Mathematical Modelling and Stability Analysis for Diabetes Predicting System

XIULING CAO
Tianjin University
Department of Mathematics
92 Weijin Road, 300072 Tianjin
CHINA
caoxiuling_0602@163.com

DANHONG LIU
Tianjin University
Department of Mathematics
92 Weijin Road, 300072 Tianjin
CHINA
liudanhong@eyou.com

SUZHI LI
TianJin Anning Hospital
Clinical Laboratory
20 Jintang Road, 300300 Tianjin
CHINA
tjlszh@163.com

YU WANG
General Hospital of Tianjin Medical University
Metabolic Department
154 Anshan Road, 300052 Tianjin
CHINA
wangyushao13@126.com

Abstract: In this paper, a model consisting of four states of diabetics is proposed for predicting the development of diabetes. Suppose that all diabetics in the model are with type 2 diabetes and that there is no input or output. While the states of diabetics can alter between each other. Under these assumptions, a model of partial differential equations is derived. The general well-posedness result of the system is obtained and the exponential stability of dynamic solution converging to the steady-state solution is proved. Herein, the steady-state solution gives the stable distribution probability of diabetics (the predicted result). Furthermore, some reliability indices are discussed. The results of some numerical simulations for the cost problem are illustrated to conclude that, if diabetics want to live a higher standard and cost less, they need to follow doctor's advice, learn more knowledge about diabetes, pay attention to their diet and so on which can make the recovery rate larger.

Key–Words: Diabetes; Modelling; Dynamic solution; Stability; Predicting.

1 Introduction

It is well-known that diabetes has been one of the chronic diseases that harm people's health seriously. With the development of society and the improvement of living standard, the morbidity of diabetes and the number of diabetics increase year by year. It has become a social problem that more and more people suffer from diabetes. Diabetes can cause many complications that are serious health threats. Thus, it brings huge economic burden to society and individuals.

There were a lot of studies on diabetes from different aspects, for example, [1] studied the states of the development of diabetes in one country, one region or various ages; [2, 3] studied the social economic burden caused by diabetes and the cost of diabetics; [4, 5] researched the drug effectiveness aiming at glycemic controlling of diabetics; [6] considered the complications problem of diabetes; [7, 8] studied the life quality problem. From the insulin diffusion point of view, some models of subcutaneous insulin kinetics were reviewed in [9]. Researchers in [10] reviewed several models relevant to the subcutaneous injection of insulin analogues in detail and pointed out that these models provide key building blocks for some important endeavors into physiological questions of insulin secretion and action.

With a deeper understanding of diabetes pathology, more and more engineers has begun to participate in the diabetes research using the control theory and technology to realize the accurate control of blood sugar. [11] gave the KADIS model describing the interaction between blood sugar and insulin. Researchers in [12] also used wiener sliding-mode control for artificial pancreas. Although there were a great deal of works had been done on diabetes from various aspects such as medicine, education, management, diet, treatment, control, and so on, they mainly aimed at a specific issue, and could not reflect the development of diabetes and change of diabetics in social point. Note that there were some paper studying systems related with reliability of machine, such as [13, 14, 15]. So we were inspired that applying the reliability models into diabetic population. [16] researched the diabetic population development of early warning modelling. The current researches lacked quantitative research on the social cost caused by diabetes, and there was no calculation approach to it.

Motivated by these questions we propose a diabetes predicting model in this paper, aiming to investigate the diabetic population in a country or a region. According to the classification of diabetes, diabetes are divided into four classes: type 1 diabetes, type 2 diabetes, gestational diabetes, special type of diabetes. Type 2 diabetes that accounts for more than 90% of the diabetic population is the most important and fastest growing class, so we only consider type 2 diabetes in this article. For the sake of simplicity, we assume that there is no new diabetics birthing or dying. We classify diabetics with type 2 diabetes into four states according to complications of diabetes: diabetics without complications of diabetes, diabetics with micro vascular complication, diabetics with major vascular complication as well as diabetics with both micro vascular complication and major vascular complication. We shall give a detailed description of it, and establish a partial differential mathematical model in Section 2. Moreover, in Section 5, we shall give a description of the total cost of the system, which will be used to measure the economic burden of society and individuals.

The rest of this paper is organized as follows: in Section 2, we describe the system and establish a mathematical model for diabetics with type 2 diabetes in detail. In Section 3, we prove the existence and uniqueness of nonnegative time-dependent solution of the system via C_0 semi-group theory of bounded linear operator. In Section 4, we give the stability result of the system based on the spectral analysis of system operator. In Section 5, we discuss some indices of reliability and the total cost of the system, and give some numerical simulations. Finally in Section 6, we conclude the present paper.

2 Mathematical modelling

In this section we shall model a diabetes predicting system. First we describe the system under consideration, and then derive a partial differential equations model by the detailed analysis.

2.1 Description of system

In the system, we regard the diabetic population as a whole machine system, the diabetics who are suffered the diabetes complications as the units which are failed, the treatment for diabetics as the repairing for units, and assume that the diabetics can be cured as health as before just like that the unit can be repaired as new as before. Then we can establish a model similar with the reliability systems under some assumptions.

The diabetes predicting system consists of all diabetics with type 2 diabetes. Since diabetes can result in many complications, we suppose that it includes mainly two classes: one is micro vascular complication that includes peripheral neuropathy, retinopathy, nephropathy, diabetic foot, and the other is major vascular complication that includes transient ischemic attacking again, angina pectoris, cerebral apoplexy, and chronic heart failure.

We assume that in the study procedure, there is no new diabetics birthing or dying. So the total number of diabetics remains a constant. We divide diabetics into four states: the state of diabetics without complications, the deterioration state of diabetics with micro vascular complication, the deterioration state of diabetics with major vascular complication, and the deterioration state of diabetics with both complications.

The system obeys the following rules:

1). The states of diabetics can transfer from one to other states according to certain probability.

2). After treatment, diabetics can recover to good states according to certain recover rate that depends on treatment time.

3). After treatment, diabetics may enter into bad states according to certain probability. The different states have distinct transfer rates, they are constants.

4). All of the random variables are independent.

2.2 Mathematical Modelling

With the assumptions above, we shall establish mathematical model for the system under consideration. We introduce the following notations:

1. States

The state 0. It is the state in which all diabetics are without complications.

The state 1. In this state, diabetics are suffering from the micro vascular complication, but without the macro vascular complication.

The state 2. In this state, diabetics are suffering from the macro vascular complication, but without the micro vascular complication.

The state 3. In this state, diabetics are suffering from both the micro vascular complication and the macro vascular complication.

2. Transfer Rate

β_i: It is the transfer rate from *the state 0* to *the state* i, $i = 1, 2, 3$.

γ_i: It is the transfer rate from *the state* i to *the state 3*, where $i = 1, 2$.

3. Recover Rate (or repair rate)

$\mu_i(x)$: It is the recovery rate from *the state i* to *the state 0*, $i = 1, 2, 3$, where x is the time of diabetics staying in *the state* i.

This means that after time x treatment, diabetics in *the state* i can recover to a better state. The recovery rate depends on the treatment time and satisfies the general distribution $F(x) = 1 - e^{-\int_0^x \mu_i(s)ds}$.

Moreover, when the elapsed time diabetics staying in *the state* i is consistent, the recovery rate can be quantified by medicine, education, management, diet, exercise, etc.

$\eta_i(x)$: It is the recovery rate from *the state 3* to *the state* i, $i = 1, 2$, where x is the time of diabetics staying in *the state 3*. The distribution is $G(x) = 1 - e^{-\int_0^x \eta_i(s)ds}$.

With above notation, we can draw the following graph:

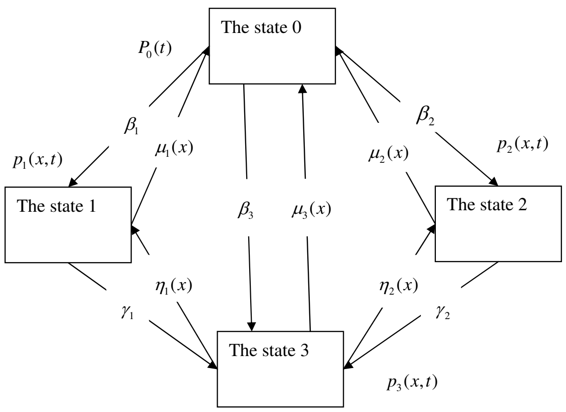

Figure 1: The state transferring graph.

Let $S(t)$ be a random variable whose value is in states set $\{0, 1, 2, 3\}$ at time t. $S(t) = i$ means that the system is in *the state* i at time t. Here we shall use $S(t)$ to indicate the state of diabetics at time t. And let $X(t)$ be a random variable whose value is in set $[0, \infty)$ at time t. Here we use $X(t)$ to denote the elapsed treatment (or repairing) time of diabetics.

Now we define the functions as follows: for $t \in (0, \infty)$,

$P_0(t)$: it is the probability of the system in *the state 0* at time t, i.e., $P_0(t) = P\{S(t) = 0\}$.

For $i = 1, 2, 3$, $p_i(x, t)\Delta x$ represents the probability of the system in *the state* i at time t and the treatment time is in interval $(x, x+\Delta x)$, i.e., $p_i(x, t)\Delta x = P\{S(t) = i, x < X(t) < x + \Delta x\}$ and the function

$$P_i(t) = \int_0^\infty p_i(x, t)dx$$

is the probability of the system in *the state* i at time t.

For function $p_i(x, t), i = 1, 2, 3$, the boundary value $p_i(0, t)$ represents at time t, the diabetics just enter into the state i from the other states without any treatment.

Under the help of above functions, the dynamic behavior of the system is governed by the following partial differential equations

$$\begin{cases} \frac{dP_0(t)}{dt} = -(\beta_1 + \beta_2 + \beta_3)P_0(t) + \sum_{i=1}^{3} \int_0^\infty \mu_i(x)p_i(x, t)dx, \\ \frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\mu_1(x)p_1(x, t) - \gamma_1 p_1(x, t), \\ \frac{\partial p_2(x,t)}{\partial t} + \frac{\partial p_2(x,t)}{\partial x} = -\mu_2(x)p_2(x, t) - \gamma_2 p_2(x, t), \\ \frac{\partial p_3(x,t)}{\partial t} + \frac{\partial p_3(x,t)}{\partial x} = -(\mu_3(x) + \eta_1(x) + \eta_2(x))p_3(x, t) \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} p_1(0, t) = \beta_1 P_0(t) + \int_0^\infty \eta_1(x)p_3(x, t)dx, \\ p_2(0, t) = \beta_2 P_0(t) + \int_0^\infty \eta_2(x)p_3(x, t)dx, \\ p_3(0, t) = \beta_3 P_0(t) + \gamma_1 \int_0^\infty p_1(x, t)dx + \gamma_2 \int_0^\infty p_2(x, t)dx \end{cases} \tag{2}$$

and initial conditions

$$P(0) = (P_0(0), p_1(x, 0), p_2(x, 0), p_3(x, 0)).$$

3 The well-posedness of the system solution

In this section, we shall discuss the well-posedness of the system described in (1) and (2). For simplicity, we use donation $\mathbb{R}^+$ to denote the real number set $[0, \infty)$ and donation $L^1(\mathbb{R}^+)$ to denote the usual Lebesgue integrable function space on $\mathbb{R}^+$.

We take the state space

$$\mathbb{X} = \mathbb{R} \times (L^1(\mathbb{R}^+))^3$$

equipped with the norm

$$\|P\| = |P_0| + \sum_{i=1}^{3} \|p_i(x)\|_{L^1}$$

for each $P = (P_0, p_1(x), p_2(x), p_3(x)) \in \mathbb{X}$. Obviously, $\mathbb{X}$ is a Banach space.

We define a subset $\mathbb{X}_+$ of $\mathbb{X}$ as

$$\mathbb{X}_+ = \{(f_0, f_1, f_2, f_3) \in \mathbb{X} \mid f_j \geq 0, j = 0, 1, 2, 3\}.$$

If $F \in \mathbb{X}_+$, we call F is a positive vector.

Further, we define an operator $\mathcal{A}$ in $\mathbb{X}$ as

$$\mathcal{A}\begin{pmatrix} P_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \end{pmatrix} = \begin{pmatrix} -(\beta_1 + \beta_2 + \beta_3)P_0 + \sum_{i=1}^{3} \int_0^\infty \mu_i(x)p_i(x)dx \\ -(\mu_1(x) + \gamma_1)p_1(x) - p_1'(x) \\ -(\mu_2(x) + \gamma_2)p_2(x) - p_2'(x) \\ -(\mu_3(x) + \eta_1(x) + \eta_2(x))p_3(x) - p_3'(x) \end{pmatrix} \tag{3}$$

with domain

$$D(\mathcal{A})=\left\{P\in\mathbb{X}\left|\begin{array}{l} p_i(x) \text{ is absolutely continuous,}\\ p_i(x),p_i'(x)\in L^1(\mathbb{R}^+),\\ \int_0^\infty \mu_i(x)p_i(x)dx \text{ is limited },\\ \int_0^\infty \eta_i(x)p_i(x)dx \text{ is limited },\\ p_1(0)=\beta_1P_0+\int_0^\infty \eta_1(x)p_3(x)dx,\\ p_2(0)=\beta_2P_0+\int_0^\infty \eta_2(x)p_3(x)dx,\\ p_3(0)=\beta_3P_0+\gamma_1\int_0^\infty p_1(x)dx\\ \qquad +\gamma_2\int_0^\infty p_2(x)dx\end{array}\right.\right\}. \quad (4)$$

Obviously, $\mathcal{A}$ is a linear operator.

With the definition of $\mathcal{A}$, the system (1) and (2) can be rewritten as an abstract equation in Banach space $\mathbb{X}$:

$$\left\{\begin{array}{l}\frac{dP(t)}{dt}=\mathcal{A}P(t), t>0,\\ P(0)=(P_0(0),p_1(x,0),p_2(x,0),p_3(x,0))^T\end{array}\right. \quad (5)$$

where

$$P(t)=(P_0(t),p_1(x,t),p_2(x,t),p_3(x,t))^T.$$

According to solvability theory of differential equation, the well-posedness of the system (1) and (2) is equivalent to the operator $\mathcal{A}$ generating a C_0 semigroup $T(t)$ on $\mathbb{X}$ (see, e.g., [19, Theorem II 6.7]).

As for operator $\mathcal{A}$, we have the following results.

Theorem 1. *Let $\mathcal{A}$ be defined by (3) and (4). Then $\mathcal{A}$ is a closed and densely linear operator in $\mathbb{X}$.*

Theorem 2. *Let $\mathcal{A}$ be defined as before. Then $\mathcal{A}$ is a dissipative operator in $\mathbb{X}$. Moreover, for any $r\in\mathbb{R}^+$, if the conditions*

$$\left\{\begin{array}{l}\sup\limits_{r\in\mathbb{R}^+}\int_r^\infty e^{-\int_r^x \mu_i(s)ds}dx=M<\infty,\\ \sup\limits_{r\in\mathbb{R}^+}\int_r^\infty e^{-\int_r^x \eta_j(s)ds}dx=N<\infty\end{array}\right. \quad (6)$$

are fulfilled. Then,

$$\begin{aligned} T &= \{\gamma\in\mathbb{C}\mid \Re\gamma>0,\ \gamma=is, s\neq 0, s\in\mathbb{R}\}\\ &\subset \rho(\mathcal{A}).\end{aligned}$$

As a direct consequence, we have the following corollary.

Corollary 3. *The spectrum $\sigma(\mathcal{A})$ is in the left-half plane, and there is no spectral point of $\mathcal{A}$ on the imaginary axis besides zero.*

Theorem 4. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before. Then $\mathcal{A}$ generates a C_0 semigroup $T(t)$ of contractions on $\mathbb{X}$. Hence (5) has unique a solution.*

Theorem 5. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before and $T(t)$ be the C_0 semigroup generated by $\mathcal{A}$. Then $T(t)$ is positive and satisfies $\|T(t)P\|=\|P\|$ for any positive vector $P\in D(\mathcal{A})$.*

Here we list the main results only and their proofs will be postponed in the appendix.

4 Stability of the system

In this section we shall discuss the stability of the system (5). We shall show that the system has a steady-state solution which is a positive vector in $\mathbb{X}$, and prove that the dynamic solution of (5) converges exponentially to the steady-state solution in the sense of norm in $\mathbb{X}$.

Theorem 6. *Let $\mathcal{A}$ be defined as before. Then $\lambda_0=0$ is a simple eigenvalue of $\mathcal{A}$. In particular, there is a positive eigenvector corresponding to λ_0.*

Proof. Let us consider the eigenvalue problem $\mathcal{A}P=0$, $P\in D(\mathcal{A})$, i.e.,

$$\left\{\begin{array}{l} -(\beta_1+\beta_2+\beta_3)P_0+\sum\limits_{i=1}^{3}\int_0^\infty \mu_i(x)p_i(x)dx=0,\\ -(\mu_1(x)+\gamma_1)p_1(x)-p_1'(x)=0,\\ -(\mu_2(x)+\gamma_2)p_2(x)-p_2'(x)=0,\\ -(\mu_3(x)+\eta_1(x)+\eta_2(x))p_3(x)-p_3'(x)=0\end{array}\right. \quad (7)$$

with boundary conditions

$$\left\{\begin{array}{l} p_1(0)=\beta_1P_0+\int_0^\infty \eta_1(x)p_3(x)dx,\\ p_2(0)=\beta_2P_0+\int_0^\infty \eta_2(x)p_3(x)dx,\\ p_3(0)=\beta_3P_0+\gamma_1\int_0^\infty p_1(x)dx\\ \qquad +\gamma_2\int_0^\infty p_2(x)dx.\end{array}\right. \quad (8)$$

Solving the differential equations in (7) yield

$$\left\{\begin{array}{l} p_1(x)=p_1(0)e^{-\int_0^x(\mu_1(s)+\gamma_1)ds},\\ p_2(x)=p_2(0)e^{-\int_0^x(\mu_2(s)+\gamma_2)ds},\\ p_3(x)=p_3(0)e^{-\int_0^x(\mu_3(s)+\eta_1(s)+\eta_2(s))ds}.\end{array}\right. \quad (9)$$

Substituting (9) into the first equation in (7) and the boundary conditions in (8), we get an algebraic equations with unknown parameter $(P_0,p_1(0),p_2(0),p_3(0))$,

$$\left\{\begin{array}{l} -(\beta_1+\beta_2+\beta_3)P_0+h_1p_1(0)\\ \qquad +h_2p_2(0)+h_3p_3(0)=0,\\ \beta_1P_0-p_1(0)+h_4p_3(0)=0,\\ \beta_2P_0-p_2(0)+h_5p_3(0)=0,\\ \beta_3P_0+(1-h_1)p_1(0)+(1-h_2)p_2(0)\\ \qquad -p_3(0)=0\end{array}\right. \quad (10)$$

where $h_j, j=1,2,\cdots,5$, are formed as following

$$\left\{\begin{array}{l} h_1=\int_0^\infty \mu_1(x)e^{-\int_0^x(\mu_1(s)+\gamma_1)ds}dx,\\ h_2=\int_0^\infty \mu_2(x)e^{-\int_0^x(\mu_2(s)+\gamma_2)ds}dx,\\ h_3=\int_0^\infty \mu_3(x)e^{-\int_0^x(\mu_3(s)+\eta_1(s)+\eta_2(s))ds}dx,\\ h_4=\int_0^\infty \eta_1(x)e^{-\int_0^x(\mu_3(s)+\eta_1(s)+\eta_2(s))ds}dx,\\ h_5=\int_0^\infty \eta_2(x)e^{-\int_0^x(\mu_3(s)+\eta_1(s)+\eta_2(s))ds}dx.\end{array}\right.$$

The determinant of coefficients matrix in (10) is

$$|D| = \begin{vmatrix} -(\beta_1+\beta_2+\beta_3) & h_1 & h_2 & h_3 \\ \beta_1 & -1 & 0 & h_4 \\ \beta_2 & 0 & -1 & h_5 \\ \beta_3 & 1-h_1 & 1-h_2 & -1 \end{vmatrix} = 0,$$

since $h_3 + h_4 + h_5 = 1$. This means that the algebraic equation (10) has a nonzero solution. A direct calculation gives

$$\begin{cases} P_0 = kp_3(0), \\ p_1(0) = (\beta_1 k + h_4)\, p_3(0), \\ p_2(0) = (\beta_2 k + h_5)\, p_3(0), \\ p_3(0) = p_3(0) \end{cases} \tag{11}$$

where

$$k = \frac{1-(1-h_1)h_4-(1-h_2)h_5}{(1-h_1)\beta_1+(1-h_2)\beta_2+\beta_3}$$

and hence $\mathcal{A}P = 0$ has non-zero solution

$$\begin{cases} P_0 = kp_3(0), \\ p_1(x) = (\beta_1 k + h_4)\, p_3(0) e^{-\int_0^x (\mu_1(s)+\gamma_1)ds}, \\ p_2(x) = (\beta_2 k + h_5)\, p_3(0) e^{-\int_0^x (\mu_2(s)+\gamma_2)ds}, \\ p_3(x) = p_3(0) e^{-\int_0^x (\mu_3(s)+\eta_1(s)+\eta_2(s))ds}. \end{cases} \tag{12}$$

Therefore, 0 is an eigenvalue of $\mathcal{A}$ of geometric multiplicity one. In particular, if $p_3(0) > 0$, then we have $P = (P_0, p_1(x), p_2(x), p_3(x)) \in \mathbb{X}_+$.

The dual operator $\mathcal{A}^*$ of the operator $\mathcal{A}$ is

$$\mathcal{A}^*Q = \begin{pmatrix} -(\beta_1+\beta_2+\beta_3)q_0 + \beta_1 q_1(0) + \beta_2 q_2(0) + \beta_3 q_3(0) \\ \mu_1(x)q_0 - (\mu_1(x)+\gamma_1)q_1(x) + \gamma_1 q_3(0) + q_1'(x) \\ \mu_2(x)q_0 - (\mu_2(x)+\gamma_2)q_2(x) + \gamma_2 q_3(0) + q_2'(x) \\ \mu_3(x)q_0 - (\mu_3(x)+\eta_1(x)+\eta_2(x))q_3(x) \\ +\eta_1(x)q_1(0) + \eta_2(x)q_2(0) + q_3'(x) \end{pmatrix}$$

with domain

$$D(\mathcal{A}^*) = \left\{ Q^T \in \mathbb{X}^* \,\middle|\, \begin{array}{l} q_i(x) \in L^\infty(\mathbb{R}^+) \text{ is absolutely continuous,} \\ q_i(0) \text{ is limited,} \\ q_i'(x) \in L^\infty(\mathbb{R}^+), i = 1,2,3. \end{array} \right\}.$$

Obviously, $Q = (1,1,1,1) \in D(\mathcal{A}^*)$ and $\mathcal{A}^*Q = 0$, so 0 is an eigenvalue of $\mathcal{A}^*$ and $Q = (1,1,1,1)$ is a corresponding eigenvector. Using (12) we get

$$(P, Q) = P_0 + \sum_{i=1}^{3} \int_0^\infty p_i(x)dx \neq 0.$$

Therefore, 0 is a simple eigenvalue of $\mathcal{A}$. □

Set $\Phi = (\phi_0, \phi_1(x), \phi_2(x), \phi_3(x))$ where

$$\begin{cases} \phi_0 = k, \\ \phi_1(x) = (\beta_1 k + h_4)\, e^{-\int_0^x (\mu_1(s)+\gamma_1)ds}, \\ \phi_2(x) = (\beta_2 k + h_5)\, e^{-\int_0^x (\mu_2(s)+\gamma_2)ds}, \\ \phi_3(x) = e^{-\int_0^x (\mu_3(s)+\eta_1(s)+\eta_2(s))ds} \end{cases} \tag{13}$$

and set $\widehat{P} = \frac{\Phi}{||\Phi||}$.

Applying the stability theorem [20], we have the following assertion.

Theorem 7. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before. Then for any $P(0) \in \mathbb{X}$, the solution $P(t) = (P_0(t), p_1(x,t), p_2(x,t), p_3(x,t)) = T(t)P(0)$ of (5) satisfies*

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} T(t)P(0) = (P(0), Q)\widehat{P}$$

where $Q = (1,1,1,1)$.

The previous theorem shows that the dynamic solution $P(t)$ converges its steady-state solution $\widehat{P}$. In what follows, we shall prove that $P(t) - (P(0), Q)\widehat{P}$ decays exponentially zero in the sense of norm in $\mathbb{X}$.

First we note that

$$\int_s^\infty e^{-\int_s^x (\gamma_i+\mu_i(\tau))d\tau} dx = \int_0^\infty e^{-\int_0^x (\gamma_i+\mu_i(\tau+s))d\tau} dx, i = 1,2$$

and

$$\int_s^\infty e^{-\int_s^x (\mu_3(\tau)+\eta_1(\tau)+\eta_2(\tau))d\tau} dx = \int_0^\infty e^{-\int_0^x (\mu_3(\tau+s)+\eta_1(\tau+s)+\eta_2(\tau+s))d\tau} dx,$$

the conditions in (6) imply

$$\begin{cases} \sup\limits_{s\geq 0} \int_0^\infty e^{-\int_0^x (\gamma_i+\mu_i(\tau+s))d\tau} dx < \infty \\ \sup\limits_{s\geq 0} \int_0^\infty e^{-\int_0^x (\mu_3(\tau+s)+\eta_1(\tau+s)+\eta_2(\tau+s))d\tau} dx < \infty. \end{cases}$$

So we can define nonnegative real number $\widehat{\eta_1}, \widehat{\eta_2}$ and $\widehat{\eta_3}$ by

$$\widehat{\eta_i} = \sup\left\{\lambda \geq 0 \,\middle|\, \sup_{s\geq 0} \int_0^\infty e^{\lambda x - \int_0^x (\gamma_i+\mu_i(\tau+s))d\tau} dx < \infty \right\}, i = 1,2,$$

and

$$\widehat{\eta_3} = \sup\left\{\lambda \geq 0 \,\middle|\, \sup_{s\geq 0} \int_0^\infty e^{\lambda x - \int_0^x (\mu_3(\tau+s)+\eta_1(\tau+s)+\eta_2(\tau+s))d\tau} dx < \infty \right\}.$$

Obviously, if $\lambda < \widehat{\eta_1}$, it holds that

$$\sup_{s\geq 0} \int_0^\infty e^{-\int_0^x (\gamma_1+\mu_1(\tau+s)-\lambda)d\tau} dx < \infty$$

and for $\lambda > \widehat{\eta_1}$, it holds that

$$\int_s^\infty e^{-\int_s^x (\gamma_1+\mu_1(\tau)-\lambda)d\tau} dx = \infty.$$

Similar result holds for $\widehat{\eta_2}$ and $\widehat{\eta_3}$.

Now we define

$$\widehat{\eta} = \min\{\widehat{\eta_1}, \widehat{\eta_2}, \widehat{\eta_3}\}.$$

The following theorem gives more exact description for spectrum of $\mathcal{A}$ using $\widehat{\eta}$.

Theorem 8. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before. Let $B(\lambda)$ be the matrix defined as in (17). Assume that $\widehat{\eta} > 0$. Then we have*

(I). $\{\lambda \in \mathbb{C} \mid \Re\lambda + \widehat{\eta} < 0\} \subset \sigma(\mathcal{A})$;

(II). $\{\lambda \in \mathbb{C} \mid \Re\lambda + \widehat{\eta} > 0, \det|B(\lambda)| \neq 0\} \subset \rho(\mathcal{A})$, and the set $\{\lambda \in \mathbb{C} \mid \Re\lambda + \widehat{\eta} > 0, \det|B(\lambda)| = 0\}$ consists of all eigenvalues of $\mathcal{A}$;

(III). For any $\delta > 0$, the number of eigenvalues of $\mathcal{A}$ in the region $\{\lambda \in \mathbb{C} \mid \Re\lambda + \widehat{\eta} \geq \delta\}$ is finite;

(IV). There exists a constant $\widehat{\omega_1} > 0$ such that there is unique an eigenvalue $\lambda_0 = 0$ in $\{\lambda \in \mathbb{C} \mid \Re\lambda \geq -\widehat{\omega_1}\}$, and hence λ_0 is strictly dominant.

Proof. Let $\lambda \in \mathbb{C}$. For any $F = (f_0, f_1, f_2, f_3) \in \mathbb{X}$, we consider the resolvent equation $(\lambda I - \mathcal{A})P = F$, which is the same as (14), whose formal solutions are given in (15) as $\lambda + \beta_1 + \beta_2 + \beta_3 \neq 0$.

(I). When $\Re\lambda + \widehat{\eta} > 0$, we denote $\Re\lambda = -\widehat{\eta} + \varepsilon$, then

$$\begin{aligned} &\int_0^\infty e^{-\int_0^x (\Re\lambda + \mu_1(\tau) + \gamma_1) d\tau} dx \\ = &\int_0^\infty e^{(\widehat{\eta} - \varepsilon)x - \int_0^x (\mu_1(\tau) + \gamma_1) d\tau} dx < \infty \end{aligned}$$

and

$$\begin{aligned} &\int_0^\infty \left| \int_0^x f_1(r) e^{-\int_r^x (\lambda + \mu_1(\tau) + \gamma_1) d\tau} dr \right| dx \\ \leq &\int_0^\infty \int_0^x |f_1(r)| e^{-\int_r^x (\Re\lambda + \mu_1(\tau) + \gamma_1) d\tau} dr dx \\ \leq &M_1(\Re\lambda) \int_0^\infty |f_1(r)| dr \\ = &M_1(\Re\lambda) \|f_1\|_{L^1} \end{aligned}$$

where

$$M_1(\Re\lambda) = \sup_{r \geq 0} \int_0^\infty e^{-\int_0^x (\Re\lambda + \mu_1(\tau + r) + \gamma_1) d\tau} dx < \infty.$$

So $p_1(x) \in L^1(\mathbb{R}^+)$. Similarly, $p_i, p_i' \in L^1(\mathbb{R}^+), i = 2, 3$. Therefore, $P \in D(\mathcal{A})$.

When $\Re\lambda + \widehat{\eta} < 0$, we denote $\Re\lambda = -\widehat{\eta} - \varepsilon$. Since $\widehat{\eta} = \min\{\eta_1, \eta_2, \eta_3\}$, without loss of generality, we can assume that $\widehat{\eta} = \eta_1$. For

$$\begin{aligned} p_1(x) = &p_1(0) e^{-\int_0^x (\lambda + \mu_1(s) + \gamma_1) ds} \\ &+ \int_0^x f_1(s) e^{-\int_s^x (\lambda + \mu_1(r) + \gamma_1) dr} ds. \end{aligned}$$

Since the term $e^{-\int_0^x (\lambda + \mu_1(s) + \gamma_1) ds}$ satisfies

$$\begin{aligned} |e^{-\int_0^x (\lambda + \mu_1(s) + \gamma_1) ds}| &= e^{-\int_0^x (\Re\lambda + \mu_1(s) + \gamma_1) ds} \\ &= e^{(\widehat{\eta} + \varepsilon)x - \int_0^x (\mu_1(s) + \gamma_1) ds}, \end{aligned}$$

by the definition of η_1, it holds that

$$\int_s^\infty e^{-\int_s^x (\Re\lambda + \mu_1(r) + \gamma_1) dr} dx = \infty.$$

Obviously, $f_1(x) = e^{-\frac{\varepsilon}{2}x} \in L^1(\mathbb{R}^+)$, but $\int_0^x f_1(s) e^{-\int_s^x (\lambda + \mu_1(r) + \gamma_1) dr} ds \notin L^1(\mathbb{R}^+)$, so $p_1(x)$ is not in $L^1(\mathbb{R}^+)$. Therefore, $\{\lambda \in \mathbb{C} \mid \Re\lambda + \widehat{\eta} < 0\} \subset \sigma(\mathcal{A})$

(II). Let $\Re\lambda + \widehat{\eta} > 0$. Then the integrals

$$\begin{cases} \int_0^\infty e^{-\int_0^x (\lambda + \mu_1(s) + \gamma_1) ds} dx, \\ \int_0^\infty e^{-\int_0^x (\lambda + \mu_2(s) + \gamma_2) ds} dx, \\ \int_0^\infty e^{-\int_0^x (\lambda + \mu_3(s) + \eta_2 + \eta_3) ds} dx \end{cases}$$

are of meaning. Substituting the formal solution into the boundary condition lead to an algebraic equation about $(P_0, p_1(0), p_2(0), p_3(0))$, see (16), whose coefficient matrix $B(\lambda)$ is given in (17).

Note that the algebraic equation is solvable if and only if $\det|B(\lambda)| \neq 0$. When $\det|B(\lambda)| \neq 0$, the equation (16) has unique a solution $(P_0, p_1(0), p_2(0), p_3(0))$. Then

$$P = (P_0, p_1(x), p_2(x), p_3(x))$$

where p_i are defined as (15) is in $D(\mathcal{A})$. So $\lambda \in \rho(\mathcal{A})$. Therefore, $\{\lambda \in \mathbb{C} \mid \Re\lambda + \widehat{\eta} > 0, \det|B(\lambda)| \neq 0\} \subset \rho(\mathcal{A})$.

When $\det|B(\lambda)| = 0$, the equation (16) has a nonzero solution

$$(P_0, p_1(0), p_2(0), p_3(0))$$

and hence the eigenvalue problem has a non-zero solution $(P_0, p_1(x), p_2(x), p_3(x))$. So, λ is an eigenvalue of $\mathcal{A}$.

(III). For any $\delta > 0$, let $\lambda \in \mathbb{C}$ satisfy $-\widehat{\eta} + \delta \leq \Re\lambda \leq 0$. Since $\det|B(\lambda)|$ is analytic in λ and $\lim_{|\Im\lambda| \to \infty} \frac{\det|B(\lambda)|}{\lambda} = 1$. Then $\det|B(\lambda)|$ has at most finite number of zeros in this strip. So the assertion is true.

(IV). Since $\lambda_0 = 0$ is an zero of $\det|B(\lambda)|$, there is no spectrum point of $\mathcal{A}$ on the imaginary axis (see Theorem 2), so the other eigenvalues of $\mathcal{A}$ are located in the region $\{\lambda \in \mathbb{C} | -\widehat{\eta} + \delta < \Re\lambda < 0\}$. Note that $\overline{\det|B(\lambda)|} = \det|B(\overline{\lambda})|$. We can set the other zeros are $z_j (j = 1, \cdots, 2k)$ where $z_{2j} = \overline{z_{2j-1}} (j = 1, \cdots, k)$. Set $\omega_1 = \min_{1 \leq j \leq k} (-\Re z_j) > 0$. Taking $0 <$

$\varepsilon < \omega_1$, and $\widehat{\omega_1} = \omega_1 - \varepsilon$, we have unique an eigenvalue λ_0 of $\mathcal{A}$ in the region $\{\lambda \in \mathbb{C} \mid \Re\lambda \geq -\widehat{\omega_1}\}$. The proof is then complete. □

Based on the result of Theorem 8, we have the following result about the exponential decay.

Theorem 9. *Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before and $T(t)$ be the C_0 semigroup generated by $\mathcal{A}$. Let $\widehat{\omega_1}$ be given as Theorem 8. Then for any initial $P(0)$ and $t \geq 0$, we have $\|P(t) - (P(0), Q)\widehat{P_0}\| \leq 2e^{-\omega_1 t}\|P(0)\|$, where $P(t) = T(t)P(0)$ and $Q = (1, 1, 1, 1)$.*

Proof. Since $\lambda_0 = 0$ is a strictly dominant eigenvalue of $\mathcal{A}$, the corresponding Riesz spectral project is

$$\begin{aligned} E(\lambda_0, \mathcal{A})F &= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \mathcal{R}(z, \mathcal{A})F dz \\ &= (F, Q)\widehat{P_0},\ \forall F \in \mathbb{X}. \end{aligned}$$

Obviously, $\|E(\lambda_0, \mathcal{A})\| = \|Q\|\|\widehat{P_0}\| = 1$.

Note that $E(\lambda_0, \mathcal{A})T(t) = T(t)E(\lambda_0, \mathcal{A})$ and $\mathbb{X} = E(\lambda_0, \mathcal{A})\mathbb{X} + (I - E(\lambda_0, \mathcal{A}))\mathbb{X}$. The semigroup $T(t)$ is dissipative on space $\mathbb{X}$, so it is in the subspace $(I - E(\lambda_0, \mathcal{A}))\mathbb{X}$. $\Re\lambda > -\widehat{\omega_1} + \delta$ implies that $R(\lambda, \mathcal{A})F = (P_0, p_1(x), p_2(x), p_3(x))$ is uniformly bounded in the region $-\widehat{\omega_1} \leq \Re\lambda \leq 0$ for $F \in (I - E(\lambda_0, \mathcal{A}))\mathbb{X}$. So,

$$\begin{aligned} & \|P(t) - (P(0), Q)\widehat{P_0}\| \\ = & \|T(t)P(0) - E(\lambda_0, \mathcal{A})T(t)P(0)\| \\ = & \|T(t)(I - E(\lambda_0, \mathcal{A}))P(0)\| \\ \leq & e^{-\widehat{\omega_1}t}\|(I - E(\lambda_0, \mathcal{A}))P(0)\| \\ \leq & 2e^{-\widehat{\omega_1}t}\|P(0)\|. \end{aligned}$$

The desired result follows. □

5 Numerical Simulation

In this section we shall discuss some indices of the system such as steady-state availability and analyze the total cost of the system. In addition, we give some numerical simulations for the total cost with parameters change.

5.1 Indices of the system and the total cost

Set $\widehat{P} = \frac{\Phi}{||\Phi||}$ where

$$\Phi = (\phi_0, \phi_1(x), \phi_2(x), \phi_3(x))$$

whose components are defined as forms in (13) and

$$\begin{aligned} \|\Phi\| = & \frac{\gamma_1\gamma_2 + h_4(1-h_1)\gamma_2(\beta_3 - \gamma_1)}{\gamma_1\gamma_2[(1-h_1)\beta_1 + (1-h_2)\beta_2 + \beta_3]} \\ & + \frac{h_5(1-h_2)\gamma_1(\beta_3 - \gamma_2)}{\gamma_1\gamma_2[(1-h_1)\beta_1 + (1-h_2)\beta_2 + \beta_3]} \\ & + \frac{\beta_1(1-h_1)\gamma_2 + \beta_2(1-h_2)\gamma_1}{\gamma_1\gamma_2[(1-h_1)\beta_1 + (1-h_2)\beta_2 + \beta_3]} \\ & + \frac{(1-h_1)(1-h_2)(\gamma_2 - \gamma_1)(h_4\beta_2 - h_5\beta_1)}{\gamma_1\gamma_2[(1-h_1)\beta_1 + (1-h_2)\beta_2 + \beta_3]} \\ & + \int_0^\infty e^{-\int_0^x(\mu_3(s)+\eta_1(s)+\eta_2(s))ds}dx. \end{aligned}$$

Then the probabilities of diabetics in steady state are

$$\begin{cases} \widehat{P}_0 = \frac{1-(1-h_1)h_4-(1-h_2)h_5}{(1-h_1)\beta_1+(1-h_2)\beta_2+\beta_3}\frac{1}{\|\Phi\|}, \\ \widehat{P}_1 = \frac{[\beta_1-\beta_1h_5(1-h_2)+\beta_2h_4(1-h_2)+\beta_3h_4](1-h_1)}{[(1-h_1)\beta_1+(1-h_2)\beta_2+\beta_3]\gamma_1}\frac{1}{\|\Phi\|}, \\ \widehat{P}_2 = \frac{[\beta_2-\beta_2h_4(1-h_1)+\beta_1h_5(1-h_1)+\beta_3h_5](1-h_2)}{[(1-h_1)\beta_1+(1-h_2)\beta_2+\beta_3]\gamma_2}\frac{1}{\|\Phi\|}, \\ \widehat{P}_3 = \frac{\int_0^\infty e^{-\int_0^x(\mu_3(s)+\eta_1(s)+\eta_2(s))ds}dx}{\|\Phi\|} \end{cases}$$

where $\widehat{P}_i, i = 0, 1, 2, 3$ represents the probability in *the state i*.

Note that diabetics in *the state 0* have stronger ability of self-care while those in *the state 1, 2* have weaker. But diabetics in *the state* 3 need others to take care of. Therefore we define the steady-state availability as the probability of diabetics in *the state 0, 1* and *2*.

Definition 10. *Let the system be defined as (5). At time t, the availability of the system is defined as the probability of the system being in the state 0, 1 and 2, i.e.,*

$$A_v(t) = \frac{P_0(t) + \int_0^\infty p_1(x,t)dx + \int_0^\infty p_2(x,t)dx}{||P(t)||}$$

where $P(t) = (P_0(t), p_1(x,t), p_2(x,t), p_3(x,t))$ is the solution of (5), and $||P(t)||$ is its norm.

If the initial value

$$P(0) = (P_0, p_1(x), p_2(x), p_3(x)) \in \mathbb{X}_+$$

satisfies $||P(0)|| = 1$, then by Theorem 5, we have $P(t) \in \mathbb{X}_+$ and $||P(t)|| = 1$. So the availability of the system at time t is

$$A_v(t) = P_0(t) + \int_0^\infty p_1(x,t)dx + \int_0^\infty p_2(x,t)dx.$$

Therefore, we have the following result.

Theorem 11. *The steady-state availability of the system is*

$$A_v = \frac{\phi_0 + \int_0^\infty \phi_1(x)dx + \int_0^\infty \phi_2(x)dx}{\|\Phi\|}$$

where

$$\begin{aligned}\|\Phi\| &= |\phi_0| + \int_0^\infty \phi_1(x)dx \\ &\quad + \int_0^\infty \phi_2(x)dx + \int_0^\infty \phi_3(x)dx.\end{aligned}$$

Now we consider the cost problem of the system. For each $i = 0, 1, 2, 3$, we assume that the cost of diabetic in *the state* i is c_i that is a constant. Usually, the costs satisfy the relation $c_0 < c_i < c_3, i = 1, 2$. At time t, the cost of the system in *the state 0* is

$$C_0(t) = \frac{c_0}{||P(t)||} P_0(t)$$

and the costs of the system in *the state* i are

$$C_i(t) = \frac{c_i}{||P(t)||} \int_0^\infty p_i(x,t)dx, \quad i = 1, 2, 3.$$

Therefore, the total cost of the system at time t is $C(t) = \sum_{i=0}^{3} C_i(t)$.

For $P(0) \in \mathbb{X}_+$ and $||P(0)|| = 1$. So $||P(t)|| = 1$, and

$$\begin{aligned}C(t) &= c_0 P_0(t) + c_1 \int_0^\infty p_1(x,t)dx \\ &\quad + c_2 \int_0^\infty p_2(x,t)dx + c_3 \int_0^\infty p_3(x,t)dx.\end{aligned}$$

Obviously, it converges when $t \to \infty$. In addition, we have the following result.

Theorem 12. *The total cost of the system in steady state is*

$$\lim_{t\to\infty} C(t) = C = \sum_{j=0}^{3} C_j$$

where

$$\begin{cases} C_0 = \frac{|\phi_0|}{\|\Phi\|} c_0, \\ C_1 = \frac{\int_0^\infty \phi_1(x)dx}{\|\Phi\|} c_1, \\ C_2 = \frac{\int_0^\infty \phi_2(x)dx}{\|\Phi\|} c_2, \\ C_3 = \frac{\int_0^\infty \phi_3(x)dx}{\|\Phi\|} c_3. \end{cases}$$

5.2 Analysis

In previous subsection, we give the total cost of diabetics undergoing different stages, which also is the social cost. Obviously, C is a function of multi-variable, c_j, $j = 0, 1, 2, 3$, γ_j, $\eta_j(x)$,$j = 1, 2$, β_j and $\mu_j(x)$, $j = 1, 2, 3$. C is affected by each variable.

Here we only consider C changes when $\mu_1(x)$ changes. For simplicity, we assume that $\eta_i(x)$, $i = 1, 2$ and $\mu_i(x)$, $i = 1, 2, 3$ are constant functions.

In addition, we suppose that

$$\begin{cases} \beta_1 = 0.1 & \beta_2 = 0.1 & \beta_3 = 0.05 \\ \gamma_1 = 0.2 & & \gamma_2 = 0.2 \\ \mu_2(x) = 0.3 & & \mu_3(x) = 0.2 \\ \eta_1(x) = 0.1 & & \eta_2(x) = 0.1 \end{cases}$$

and

$$\begin{cases} c_0 = 4773.83, & c_1 = 15101.63, \\ c_2 = 18836.98, & c_3 = 44465.42 \end{cases}$$

where the cost values c_i, $i = 0, 1, 2, 3$ are taken from [24].

We consider C changes as μ_1 verifies from 0.1 to 1.

We choose the step length $h = 0.005$. Using Matlab to draw the picture. The following Figure 2 to Figure 6 describe the variety of the steady state total costs with μ_1 changes.

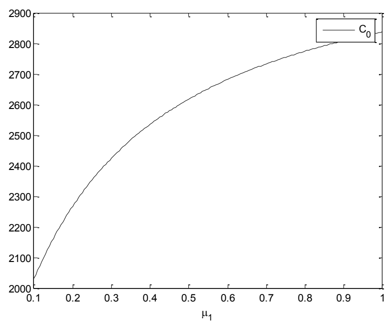

Figure 2: C_0 changes when $\mu_1(x)$ changes.

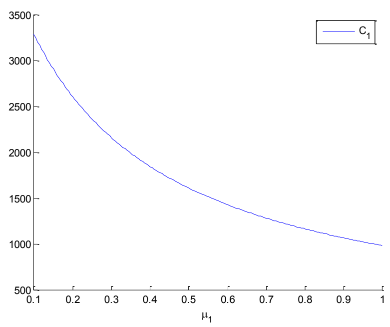

Figure 3: C_1 changes when $\mu_1(x)$ changes.

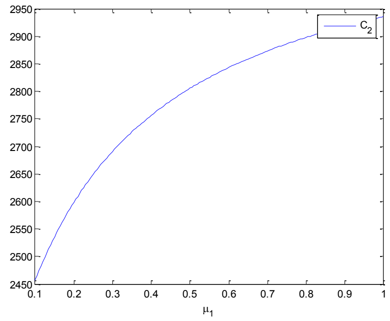

Figure 4: C_2 changes when $\mu_1(x)$ changes.

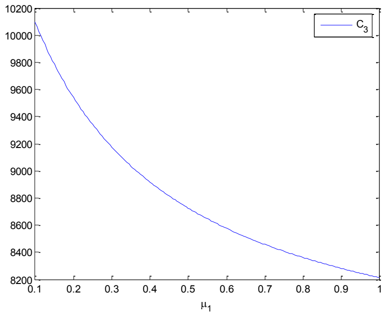

Figure 5: C_3 changes when $\mu_1(x)$ changes.

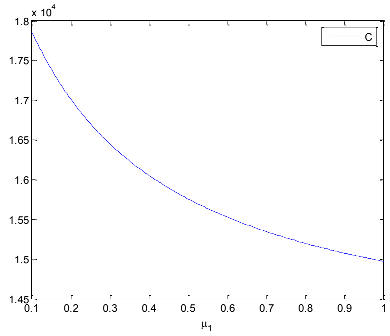

Figure 6: C changes when $\mu_1(x)$ changes.

From Figure 2 to Figure 6 we can see that the following facts are true:

(1) C_0 increases as μ_1 increases.

Obviously, when μ_1 increases, diabetics in *the state 1* will return to *the state 0*, and hence the probability in *the state 0* will increase, so the cost of state 0 increases with μ_1 increasing.

(2) C_1 decreases when μ_1 increases.

When μ_1 increases, diabetics in *the state 1* have returned to *the state 0*, the number of diabetics in *the state 1* decreases and hence the probability of *the state 1* will decrease, so the cost of *the state 1* decreases as μ_1 increasing.

(3) C_2 increases when μ_1 increases.

This is a relevance effect of states change. When μ_1 increases, the number of diabetics in *the state 2* might increase, so does the probability in *the state 2*. Hence the cost of *the state 2* will increase with μ_1 increasing.

(4) C_3 decreases when μ_1 increases.

This also is a relevance effect of states change. When μ_1 increases, the probability in *the state 1* decreases, and the probability in *the state 2* increases. As a result, the probability of *the state 3* will decrease as μ_1 increasing, so does the cost of *the state 3*.

(5) C decreases when μ_1 increases.

As a whole effects, C decreases when μ_1 increases. This is the result we expect.

The numerical simulation result shows that the total cost of the system will drop when the recovery rate $\mu_1(x)$ increases, we can see clearly in Figure 6, for example: when $\mu_1(x) = 0.225$, the total cost is $1.6849 * 10^4$, when $\mu_1(x) = 0.47$, the total cost is $1.5836*10^4$, the total cost greatly reduces when $\mu_1(x)$ increases.

Herein we considered one case only; similarly, we can discuss other cases. In our daily life, we have to make a decision in accordance with the actual situation. Numerical simulation will provide us a method to judge how to decrease the total cost of the system.

6 Conclusion

In the present paper we established a mathematical model for the diabetes predicting system, according to the treatment feature, the treatment process and the changing of states of diabetes. Under certain conditions, we proved the well-posedness of the system. Moreover, the system had a steady-state solution, and the dynamic solution of the system converged exponentially to its steady-state solution. The steady-state solution gave the probability distribution of diabetics in the steady state. Furthermore we studied the health indices of diabetics in the system and analyzed the total cost of the system when the parameter changes. The result showed that the total cost of the system will decrease if the recovery rate $\mu_1(x)$ increases, which was the result we expect.

Note that the recovery rate $\mu_1(x)$ can be quantified by medicine, education, management, diet, exercise, etc, such as diabetics following doctor's advice, learning more knowledge about diabetes, paying attention to their diet and so on. These factors made the recovery rate $\mu_1(x)$ become larger. So we suggest diabetics do some things on these aspects to relieve the economic pressure of the society and individuals and to improve the life quality. Our predicting model for diabetics and the system cost give an approach to measure probability of diabetics and cost in different states. These quantitative indices can be referenced for diabetics and doctors. In the future, we will determine the relation between the recovery rate and diabetes management further including medicine, education, diet, exercise, etc.

Appendix: The proofs of main results in Section 3

The proof of Theorem 1 We divide the proof into the following several steps.

Step 1. $D(\mathcal{A})$ is dense in $\mathbb{X}$.

We should prove that $\forall\varepsilon > 0, \forall F \in \mathbb{X}$, we can find a $P \in D(\mathcal{A})$ such that $\|F - P\| < \varepsilon$.

For any $F = (f_0, f_1(x), f_2(x), f_3(x)) \in \mathbb{X}$ fixed, let us condition a $P \in \mathbb{X}$,

$$\|F-P\|_{\mathbb{X}} = |f_0 - P_0| + \sum_{i=1}^{3}\int_0^\infty |f_i(x) - p_i(x)|dx.$$

We can take $P_0 = f_0$. Since $F \in \mathbb{X}$, then $\forall\varepsilon > 0$, there exist constants $G_i > 0, i = 1, 2, 3$ such that $\int_{G_i}^\infty |f_i(x)dx| < \frac{\varepsilon}{9}, i = 1, 2, 3.$

By the absolute continuity of the integral, we can choose positive constants

$$\begin{cases} \delta_i < \frac{\varepsilon}{18(1+|\beta_i P_0|+\|\eta_i p_3\|)}, i = 1, 2, \\ \delta_3 < \frac{\varepsilon}{18(1+|\beta_3 P_0|+\|\gamma_1 p_1\|+\|\gamma_2 p_2\|)} \end{cases}$$

such that $\int_0^{\delta_i} |f_i(x)|dx < \frac{\varepsilon}{18}, i = 1, 2, 3.$ We define functions by

$$p_1(x) = \begin{cases} \beta_1 P_0 + \int_0^\infty \eta_1(x)p_3(x)dx, & 0 \le x \le \delta_1, \\ g_1(x), & \delta_1 \le x \le G_1, \\ 0, & x \ge G_1, \end{cases}$$

$$p_2(x) = \begin{cases} \beta_2 P_0 + \int_0^\infty \eta_2(x)p_3(x)dx, & 0 \le x \le \delta_2, \\ g_2(x), & \delta_2 \le x \le G_2, \\ 0, & x \ge G_2, \end{cases}$$

and

$$p_3(x) = \begin{cases} \beta_3 P_0 + \sum\limits_{i=1}^{2}\gamma_i \int_0^\infty p_i(x)dx, & 0 \le x \le \delta_3, \\ g_3(x), & \delta_3 \le x \le G_3, \\ 0, & x \ge G_3 \end{cases}$$

where $g_i(G_i) = 0, g_i(\delta_i) = p_i(0), g_i(x)$ is continuously differential and $g_i(x)$ satisfies $\int_0^\infty |f_i(x) - g_i(x)|dx < \frac{\varepsilon}{9}$. Obviously, $p_i(x), p_i'(x) \in L^1(\mathbb{R}^+)$ and

$$\begin{aligned}\int_0^\infty |f_i(x) - p_i(x)|dx &= \int_0^{\delta_i} |f_i(x) - p_i(x)|dx \\ &+\int_{\delta_i}^{G_i} |f_i(x) - p_i(x)|dx + \int_{G_i}^{\infty} |f_i(x) - p_i(x)|dx \\ &\le \frac{\varepsilon}{3}.\end{aligned}$$

Let p_i be defined as above, and let

$$P = (P_0, p_1(x), p_2(x), p_3(x)).$$

By our construction, we have $P \in D(\mathcal{A})$ and

$$\begin{aligned} &\|F-P\| \\ = \ & |P_0 - f_0| + \sum_{i=1}^{3}\int_0^\infty |f_i(x) - p_i(x)|dx \\ = \ & \sum_{i=1}^{3}\left(\int_0^{\delta_i} |f_i(x) - p_i(x)|dx\right. \\ + \ & \left.\int_{\delta_i}^{G_i} |f_i(x)-p_i(x)|dx+\int_{G_i}^{\infty} |f_i(x)-p_i(x)|dx\right) \\ \le \ & \sum_{i=1}^{3}\left(\int_0^{\delta_i} |f_i(x)|dx + \int_0^{\delta_i} |p_i(x)|dx\right. \\ & \left.+\int_{\delta_i}^{G_i} |f_i(x) - p_i(x)|dx + \int_{G_i}^{\infty} |f_i(x)|dx\right) \\ < \ & \varepsilon. \end{aligned}$$

Therefore $\overline{D(\mathcal{A})} = \mathbb{X}$.

Step 2. $\mathcal{A}$ is a closed linear operator.

Obviously, $\mathcal{A}$ is a linear operator in $\mathbb{X}$. We only need to prove $\mathcal{A}$ is a closed operator. By the definition of closed operator, $\mathcal{A}$ is closed if and only if when

$$P^{(m)} \in D(\mathcal{A}), P^{(m)} \to P^{(0)}, \mathcal{A}P^{(m)} \to F,$$

then $P^{(0)} \in D(\mathcal{A})$ and $\mathcal{A}P^{(0)} = F$.

Now we suppose that there is a sequence $P^{(m)} \in D(\mathcal{A})$ satisfying $P^{(m)} \to P^{(0)}$ and $\mathcal{A}P^{(m)} \to F$. Let

$$\begin{cases} P^{(m)} = (P_0^{(m)}, p_1^{(m)}(x), p_2^{(m)}(x), p_3^{(m)}(x)), \\ P^{(0)} = (P_0^{(0)}, p_1^{(0)}(x), p_2^{(0)}(x), p_3^{(0)}(x)), \\ F = (f_0, f_1(x), f_2(x), f_3(x)). \end{cases}$$

Thus we have that when $m \to \infty$

$$P_0^{(m)} \to P_0, \int_0^\infty |p_i^{(m)}(x) - p_i(x)|dx \to 0, i = 1, 2, 3.$$

and

$$\begin{cases} |-(\beta_1+\beta_2+\beta_3)P_0^{(m)} + \sum\limits_{i=1}^{3}\int_0^\infty \mu_i(x)p_i^{(m)}(x)dx - f_0| \to 0, \\ \int_0^\infty |-(\mu_1(x)+\gamma_1)p_1^{(m)}(x) - (p_1^{(m)}(x))' - f_1(x)|dx \to 0, \\ \int_0^\infty |-(\mu_2(x)+\gamma_2)p_2^{(m)}(x) - (p_2^{(m)}(x))' - f_2(x)|dx \to 0, \\ \int_0^\infty |-(\mu_3(x)+\eta_1(x)+\eta_2(x))p_3^{(m)}(x) \\ \quad -(p_3^{(m)}(x))' - f_3(x)|dx \to 0. \end{cases}$$

For any $x \in \mathbb{R}^+$, the function $\chi_{[0,x]} \in L^\infty(\mathbb{R}_+)$ is a bounded and linear functional of $L^1(\mathbb{R}_+)$. Then for each $i = 1, 2, 3, \chi_{[0,x]}(AP^{(m)})_i$ has meaning and $\lim\limits_{m\to\infty}(AP^{(m)})_i = \chi_{[0,x]}(f_i)$ due to the continuity. Thus we get

$$\begin{cases} \lim\limits_{m\to\infty}\int_0^x[-(\mu_1(\tau)+\gamma_1)p_1^{(m)}(\tau)-(p_1^{(m)}(\tau))'-f_1(\tau)]d\tau \\ \quad =\int_0^x f_1(\tau)d\tau, \\ \lim\limits_{m\to\infty}\int_0^x[-(\mu_2(\tau)+\gamma_2)p_2^{(m)}(\tau)-(p_2^{(m)}(\tau))'-f_2(\tau)]d\tau \\ \quad =\int_0^x f_2(\tau)d\tau, \\ \lim\limits_{m\to\infty}\int_0^x[-(\mu_3(\tau)+\eta_1(\tau)+\eta_2(\tau))p_3^{(m)}(\tau)-(p_3^{(m)}(\tau))' \\ \quad -f_3(\tau)]d\tau=\int_0^x f_3(\tau)d\tau. \end{cases}$$

From above we can get

$$\begin{cases} p_1(x)-p_1(0)+\int_0^x(\mu_1(\tau)+\gamma_1)p_1(\tau)d\tau+\int_0^x f_1(\tau)d\tau=0, \\ p_2(x)-p_2(0)+\int_0^x(\mu_2(\tau)+\gamma_2)p_2(\tau)d\tau+\int_0^x f_2(\tau)d\tau=0, \\ p_3(x)-p_3(0)+\int_0^x(\mu_3(\tau)+\eta_1(\tau)+\eta_2(\tau))p_3(\tau)d\tau \\ \quad +\int_0^x f_3(\tau)d\tau=0. \end{cases}$$

The above formulas show that $p_i^{(0)}, i = 1, 2, 3$ are absolute continuous and

$$\begin{cases} -(\mu_1(x)+\gamma_1)p_1^{(0)}(x)-(p_1^{(0)}(x))'=f_1(x), \\ -(\mu_2(x)+\gamma_2)p_2^{(0)}(x)-(p_2^{(0)}(x))'=f_2(x), \\ -(\mu_3(x)+\eta_1(x)+\eta_2(x))p_3^{(0)}(x) \\ \qquad -(p_3^{(0)}(x))'=f_3(x) \end{cases}$$

which implies that $(p_i^{(0)}(x))' \in L^1(\mathbb{R}^+), i = 1, 2, 3.$

Since $P^{(m)} \in D(\mathcal{A})$, we have

$$\begin{cases} p_1^{(m)}(0)=\beta_1P_0^{(m)}+\int_0^\infty\eta_1(x)p_3^{(m)}(x)dx, \\ p_2^{(m)}(0)=\beta_2P_0^{(m)}+\int_0^\infty\eta_2(x)p_3^{(m)}(x)dx, \\ p_3^{(m)}(0)=\beta_3P_0^{(m)}+\gamma_1\int_0^\infty p_1^{(m)}(x)dx \\ \qquad +\gamma_2\int_0^\infty p_2^{(m)}(x)dx. \end{cases}$$

Using the fact that

$$\mu_1(x),\mu_2(x),\mu_3(x),\eta_1(x),\eta_2(x)\in L^\infty(\mathbb{R}^+),$$

we get

$$\begin{cases} p_1^{(0)}(0)=\beta_1P_0^{(0)}+\int_0^\infty\eta_1(x)p_3^{(0)}(x)dx, \\ p_2^{(0)}(0)=\beta_2P_0^{(0)}+\int_0^\infty\eta_2(x)p_3^{(0)}(x)dx, \\ p_3^{(0)}(0)=\beta_3P_0^{(0)}+\gamma_1\int_0^\infty p_1^{(0)}(x)dx \\ \qquad +\gamma_2\int_0^\infty p_2^{(0)}(x)dx. \end{cases}$$

Furthermore

$$\begin{aligned} &\lim_{m\to\infty}\sum_{i=1}^3\int_0^\infty\mu_i(x)p_i^{(m)}(x)dx \\ =\ & f_0+(\beta_1+\beta_2+\beta_3)P_0^{(0)}. \end{aligned}$$

Because of

$$\begin{aligned} &\lim_{m\to\infty}\sum_{i=1}^3\int_0^T\mu_i(x)p_i^{(m)}(x)dx \\ =\ & \sum_{i=1}^3\int_0^T\mu_i(x)p_i^{(0)}(x)dx, \forall T>0, \end{aligned}$$

we have

$$\begin{aligned} &\lim_{m\to\infty}\sum_{i=1}^3\int_0^\infty\mu_i(x)p_i^{(m)}(x)dx \\ =\ & \sum_{i=1}^3\int_0^\infty\mu_i(x)p_i^{(0)}(x)dx. \end{aligned}$$

So

$$-(\beta_1+\beta_2+\beta_3)P_0+\sum_{i=1}^3\int_0^\infty\mu_i(x)p_i^{(0)}(x)dx=f_0.$$

Therefore, we have $P^{(0)} \in D(\mathcal{A})$ and $\mathcal{A}P^{(0)} = F$. So, $\mathcal{A}$ is a closed and densely defined linear operator in $\mathbb{X}$. □

The proof of Theorem 2 By the direct verification, the dual space of $\mathbb{X}$ is

$$\mathbb{X}^*=\mathbb{R}\times[L^\infty(\mathbb{R}_+)]^3$$

with the norm for $F \in \mathbb{X}^*$

$$\|F\|=\max\{|f_0|,\|f_1\|,\|f_2\|,\|f_3\|\}.$$

Step 1. $\mathcal{A}$ is a dissipative operator in $\mathbb{X}$.

For any $P \in D(\mathcal{A})$, we define

$$Q=(q_0,q_1(x),q_2(x),q_3(x))$$

where

$$q_0=\|P\|sign(P_0), q_i(x)=\|P\|sign(p_i(x)).$$

Obviously, $Q \in \mathbb{X}^*$ and $Q \in \mathcal{F}(P) = \{Q \in \mathbb{X}^*|(P,Q) = \|P\|^2 = \|Q\|^2\}$. Moreover, because

$$\int_0^\infty p_i'(x)sign(p_i(x))dx=-|p_i(0)|, i=1,2,3,$$

and using the boundary condition in $D(\mathcal{A})$ we get

$$\begin{aligned} \frac{(\mathcal{A}P,Q)}{\|P\|} &= -(\beta_1+\beta_2+\beta_3)|P_0| \\ &+\sum_{i=1}^3\int_0^\infty\mu_i(x)p_i(x)sign(P_0)dx \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int_0^\infty(\mu_i(x)+\gamma_i)|p_i(x)|dx\\
&-\int_0^\infty(\mu_3(x)+\eta_1(x)+\eta_2(x))|p_3(x)|dx\\
&+\sum_{i=1}^{3}|p_i(0)|\\
\leq\ & \sum_{i=1}^{3}\int_0^\infty\mu_i(x)(p_i(x)sign(P_0)-|p_i(x)|)dx\\
\leq\ & 0.
\end{aligned}
$$

Therefore, $\mathcal{A}$ is dissipative.

Step 2. Set $T=\{\lambda\in\mathbb{C}|\Re\lambda\geq 0,\lambda\neq 0\}$*, then* $T\subset\rho(\mathcal{A})$.

For any $F\in\mathbb{X}$ and $\lambda\in T$, we consider the resolvent equation $(\lambda I-\mathcal{A})P=F$, that is

$$
\begin{cases}
(\lambda+\beta_1+\beta_2+\beta_3)P_0-\sum\limits_{i=1}^{3}\int_0^\infty\mu_i(x)p_i(x)dx=f_0,\\
(\lambda+\mu_1(x)+\gamma_1)p_1(x)+p_1'(x)=f_1(x),\\
(\lambda+\mu_2(x)+\gamma_2)p_2(x)+p_2'(x)=f_2(x),\\
(\lambda+\mu_3(x)+\eta_1(x)+\eta_2(x))p_3(x)+p_3'(x)=f_3(x)
\end{cases}
\tag{14}
$$

with boundary conditions

$$
\begin{cases}
p_1(0)=\beta_1P_0+\int_0^\infty\eta_1(x)p_3(x)dx,\\
p_2(0)=\beta_2P_0+\int_0^\infty\eta_2(x)p_3(x)dx,\\
p_3(0)=\beta_3P_0+\gamma_1\int_0^\infty p_1(x)dx+\gamma_2\int_0^\infty p_2(x)dx.
\end{cases}
$$

At first, we solve

$$
\begin{aligned}
p_1(x)\ =\ & p_1(0)e^{-\int_0^x(\lambda+\mu_1(s)+\gamma_1)ds}\\
&+\int_0^x f_1(s)e^{-\int_s^x(\lambda+\mu_1(r)+\gamma_1)dr}ds.
\end{aligned}
$$

When $\Re\lambda>0$,

$$
\begin{aligned}
&\int_0^\infty\left|\int_0^x f_1(s)e^{-\int_s^x(\lambda+\mu_1(r)+\gamma_1)dr}ds\right|dx\\
\leq\ & \int_0^\infty|f_1(s)|ds\int_s^\infty e^{-\int_s^x(\lambda+\mu_1(r)+\gamma_1)dr}dx\\
\leq\ & \frac{1}{\Re\lambda+\gamma_1}\|f_1\|_{L^1}\leq\frac{1}{\Re\lambda}\|f_1\|_{L^1}.
\end{aligned}
$$

When $\lambda=is, s\neq 0$,

$$
\begin{aligned}
&\int_0^\infty|p_1(x)|dx\\
\leq\ & |p_1(0)|\int_0^\infty e^{-\int_0^x(\lambda+\mu_1(s)+\gamma_1)ds}dx\\
&+\int_0^\infty|f_1(s)|ds\int_s^\infty e^{-\int_s^x(\lambda+\mu_1(r)+\gamma_1)dr}dx.
\end{aligned}
$$

By the assumption (3), we have

$$
\int_0^\infty|f_1(s)|ds\int_s^\infty e^{-\int_s^x(\lambda+\mu_1(r)+\gamma_1)dr}dx<\infty.
$$

Therefore, $p_1(x)\in L^1(\mathbb{R}^+)$, and

$$
\begin{aligned}
&\int_0^\infty\mu_1(x)p_1(x)dx\\
=\ & \int_0^\infty f_1(x)dx+p_1(0)-(\lambda+\gamma_1)\int_0^\infty p_1(x)dx
\end{aligned}
$$

is finite. So we can prove $p_1(x),p_1'(x)\in L^1(\mathbb{R}^+)$.

Similarly, we can prove that

$$
p_2(x),p_2'(x),p_3(x),p_3'(x)\in L^1(\mathbb{R}^+).
$$

A direct calculation gives

$$
\begin{cases}
P_0=\frac{f_0+\sum\limits_{i=1}^{3}\int_0^\infty\mu_i(x)p_i(x)dx}{\lambda+\beta_1+\beta_2+\beta_3},\\
p_1(x)=p_1(0)e^{-\int_0^x(\lambda+\mu_1(s)+\gamma_1)ds}\\
\qquad+\int_0^x f_1(s)e^{-\int_s^x(\lambda+\mu_1(r)+\gamma_1)dr}ds,\\
p_2(x)=p_2(0)e^{-\int_0^x(\lambda+\mu_2(s)+\gamma_2)ds}\\
\qquad+\int_0^x f_2(s)e^{-\int_s^x(\lambda+\mu_2(r)+\gamma_2)dr}ds,\\
p_3(x)=p_3(0)e^{-\int_0^x(\gamma+\mu_3(s)+\eta_1(s)+\eta_2(s))ds}\\
\qquad+\int_0^x f_3(s)e^{-\int_s^x(\lambda+\mu_3(r)+\eta_1(r)+\eta_2(r))dr}ds.
\end{cases}
\tag{15}
$$

Next, we substitute (15) into the boundary conditions and get algebraic equations about $(P_0,p_1(0),p_2(0),p_3(0)$, i.e.,

$$
\begin{cases}
(\lambda+\beta_1+\beta_2+\beta_3)P_0-\sum\limits_{i=1}^{3}p_i(0)\int_0^\infty\mu_i(x)g_idx\\
\qquad=\sum\limits_{i=1}^{3}\int_0^\infty\mu_i(x)g_{i+3}dx+f_0,\\
-\beta_1P_0+p_1(0)-p_3(0)\int_0^\infty\eta_1(x)g_3dx\\
\qquad=\int_0^\infty\eta_1(x)g_6dx,\\
-\beta_2P_0+p_2(0)-p_3(0)\int_0^\infty\eta_2(x)g_3dx\\
\qquad=\int_0^\infty\eta_2(x)g_6dx,\\
-\beta_3P_0-\gamma_1\int_0^\infty g_1dxp_1(0)-\gamma_2\int_0^\infty g_2dxp_2(0)\\
\qquad+p_3(0)=\gamma_1\int_0^\infty g_4dx+\gamma_2\int_0^\infty g_5dx
\end{cases}
\tag{16}
$$

where

$$
\begin{cases}
g_1=e^{-\int_0^x(\lambda+\mu_1(\tau)+\gamma_1)d\tau},\\
g_2=e^{-\int_0^x(\lambda+\mu_2(\tau)+\gamma_2)d\tau},\\
g_3=e^{-\int_0^x(\lambda+\mu_3(\tau)+\eta_1(\tau)+\eta_2(\tau))d\tau},\\
g_4=\int_0^x f_1(r)e^{-\int_r^x(\lambda+\mu_1(\tau)+\gamma_1)d\tau}dr,\\
g_5=\int_0^x f_2(r)e^{-\int_r^x(\lambda+\mu_2(\tau)+\gamma_2)d\tau}dr,\\
g_6=\int_0^x f_3(r)e^{-\int_r^x(\lambda+\mu_3(\tau)+\eta_1(\tau)+\eta_2(\tau))d\tau}dr.
\end{cases}
$$

Set $B(\lambda)$ denote matrix

$$
\left(\begin{array}{cc}
\lambda+\beta_1+\beta_2+\beta_3 & -\int_0^\infty\mu_1(x)g_1dx\\
-\beta_1 & 1\\
-\beta_2 & 0\\
-\beta_3 & -\gamma_1\int_0^\infty g_1dx
\end{array}\right.
$$

$$\begin{matrix} -\int_0^\infty \mu_2(x)g_2dx & -\int_0^\infty \mu_3(x)g_3dx \\ 0 & -\int_0^\infty \eta_1(x)g_3dx \\ 1 & -\int_0^\infty \eta_2(x)g_3dx \\ -\gamma_2\int_0^\infty g_2dx & 1 \end{matrix} \Bigg). \quad (17)$$

Since we have the following inequalities

$$\begin{cases} |-\beta_1-\beta_2-\beta_3| < |\lambda+\beta_1+\beta_2+\beta_3|, \\ |-\int_0^\infty \mu_1(x)g_1dx - \gamma_1\int_0^\infty g_1dx| < 1, \\ |-\int_0^\infty \mu_2(x)g_2dx - \gamma_2\int_0^\infty g_2dx| < 1, \\ |-\int_0^\infty \mu_3(x)g_3dx - \int_0^\infty \eta_1(x)g_3dx \\ \quad -\int_0^\infty \eta_2(x)g_3dx| < 1, \end{cases}$$

so $B(\lambda)$ is a column strictly diagonal dominant. So the equation (16) has unique a solution $(P_0, p_1(0), p_2(0), p_3(0))$. Therefore, for each $F \in \mathbb{X}$ there is unique a solution

$$P = (P_0, p_1(x), p_2(x), p_3(x)) \in D(\mathcal{A})$$

such that $(\lambda I - \mathcal{A})P = F$. The closed operator theorem asserts that $(\lambda I - \mathcal{A})^{-1}$ exists and is bounded. Hence, $T \subset \rho(\mathcal{A})$. $\square$

The proof of Theorem 4 Since $\mathcal{A}$ is a dissipative operator and $(0,\infty) \subset \rho(\mathcal{A})$(see Corollary 3), the Lumer-Phillips Theorem (see, [17]) asserts that $\mathcal{A}$ generates a C_0-semigroup $T(t)$ of contractions on $\mathbb{X}$. Hence the Cauchy problem has unique a solution.

The proof of Theorem 5 Let $\mathbb{X}$ and $\mathcal{A}$ be defined as before, and $T(t)$ be the C_0 semigroup generated by $\mathcal{A}$. We regard $\mathbb{X}$ as a real Banach space, and the positive cone defined by

$$\mathbb{X}_+ = \{F = (f_0, f_1(x), f_2(x), f_3(x)) \in \mathbb{X} \\ |f_i \geq 0, i = 0,1,2,3\}.$$

It is a closed convex cone, also called positive cine. A bounded linear operator T is called a positive operator if $T\mathbb{X}_+ \subset \mathbb{X}_+$. In what follows, we shall show $T(t)$ is positive semigroup and satisfies $\|T(t)P\| = \|P\|$ for any positive vector $P \in D(\mathcal{A})$.

Step 1. $T(t)$ is positive semigroup.

According to the positive semigroup theory (see [21]), $\mathcal{A}$ generates a positive C_0-semigroup of contractions if and only if $\mathcal{A}$ is a dispersive and $\mathfrak{R}(I - \mathcal{A}) = \mathbb{X}$. Since Theorem 3.2 has asserted that $\mathfrak{R}(I - \mathcal{A}) = \mathbb{X}$, we only need to prove $\mathcal{A}$ is a dispersive operator.

For any $P \in D(\mathcal{A})$, we take

$$Q = \|P\|(q_0, q_1(x), q_2(x), q_3(x))$$

where

$$q_0 = sign_+(P_0), q_i(x) = sign_+(p_i(x)), i = 1,2,3$$

where

$$sign_+(p_i) = \begin{cases} 1, p_i > 0 \\ 0, p_i \leq 0 \end{cases}$$

Similar to calculation in the proof of Theorem 2, we can prove that $(\mathcal{A}P, Q) \leq 0$, so $\mathcal{A}$ is dispersive operator. Therefore, $T(t)$ is a positive operator for all $t \geq 0$.

Step 2. For any $F \in \mathbb{X}_+$, it holds that $\|T(t)F\| = \|F\|$.

Due to $\overline{D(\mathcal{A})} = \mathbb{X}$, we only need to prove $\|T(t)P\| = \|P\|$ for any positive vector $P \in D(\mathcal{A})$. Let $P \in D(\mathcal{A}) \cap \mathbb{X}_+$. For $t \geq 0$, we get $T(t)P \in D(\mathcal{A}) \cap \mathbb{X}_+$ since $T(t)$ is positive. Let

$$P(t) = (P_0(t), p_1(x,t), p_2(x,t), p_3(x,t)) = T(t)P.$$

Then it satisfies the equation $\frac{d}{dt}T(t)P = \mathcal{A}P$, or equivalently the differential equation (1). Due to $T(t)P \geq 0$ we have

$$\|P(t)\|_{\mathbb{X}} = |P_0(t)| + \int_0^\infty p_1(x,t)dx \\ + \int_0^\infty p_2(x,t)dx + \int_0^\infty p_3(x,t)dx$$

and

$$\frac{d}{dt}\|P(t)\| = \frac{d}{dt}P_0(t) + \frac{d}{dt}\int_0^\infty p_1(x,t)dx \\ + \frac{d}{dt}\int_0^\infty p_2(x,t)dx + \frac{d}{dt}\int_0^\infty p_3(x,t)dx \\ = 0,$$

so $\|P(t)\|$ is a constant. By the continuity of $T(t)P$, we get $\|T(t)P = \| = \|P\|$. $\square$

Acknowledgements: The research is supported by the Natural Science Foundation of China grant NSFC-61174080, 61104130.

References:

[1] International Diabetes Federation, *Diabetes Atlas,* sixth edition, 2013.

[2] L. Tang, X .B. Chen, H. Y. Chen, The control of fasting plasma glucose and HbA_1c multifactor analysis for medical cost in patients with type 2 diabetes mellitus, *Chinese Journal of Diabetes,* 12, 2004, pp. 205-208.

[3] T. Dall, Y. D. Zhang, et al. Economic Costs of Diabetes in the U.S. in 2007, *Deabetes Care,* 31, 2008, pp. 596-615.

[4] S. E. Nissen, K. Wolski, Rosiglitazone, An Updated Meta-analysis of Risk for Myocardial Infarction and Cardiovascular Martality, *Arcjives of Internal Medicine,* 170, 2010, pp. 1191-1201.

[5] J. Wang, D. J. Zou, Cardiovascular benefits and risks in oral antidiabetic treatment of type 2 diabetes, *Chinese Journal of Practical Internal Medicine,* 31, 2011, pp. 340-342.

[6] E. M. Kohner, S. J. Aldington, I. M. Stratton, et al. United Kingdom Prospective Diabetes Study 30: diabetic retinopathy at diagmodid of non-insulin-dependent diabetes mellitus and associated risk factors, *Arch Ophthalmol,* 116, 1998, pp. 297-303.

[7] L. Ribu, Br. Hanestad, T. Moum, et al. A comparison of the health-related quality of life in patients with diabetic foot ulcers, with a diabetes group and a non-diabetes group from the general population, *Qual Life Res,* 16, 2007, pp. 179-189.

[8] R. Mistra, J. Lager, Predictors of quality of life among adults with type 2 diabetes mellitus, *Diabetes Complications,* 22, 2008, pp. 217-223.

[9] G. Nucci, C. Cobelli, Models of subcutaneous insulin kinetica. A critical review, *Computer Methods and Programs in Biomedicine,* 62, 2000, pp. 249-257.

[10] J. X. Li, J. D. Johnson, Mathmatical models of subcutaneous injection of insulin analogues: A mini-review, *Discrete and Contiunous Dynamic System-B,* 12, 2009, pp. E401-E414.

[11] E. Salzsieder, L. Vogt, K.-D. Kohnert et al. Model-based decision support in diabetes care, *Computer methods and programs in biomedicine,* 102, 2011, pp. 206-218.

[12] A. Abu-Rmileh, W. Garcia-Gabin, Wiener sliding-mode control for artificial pancreas: A new nonlinear approach to glucose regulation, *Computer methods and programs in biomedicine,* 107, 2012, pp. 327-340.

[13] L. X. Ma, G.Q.Xu, Nikos E. Mastorakis, Analysis of a deteriorating cold standby system with priority, *WSEAS Transaction on Mathematics,* 2, 2011, pp. 84-94.

[14] D. H. Liu, G. Q. Xu, Nikos E. Mastorakis, Reliability Analysis of a Deteriorating System with Delayed Vacation of Repairman, *WSEAS Transaction on Mathematics,* 12, 2011, pp. 413-424.

[15] W. Z. Yuan, G. Q. Xu, Spectral analysis of a two unit deteriorating standby system with repair, *WSEAS Transaction on Mathematics,* 4, 2011, pp. 125-138.

[16] D. H. Liu, G. Q. Xu, Study on the Diabetic Population Development of Early Warning Modeling, *Acta Mathematica Scientia,* 34, 2014, pp. 495-511.

[17] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*. Springer, New York, 1983.

[18] H. B. Xu, W. H. Guo, J. Y. Yu, G. T. Zhu, The Asympototic Stability of a Series Repairable System, *Acta Mathematical Applicata Sinca,* 29, 2006, pp. 46-52.

[19] K. J. Engel, R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Gradute Texts in Math, 194, Springer-Verlag, 2000.

[20] W. Arendt, C. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. AMS,* 306, 1988, pp. 837-852.

[21] R. Nagel, *One-parameter Semigroup of Positive Operators, Lecture Notes in Mathematics*, Springer, New York, 1986.

[22] L. N. Guo, H. B. Xu, C.Gao and G.T.Zhu, Stability analysis of a new kind series system, *IMA Journal of Applied Mathematics,* 75, 2010, pp. 439-460.

[23] H. Y. Wang, G. Q. Xu and Z. J. Han, Modeling of health status on given public and its analysis of well-posedness, *Journal of Systems Engineering,* 23, 2007, pp. 385-391.

[24] X. B. Chen, L. Tang, H. Y. Chen, Assessing the impact of complications on the costs of Type 2 diabetes in urban China, *China Journal Diabetes,* 11, 2003, pp. 238-241.