On a two-variable Functional Equation arising from Databases

EL-SAYED El-HADY[1,2]
[1]Innsbruck University
Institute of Mathematics
Innsbruck, A-6020
AUSTRIA
[2]Suez Canal University
Ismailia, 41522
EGYPT
el-sayed.el-hady@uibk.ac.at

WOLFGANG FÖRG-ROB
Innsbruck University
Institute of Mathematics
Innsbruck, A-6020
AUSTRIA
wolfgang.foerg-rob@uibk.ac.at

MOHAMED MAHMOUD
Al-Azhar University, Cairo
EGYPT
mawmahmoud11@hotmail.com

Abstract: Functional equations offer a tool for narrowing the models used to describe many phenomena. Recently, a certain class of functional equations stems when obtaining the generating functions of queueing systems distributions. This paper has been motivated by an issue considered by L. Flatto and S. Hahn in [*SIAM Journal on Applied Mathematics* 44(5)*,* (1984)*, 1041–1053*]. The functional equation obtained there has been converted into a set of conditions on the two-unknowns, which in turn lead to the determination of the main unknown. Another motivation of solving such functional equation comes from the fact that the underlying queueing system has applications in the inventory control of database systems. Unfortunately that solution seems to be a bit too general with many technical assumptions. In this paper we introduce a solution using boundary value problem approach. Our solution is validated, obtained by assuming full symmetry on the system under study, and by the reduction to a Riemann-Hilbert boundary value problem.

Key–Words: Queueing systems, Functional equations, Boundary value problems, Inventory control of database systems.

1 Introduction

Functional equations (FEs) are defined as the equations where the unknowns are functions rather than simple variables [1, 2, 3, 4, 5, 6, 7]. They are more than 200 years old subject of mathematics, but their theory has flourished principally through the work of the prolific mathematician J. Aczél [7, 9, 10, 11, 12] who identified many of their classes, illustrating efficient methods for their solutions as well as criteria for the existence and uniqueness of those solutions [8]. FEs arise abundantly in models of various fields, such as population ethics [13], astronomy [14], neural networks [15], economics [32], digital filtering [33], and the experimental sciences [16].

Specifically, each of these models can be formulated so as to eventually lead to an FE that can yield precise quantitative relationships. FEs can be in one variable or two variables, depending on whether the underlying model is one-dimensional or two dimensional. There is no universal solution technique for these FEs, but rather almost each equation is solved differently than the others. The problem of finding some performance measures to some communication, and networks systems comes with a certain interesting class of functional equations.

In general using the literature there are many approaches available to solve such problem e.g., experimental approach, numerical approach, simulation approach, and the analytical approach see [17]. It is the right place to state that Malyshev [28] pioneered the approach of solving functional equations using the theory of boundary value problems in early 1970s.

The idea of reducing functional equations for the generating function to a standard Riemann-Hilbert boundary value problem stems from the work of Fayolle and Iasnogorodski [19] on two parallel $M/M/1$ queues with coupled processors. Extensive treatments of the boundary value technique for functional equations can be found in Cohen and Boxma [20] and Fayolle et al. [21]. In particular, the following general class of two-variable functional equations

$$\begin{aligned} C_1(x,y)P(x,y) = C_2(x,y)P(x,0) + C_3(x,y)P(0,y) \\ +C_4(x,y)P(0,0), \end{aligned} \tag{1}$$

where $C_i(x,y), i = 1,2,3,4$ are given polynomials in two complex variables x, y, arises from different communication and networks systems. The unknown functions $P(x,y), P(x,0), P(0,y)$ are defined as follows

$$P(x,y) = \sum_{m,n=0}^{\infty} p_{m,n} x^m y^n, |x| \leq 1, |y| \leq 1$$

$$P(x,0) = \sum_{m=0}^{\infty} p_{m,0} x^m, |x| \leq 1,$$

and

$$P(0,y) = \sum_{n=0}^{\infty} p_{0,n} y^n, |y| \leq 1$$

for some sequences of interest $p_{m,n}, p_{m,0}, p_{0,n}$ respectively.

Similar equations appears in [22, 23, 24] such equations have been solved by using the theory of boundary value problems [20]. A simple but similar equation appears in [25] has been solved by using Rouchés theorem, and applying the normalization condition. A whole class of similar functional equations are given in [26]. One more form of the above functional equation has appeared in [27] in the context of analyzing a multiprogrammed computer. The technique used there is that of analytic continuation. A survey paper [29] gives an array of similar functional equations and techniques for their solution.

As an application of such queues the authors mentioned the inventory control of data base systems. A common inaccuracy problem, which stems in such systems is the deviation of the number of items recorded in the data base from the number of items present in stock. The deviation is due to the fact that the updating in the system may be made at times different from those when an item is actually added or removed from stock. If the deviation is due to removals only, then the actual and recorded number of items may be treated as two parallel queues created by the same arrival process, each having its own servicing time distribution.

This article is mainly concerned with a solution of a two-variable FE arising from a double queue model originally published in [18] using boundary value problem. The sequel of the paper is laid as follows: In the next section we recall the functional equation from the original article [18] together with the solution as originally given in [18] but in brief, in section (3) we solve the equation by reduction to Riemann-Hilbert boundary value problem, in section (4) we compute some expectation using the corresponding generating function, and in section (5) we conclude our work.

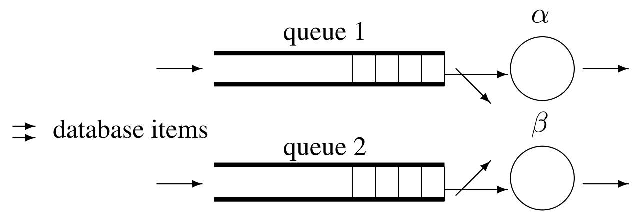

Figure 1: queueing system arising in the inventory control of database systems

2 The functional equation

This equation arises [18] from a double queue model, illustrated in Figure (1), where the arriving customers simultaneously place two demands handled independently by two servers. The arrivals are assumed to be a Poisson process with unit mean, and the two servers have exponential service times with rates α, β, with the stability condition $1 < \alpha \leq \beta$. The probability generating function (PGF) $P(x,y)$ of the two-dimensional distribution characterizing the system yields the two-variable FE

$$Q(x,y)P(x,y) = N(x,y), \tag{2}$$

where

$$Q(x,y) = (1+\alpha+\beta)xy - \alpha y - \beta x - x^2y^2,$$

and

$$N(x,y) = \beta x(y-1)P(x,0) + \alpha y(x-1)P(0,y).$$

Equation (2) has been solved using the analytic continuation. First it can be rewritten as follows

$$\begin{aligned}((1+\alpha+\beta)xy - \alpha y - \beta x - x^2y^2)P(x,y) \\ = \beta x(y-1)P(x,0) + \alpha y(x-1)P(0,y). \quad (3)\end{aligned}$$

It is obvious that the above equation is a special case of the general class of equations given by (1) where

$$C_4(x,y) = 0,$$

$$C_1(x,y) = ((1+\alpha+\beta)xy - \alpha y - \beta x - x^2y^2),$$

$$C_2(x,y) = \beta x(y-1),$$

and

$$C_3(x,y) = \alpha y(x-1).$$

The authors in [18] solved (3) by parameterizing the curve defined by $Q(x,y) = 0$ by a pair of elliptic functions $x = x(t), y = y(t)$. The functional equation for $P(x,y)$ is converted into a set of conditions on $P(x(t),0), P(0,y(t))$, which in turn leads to the determination of $P(x,y)$ in the form

$$P(x,0) = \frac{\beta-1}{\beta}\frac{A(x)}{A(1)} \qquad |x| \leq 1,$$

where the function $A(x)$ is given by

$$A(x) = \frac{\sqrt{a_3-x}+\sqrt{a_3-1}}{[\sqrt{a_3-x}+\sqrt{a_3-\alpha/\beta}][\sqrt{a_3-x}-\sqrt{a_3-\alpha}]},$$

where a_3 is some real number greater than one, and the other unknown is given by

$$P(0,y) = \frac{\alpha-1}{\alpha}\frac{B(y)}{B(1)} \qquad |y| \leq 1,$$

where the function $B(y)$ is given by

$$B(y) = \frac{\sqrt{a_3'-y}+\sqrt{a_3'-1}}{[\sqrt{\acute{a}_3-y}+\sqrt{\acute{a}_3-\beta/\alpha}][\sqrt{\acute{a}_3-y}+\sqrt{\acute{a}_3-\beta}]},$$

where $\acute{a}_3$ is some real number greater than one, for $\alpha=\beta$ the solution is given by

$$P(x,0) = P(0,x) = \frac{(\alpha-1)^{3/2}}{\alpha(\alpha-x)^{1/2}}$$

3 The Boundary Value Problem Model of Eq.(2) and its Solution

In order to solve equation (2), the main idea stems from the fact that the main unknown function $P(x,y)$ is an analytic function in the unit disk, this means that if the $Q(x,y)$ is zero then also the right hand side containing the other unknowns must be zero. Let

$$Q(x,y) := (1+\alpha+\beta)xy-\alpha y-\beta x-x^2y^2 = 0 \quad (4)$$

then also

$$\beta x(y-1)P(x,0)+\alpha y(x-1)P(0,y) = 0, \quad (5)$$

The solution of the original functional equation (2) is now reduced to the solution of the functional equation (5). Assume full symmetry in the system under study, i.e., let $\alpha=\beta$ so that equation (5) can be rewritten as follows

$$\alpha x(y-1)P(x,0)+\alpha y(x-1)P(0,y) = 0, \quad (6)$$

dividing the above equation by $(x-1)(y-1)\neq 0$ to get

$$\frac{\alpha x}{x-1}P(x,0)+\frac{\alpha y}{y-1}P(0,y) = 0. \quad (7)$$

Introduce the function

$$g(x) := \frac{\alpha x}{x-1}P(x,0),$$

then equation (7) can be rewritten as

$$g(x)+g(y) = 0, \quad (8)$$

where the function $g(.)$ is an analytic function except for a simple pole at 1.
Now we reduced the solution of the main functional equation to the solution of (8) in $\{(x,y) : Q(x,y) = 0\}$. But $Q(x,y) = 0$ offers a very large number of ordered pairs, in our symmetric case it is natural to consider the set:

$$M^* := \{(x,\tilde{x}) : Q(x,\tilde{x}) = 0\},$$

where $\tilde{x}$ is the complex conjugate of x. Using this special set we can rewrite equation (8) in the form

$$g(x)+g(\tilde{x}) = 0 \rightarrow \Re g(x) = 0, \quad (9)$$

where $\Re$ sands or the real part of the complex variable. Now we have a boundary value problem: The problem of determining a function $g(.)$ which satisfies:

- Analytic everywhere except for a simple pole at 1
- $\Re g(.) = 0$ on $M \setminus \{1\}$
- $\lim_{x\to 1}(x-1)g(x) = \alpha-1$

In order to solve the boundary value problem constructed, let ϕ, with inverse ψ, be the conformal mapping of the unit disk onto the region bounded by M^* with normalization conditions $\phi(0) = K$, $\phi(1) = 1$ for some real number K. This mapping exists by the Riemann mapping theorem see [30], the curve M^* is simply connected only for large values of α. Define $h(w) := g(\phi(w))$. We then obtain a relatively simple Riemann Hilbert boundary value problem with a pole, for $h(.)$ on the unit circle D (actually, it is a Dirichlet problem with a pole see [20]):

- $\Re h(w) = 0$, on $w \in D \setminus \{1\}$
- $\lim_{w\to 1}(w-1)h(w) = \frac{\alpha-1}{\acute{\phi}(1)}$, where $\acute{\phi}(.) = \frac{d\phi}{dx}$

with $h(.)$ analytic on D, continuous on $\overline{D}\setminus\{1\}$. The solution of this boundary value problem is

$$h(w) = \frac{1}{2}\frac{\alpha-1}{\acute{\phi}(1)}\frac{w+1}{w-1}, \quad w \in D$$

which determines

$$g(x) = h(\psi(x)) = \frac{1}{2}\frac{\alpha-1}{\acute{\phi}(1)}\frac{\psi(x)+1}{\psi(x)-1},$$

inside the curve M; Substitution in the original equation finally yields

$$P(x,y) = (\alpha-1)\psi'(1)\frac{(x-1)(y-1)}{(\psi(x)-1)(\psi(y)-1)} \times \frac{\psi(x)\psi(y)-1}{(1+2\alpha)xy-\alpha y-\alpha x-x^2y^2} \qquad (10)$$

The above equation represents a possible solution to the original equation, noting that we validate that it is a possible solution because it satisfies the normalization condition $P(1,1) = 1$ through applying the L'Hôpital's rule. In the subsequent subsections we investigate the conformal mapping $\psi(.)$, and the contour M^* respectively.

3.1 The function $\psi(.)$

In order to find an explicit form to the function ψ, we compare our solution with the solution obtained in the original paper [18] to get:
The original one $P(x,0)$ for $\alpha = \beta$ is given by:

$$P(x,0) = \frac{(\alpha-1)^{3/2}}{\alpha(\alpha-x)^{1/2}}. \qquad (11)$$

Using our solution (10), the function $P(x,0)$ for $\alpha = \beta$ is given by:

$$P(x,0) = (\alpha-1)\acute{\psi}(1)\frac{(x-1)(-1)}{(\psi(x)-1)(\psi(0)-1)} \times \frac{\psi(x)\psi(0)-1}{-\alpha x}. \qquad (12)$$

If we equate the two solutions we get that

$$\sqrt{\frac{\alpha-1}{\alpha-x}} = \acute{\psi}(1)\frac{(x-1)}{(\psi(x)-1)(\psi(0)-1)} \times \frac{\psi(x)\psi(0)-1}{x},$$

solving for $\psi(x)$ to get in the final form

$$\psi(x) = \frac{x\sqrt{\frac{\alpha-1}{\alpha-x}}(\psi(0)-1) - \acute{\psi}(1)(x-1)}{x\sqrt{\frac{\alpha-1}{\alpha-x}}(\psi(0)-1) - \psi(0)\acute{\psi}(1)(x-1)}.$$

3.2 The curve M^*

The curve defined by $M^* := \{(x,\tilde{x}) : Q(x,\tilde{x}) = 0\}$, can be rewritten as follows

$$Q(x,\tilde{x}) = (1+2\alpha)x\tilde{x} - \alpha\tilde{x} - \alpha x - x^2(\tilde{x})^2 = 0,$$

which can be rewritten as

$$Q(x,\tilde{x}) = (1+2\alpha)|x|^2 - 2\alpha(\Re x) - (|x|^2)^2 = 0.$$

If we assume that $x = a+ib$ the curve can be written as

$$a^4+b^4+2a^2b^2-(1+2\alpha)a^2-(1+2\alpha)b^2+2\alpha a = 0,$$

whenever α is large enough the curve M^* describes a simply connected domain. So it is possible to map the interior of the curve M^* conformally to the unit disk in the case that α is large enough.

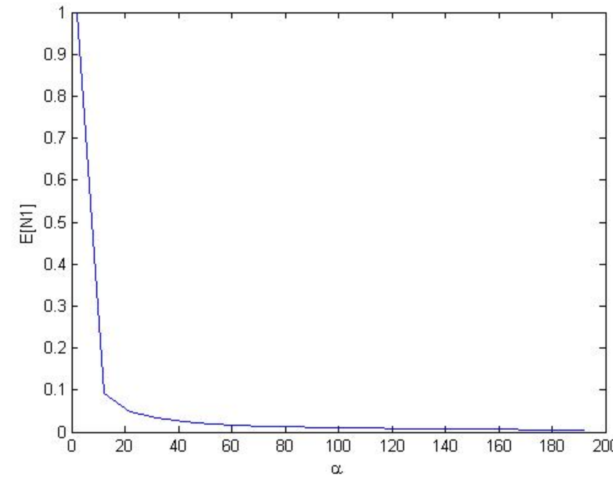

Figure 2: The expected number of database items vs. the service rate

4 Expectations

In this section we find the expected number of files in the database in one of the queues using the corresponding generating functions. It is easy to see using (3) that the generating function of the number of files in the first queue is given by

$$P(x,1) = \sum_{m=0}^{\infty} P(N_1 = m)x^m = \frac{\alpha(x-1)P(0,1)}{((1+\alpha+\beta)x-\alpha-\beta x-x^2)}, \qquad (13)$$

where N_1 is the number of files in the first queue. Using the normalization condition $P(1,1) = 1$ in the above equation to find $P(0,1)$ after some nontrivial manipulations including applying l'Hospitals rule to get

$$P(0,1) = \frac{\alpha-1}{\alpha}.$$

From (13) we will compute the expected number of files in queue 1. It is well known see e.g. [31] that the expected number of packets in queue 1 is given by

$$E[N_1] = \frac{\partial}{\partial x}P(x,1)\mid_{x=1} = P(0,1)\frac{\partial}{\partial x}\frac{\alpha(x-1)}{((1+\alpha+\beta)x-\alpha-\beta x-x^2)} = \frac{0}{0},$$

therefore by applying the L'Hôpital's rule we get

$$E[N_1] = \frac{1}{\alpha-1}. \qquad (14)$$

The above equation represents the expected number of database items in one of the queues. It is clear from (14) that the higher the service rate the lower the expected number of items will be in the first queue which clearly makes perfect sense. By symmetry we can compute the expected number of database items in the second queue using the same procedure. In figure (2) we plot the expected number of database items $E[N_1]$ versus the service rate α. It is clear from the figure that when the service rate is large enough then the expected number of waiting items in the queue will be sufficiently small which practically makes sense.

5 Conclusion

In this article, we managed to introduce a solution of a two-variable functional equation arising from a queueing model. This is done by considering the fully symmetric case on the underlying system and by reduction to a boundary value problem. Another contribution is using generating functions to find some expectations of interest. Possible extension of this work could be to use boundary value problems to find a solution to the asymmetric case. Another extension is to find a general solution methodology to such interesting class of equations.

Acknowledgements: The first author is a channel-scholarship holder. Such a scholarship is fully funded by the *Egyptian ministry of higher education and scientific research, Cairo, EGYPT* during the period January, 2013 till January, 2015. The author also gets a scholarship from Innsbruck University from March, 2015 till August, 2015.

References:

[1] P.–K. Sahoo, P. Kannappan, *Introduction to Functional Equations,* Taylor and Francis Group, LLC, 2011.

[2] M. Kuczma, A survey of the theory of functional equations *Matematika i Fizika-Serie: Mathematiques et physique*, 1964, pp. 1–64.

[3] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities,* Birkhcäuser, 2009.

[4] T.–M. Rassias, J. Brzdek, *Functional Equations in Mathematical Analysis,* Springer Optimization and Its Applications, 52, 2012.

[5] Z. Daroczy, Z. Pales, *Functional Equations Results and Advances,* Advances in Mathematics, 3, 2002.

[6] C.–G. Small, *Functional equation and how to solve them*. Springer, Waterloo, Canada, 2007.

[7] J. Aczél, *Lectures on functional equations and their applications*. Academic Press, New York, 1966.

[8] J. Aczél, R. Luce, Functional equations in the behavioral sciences. *Math. Japonica,* 52, no. 3, 2000, pp. 469–512.

[9] J. Aczél, *Functional Equations: History*. Applications and Theory, 1984.

[10] J. Aczél, *Short Course on Functional Equations: Based Upon Recent Applications to the Social and Behavioural Sciences,* 3, Springer, 1987.

[11] J. Aczél, and J. G. Dhombres *Functional equations in several variables,* 31, Cambridge University Press, 1989.

[12] J. Aczél, and J. G. Dhombres, Some recent applications of functional equations to the social and behavioral sciences. Further problems, *Aggregating clones, colors, equations, iterates, numbers, and tiles*, 1995, pp. 38–49.

[13] C. Blackorby, W. Bossert, D. Donaldson, Functional equations and population ethics. *Aequationes Math.,* 58, no. 3, 1999, pp. 272–284.

[14] H. Fripertinger, J. Schwaiger, Some applications of functional equations in astronomy. *Grazer Math. Ber.*, no. 344, 2008, pp. 1–6.

[15] L. Kindermann, A. Lewandowski, P. Protzel, A framework for solving functional equations with neural networks. *Neural Information Processing,* 2, 2001, pp. 1075–1078.

[16] J.–O. Ramsay, B.–W. Silverman, Functional equations in the behavioral sciences. *International Encyclopedia of the Social & Behavioral Sciences*, Springer, New York, 1996, pp. 5828–5833.

[17] J. Walraevens, Discrete-time queueing models with priorities. *Ph-d Thesis*, Gent University, Ghent, Belgium, 2005.

[18] L. Flatto, S. Hahn, Two Parallel Queues Created by Arrivals with Two Demands I. *SIAM Journal on Applied Mathematics,* 44, 1984, pp. 1041-1053.

[19] G. Fayolle, R. Iasnogorodski, Two coupled processors: the reduction to a Riemann-Hilbert problem.Z. *Wahrscheinlichkeitstheorie Verwandte Gebiete,* 47, 1979, pp. 325–351.

[20] J.–W. Cohen, O.–J. Boxma, *Boundary Value Problems in Queueing System Analysis,* North-Holland, Amsterdam, 1983.

[21] G. Fayolle, R. Iasnogorodski, V. Malyshev, *Random Walks in the Quarter Plane.* Springer, New York, 1999.

[22] J. Resing L. Ormeci, A tandem queueing model with coupled processors, *Operations Research Letters,* 31, 2003, pp. 383–389.

[23] F. Guillemin, J.–S.–H. van Leeuwaarden, Rare event asymptotic for a random walk in the quarter plane. *Queueing Syst.* 67, 2011, pp. 1–32.

[24] I.–J. B. F. Adan, O.–J. Boxma, J.–A. Resing, Queueing models with multiple waiting lines. *Queueing Systems,* 37, 2001, pp. 65–98.

[25] H. Nassar, H. Al Mahdi, Queueing analysis of an ATM multimedia multiplexer with non-pre-emptive priority, *IEE Proc. Commun.,* 150, 2003, pp. 189–196.

[26] G. Fayolle, On Functional Equations of One and Two Complex Variables Arising in the Analysis of Stochastic Models, *In Mathematical Computer Performance and Reliability, G. Iazeolla, P.Courtois and A.Hordijk (eds.),* 1984, pp. 55–75.

[27] I. Adan, J. Wessels, W. Zijm, Analyzing Multiprogramming Queues by Generating Functions, *SIAM J. Appl. Math,* 53, 1993, pp. 1123–1131.

[28] VA. Malyshev, An analytical method in the theory of two-dimensional positive random walks, *Siberian Mathematical Journal,* 13, 1972, pp. 917–929.

[29] O. Boxma, G. Koole, Z. Liu, Queueing thoeretic Solution Methods for Models of Parallel and Distributed Systems, *In Performance Evaluation of Parallel and Distributed Sytems Solution Methods,* O. Boxma and G. Koole (eds.), 1994, pp. 124.

[30] S. Ponnusamy, H. Silverman, *Complex Variables with Applications,* Birkhäuser, 2006.

[31] H. Nassar and M. Ahmed, Performance analysis of an ATM buffered switch transmitting two-class traffic over unreliable channels,*Int.J.Electron. Commun.(AEÜ),* 57, 2003, pp. 190–200.

[32] W. Gehrig, Functional equation methods applied to economic problems: some examples, *Functional Equations: History, Applications and Theory,* Springer, 1985, pp. 33–52.

[33] P.–K. Sahoo, and L. Székelyhidi, On a functional equation related to digital filtering, *aequationes mathematicae,* 62, 2001, pp. 280–285.