac{1}{(3,q+1)} q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1) \mid p!$. It follows that $p \mid q^t - 1$ with $t = 12, 8$, or $p \mid q^k + 1$ with $k = 9, 5$.

   Let $t = 12$. If q is odd, then $2 \mid q-1$ and $2 \mid q+1$. It follows from Lemma 9, that $|S|_2 = 2^7 \cdot (q-1)_2^4 \cdot (q+1)_2^2$ and $\exp(|S|, 2) \ge 15$. On the other hand, $12 \le p-1$ by Remark 16. By Lemma 9, $15 \le \exp(|S|, 2) \le 16$ and so $q = 3$.

Order consideration rules out. If q is even, then by Lemma 9, $12 \le p-1$ and hence $36 \le 12+4$, a contradiction. Similarly we can rule out "$t = 8$".

Let $t = 9$. If q is odd, then similarly we have that $\exp(|S|, 2) \ge 15$. On the other hand, $12 \le p-1$. Thus we also have $q = 3$ and so $p = 703$. Order consideration rule out. If q is even, $36 \le 12 + 4$, a contradiction. Similarly, we can rule "$t = 5$".

7. $S \cong^2 B_2(q)$ with $q = 2^{2m+1}$.

   It follows that $q^2(q^2+1)(q-1) \mid (p+4)!$. Thus $p \mid q^2+1$ or $p \mid q-1$. Let $p \mid q^2+1$. We can assume that $p = q^2+1$ and hence, $m = 0$. By [9, pp. xv], $S \cong 5 : 4$ is soluble, a contradiction. It is easy to rule out when $p \mid q-1$. Similarly $S \ncong^2 F_4(2^{2m+1})$.

8. $S \cong^2 G_2(q)$, $q = 3^{2n+1}$ with $n \ge 1$.

   We see that $q^3(q^3+1)(q-1) \mid (p+4)!$. It follows that $p \mid q^3+1$ or $p \mid q-1$. If $p \mid q^3+1$, then we can assume that $p = \frac{q^3+1}{4}$ and so, $6n+3 \mid \frac{q^2+9}{2}$. It follows that $n = 1$ and $p = 73$. We can rule out this case by order consideration. If $p \mid q-1$ and $r \mid q$, then there exists a Frobenius group of $r \cdot p$ with a Kernel of order r and a complement of order p respectively, and so there is an element of order $r \cdot p$, which contradicts the fact that $\deg(p) = 1$.

9. $S \cong^3 D_4(q)$.

   We have that $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1) \mid (p+4)!$. In this case, since G has a Sylow p-subgroup of order p, then $p \mid q^8+q^4+1$, or $q \mid q^6-1$. If $p \mid q^8+q^4+1$, then by Remark 16, $12 \mid p-1$. If q is odd, then $12 \mid 6$, a contradiction.

   If $p \mid q^6-1$, then $6 \mid p-1$ and similarly, we also can rule out.

   Similarly we can rule out this case "$p \mid q^2-1$".

10. $S \cong A_n(q)$ with $n \ge 1$.

    It is easy to get that $\frac{1}{(n+1,q-1)} q^{n(n+1)/2} \prod_{i=1}^{n}(q^{i+1}-1) \mid (p+4)!$. It follows that $p \mid \prod_{i=1}^{n}(q^{i+1}-1)$ and so $p \mid q^{t+1}-1$ for some integer $t = n, n-1$.

    Let $t = n-1$. Then $p \mid q^n-1$ and so $n \le p-1$. If q is odd, then by Lemma 12 $|S|_2 = (q-1)_2^n \cdot \prod_{i=1}^{n}(i+1)_2$ and hence $\exp(|S|, 2) \ge \frac{3n}{2}$. By Lemma 9, we conclude that $\frac{3n}{2} \le n+4$ and $n \le 6$. Order consideration can rule out this case. If q is even, then similarly, $\exp(|S|, 2) \ge \frac{n(n+1)}{2}$ and hence, $\frac{n(n+1)}{2} \le n+4$. Thus we get that $n \le 3$, order consideration rules out.

    Let $t = n$. Then similarly we can rule out as "$t = n-1$".

This completes the proof of the lemma. □

Lemma 24. $G \cong A_{p+4}$.

Proof: By Lemma 23,

$$A_{p+4} \le \overline{G} \le \mathrm{Aut}(A_{p+4}) \cong S_{p+4}.$$

If $\overline{G} \cong S_{p+4}$, then there exists an element $\overline{x}$ of $\overline{G}$ with

$$\overline{x}^{\overline{G}} = \frac{(p+4)!}{4p}$$

which contradicts Lemma 19.

So $\overline{G} \cong A_{p+4}$. Then we define the normal series $1 \le K \le G$ into the chief ones. We prove that $K = 1$. By Lemma 22, $\pi(K) \subseteq \{2, 3\}$.

If K is a 2-group. In this case, let $|\overline{x}| = p$. Then

$$\frac{(p+4)!}{6p} \mid |\overline{x}^{\overline{G}}|.$$

By Lemma 19,

$$|x^G| = |\overline{x}^{\overline{G}}| = \frac{(p+4)!}{6p}.$$

Then x centralizes K and hence $K \le C_G(x)$. It follows that there is an element y of order $2 \cdot p$ having $|y^G| = \frac{(p+4)!}{2 \cdot p}$. It follows from Lemma 5 that, $|x^G| \mid |y^G|$, a contradiction.

If K is a 3-group. Then similarly as the case "K be a 2-group", we have that $|x^G| = |\overline{x}^{\overline{G}}|$ is maximal in $N(G)$ and $C_G(x)$ is abelian. So by Lemma 1.12 of [11], $K \le Z(G) = 1$.

Therefore $K = 1$ and $G \cong A_{p+4}$. This completes the proof of the Lemma □

The main theorem is proved.

4 Some applications

Y. Chen and G. Chen in [8] proved that the group A_{10} can be characterized by its order and two special conjugacy classes sizes. Then obviously, we also have the following result.

Corollary 25. *Let G be a finite group with trivial center. Assume that $N(G) = N(A_{p+4})$ and $|G| = |A_{p+4}|$. Then $G \cong A_{p+4}$.*

We know that the alternating groups A_n with $n = 10, 16, 22, 26$, are characterized by $N(G)$. Then by [6, 7, 14, 20, 30] and our main theorem, we have the following.

Corollary 26. *Let G be a finite group with trivial center. Assume that $N(G) = N(A_n)$ with $n = p, p+1, p+2, p+3, p+4$. Then $G \cong A_n$.*

Shi gave the following conjecture.

Conjecture [25] Let G be a group and H a finite simple group. Then $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.

Then we have the following corollary.

Corollary 27. *Let G is a group and $p \geq 5$ is a prime. Then $G \cong A_n$ where $n = p, p+1, p+2, p+3, p+4$ if and only if $\omega(G) = \omega(A_n)$ and $|G| = |A_n|$.*

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