# Exponential stabilization of 1-d wave network with one circuit

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*Abstract:* In this paper, we consider a complex network of strings. Suppose that the network is comprised of eight strings with a fixed vertex, and other exterior vertices that are imposed velocity feedback controller. The displacement is not continuous at one interior node, while at the other interior nodes continuity holds and the force is not balanced at all interior nodes. We design controllers for the nodes with discontinuous displacement and with unbalanced force. We show that the operator determined by the closed loop system has a compact resolvent and generates $C_0$ semigroup in an appropriate Hilbert space. Under certain condition, we prove that the closed loop system is asymptotically stable. We also show that there is a sequence of generalized eigenvectors of the system operator, which forms a Riesz basis. Hence the spectrum determined growth condition holds. If the imaginary axis is not an asymptote of the spectrum, then the closed loop system is exponentially stable.

*Key–Words:* String; Network; Stabilization; Feedback control; $C_0$ Semigroup

## 1 Introduction

The dynamical behavior of networks and their control problems, which appear widely in engineering (see [1, 2]), are hot issues of great interest in engineering and applied mathematics. Early works on networks mainly considered the networks described by Ordinary Differential Equations (ODE for short), which is called the Point-to-Point networks system. As an application of networks, the ODE network systems have given nice results for some actual problems. However, some kinds of structures, such as multi-link flexible structure (see [1]), electron scattering and neural impulses (see [3, 4, 5]), cannot be suitably described by ODE networks, since for such kind of networks, not only the global dynamic behavior but also the interaction and transmission of effects at the nodes have to be taken into account due to the flexibility of individual elements. Therefore, the networks described by Partial Differential Equations (PDE for short) are proposed. In the past decades, with the wide use of flexible material in engineering, various physical models of multi-link flexible structures, for instance, trusses, frames, robot arms, solar panels, antennae and deformable mirrors that are consisting of finitely many interconnected flexible elements such as strings, beams, plates and shells, have been mathematically studied. For more applications of flexible structures networks we refer to [1, 6, 7, 8] and the references therein.

Because of the importance of the PDE networks in practice, many mathematicians have devoted to study the control problem of networks such as strings networks and beams networks, a lot of nice results have been obtained. For example, Rolewicz [9] proved that networks are not exact controllable under some geometrical conditions, which may be the earliest results of control problems for flexible structure networks. Chen et al. [2] dealt with the stabilization problem for serially connected beams by the energy multiplier method. Using the similar method, [10] got the exponential stabilization of a long chain of coupled vibrating strings. Ammari et al. [11, 12] and [13] discussed the stabilization problem of tree-shaped and star-shaped of elastic strings and asserted that the networks are asymptotically stable under some conditions. A similar method was used to consider the energy decay of elastic Euler-Bernoulli beams with star-shaped and tree-shaped network configuration (see [14]). By virtue of Hamiltons principle, Schmidt [15] derived a nonlinear system of partial differential equations for networks of vibrating strings and obtained a controllability result for the linearized coupled wave equations.

Compared with rigid structures, the control problems of flexible structures are more complicated because the system we want to control is described by partial differential equations (PDEs) which must be discussed in an infinite dimensional space. It will become a difficult problem to give an analytic solution although it may be simple in rigid structures. By the great efforts of many mathematicians and engineers, there have obtained many nice results on the control problems of flexible structures. For example, Dáger and Zuazua in [16, 17] studied the controllability of star-shaped and tree-shaped networks of string and in [18] Dáger concerned with observation and control of vibrations in tree-shaped networks of strings; Leugering et al in [19] studied the exact controllability of networks of strings and beams by using the multiplier method and in [20, 21] studied the domain decomposition of optimal control problems for dynamic networks of elastic strings and beams; Deckoninck and Nicaise in [22, 23] studied control and eigenvalue problems of networks of Euler-Bernoulli beams. The stabilization of an elastic chain system, as one of the simplest network structures, also has been studied by many researchers. For instance, Liu et al [10] studied the exponential stability of a long chain coupling vibrating strings; Xu and Han in [24, 25] studied the stabilization and Riesz basis property of serially connected Timoshenko beams. For the elastic network structures, Wang et al in [26] studied Riesz bases and stabilization for tree-shaped Euler-Bernoulli beams containing three beams. However, there was few results concerned with the stabilization and Riesz basis property of the complex networks with circuits. The aim of this paper is to study a complex network of strings with one circuit. In particular, we are interested in the stabilization, Riesz basis property and spectrum determined growth condition.

Let $G = (V, E)$ be a graph with vertices $V = \{a_1, a_2, \cdots, a_8\}$ and edges $E = \{s_1, s_2, \cdots, s_8\}$. The nodes $a_2, a_3, a_4$ and $a_7$ are interior nodes of $G$, and the vertices $a_1, a_5, a_6$ and $a_8$ are external nodes of $G$ (or called boundary of $G$). The edges $s_1, s_2, \cdots, s_8$ are connected by $a_1$ and $a_2$, $a_2$ and $a_3$, $a_2$ and $a_4$, $a_3$ and $a_5$, $a_4$ and $a_6$, $a_3$ and $a_7$, $a_4$ and $a_7$,$a_7$ and $a_8$,respectively. It is shown as in the following Figure 1. Now we suppose that the elastic structure is expanded on the graph $G$, whose equilibrium position coincides with $G$. Suppose that the elastic structure at node $a_1$ is fixed and at $a_5, a_6, a_8$ are free. Denote displacement of the elastic structure by $y_j(x,t)$ on the $j$-th edge at position $x \in s_j$ and at time $t$, $j = 1, 2, \cdots, 8$ respectively. The notation $y_x(x,t)$ and $y_t(x,t)$ denote the partial differential with respect to $x$ and $t$, respectively.

The motion of the elastic structure on edges $s_j$ is governed by partial differential equation

$$T_j y_{j,xx}(x,t) = m_j y_{j,tt}(x,t),$$

where $j = 1, 2, \cdots, 8$, and $m_j$ and $T_j$ are the mass densities and tensions, respectively.

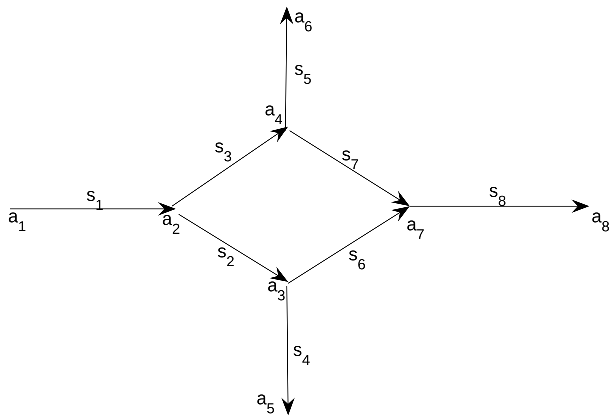


Figure 1: An elastic structure on the graph $G$

For this elastic structure, we impose the following geometric and dynamic conditions:

1) At the interior nodes $a_2$, $a_3$ , and $a_4$, displacement of the structure satisfy the continuity condition, but there are some exterior forces on these nodes, i.e.,

Geometric conditions

$$y_1(1,t) = y_2(0,t) = y_3(0,t);$$

$$y_2(1,t) = y_4(0,t) = y_6(0,t);$$

$$y_3(1,t) = y_5(0,t) = y_7(0,t);$$

at the interior nodes $a_7$ satisfy the following condition

$$y_6(1,t) + y_7(1,t) = y_8(0,t);$$

and forces conditions

$$T_1 y_{1,x}(1,t) - T_2 y_{2,x}(0,t) - T_3 y_{3,x}(0,t) = u_1(t);$$

$$T_2 y_{2,x}(1,t) - T_4 y_{4,x}(0,t) - T_6 y_{6,x}(0,t) = u_2(t);$$

$$T_3 y_{3,x}(1,t) - T_5 y_{5,x}(0,t) - T_7 y_{7,x}(0,t) = u_3(t);$$

$$T_6 y_{6,x}(1,t) - T_8 y_{8,x}(0,t) = u_6(t);$$

$$T_7 y_{7,x}(1,t) - T_8 y_{8,x}(0,t) = u_7(t);$$

where $u_k(t), k = 1, 2, 3, 6, 7$ are external exciting lateral forces.

2) At the external vertices $a_5$,$a_6$ and $a_8$, the elastic structure satisfies the dynamic conditions

$$T_j y_{j,x}(1,t) = u_j(t), j = 4, 5, 8$$

where $u_k(t), k = 4, 5, 8$ are external exciting lateral force.

In order to control this system, at interior nodes $a_2, a_3, a_4, a_7$ and the exterior vertices $a_5$,$a_6$ and $a_8$, we adopt velocities feedback controls, i.e.,

$$u_k(t) = -\alpha_k y_{k,t}(1,t), k = 1, \cdots, 8$$

In addition, we assume that the initial position of the system is given by

$$y_j(x,0) = y_{j,0}(x), y_{j,t}(x,0) = y_{j,1}(x)$$

Thus, the closed form of the complex network system is described by

$$\begin{cases} T_j y_{j,xx}(x,t) = m_j y_{j,tt}(x,t), \\ y_1(0,t) = 0, \\ y_1(1,t) = y_2(0,t) = y_3(0,t), \\ y_2(1,t) = y_4(0,t) = y_6(0,t), \\ y_3(1,t) = y_5(0,t) = y_7(0,t), \\ y_6(1,t) + y_7(1,t) = y_8(0,t), \\ T_1 y_{1,x}(1,t) - T_2 y_{2,x}(0,t) - T_3 y_{3,x}(0,t) \\ = -\alpha_1 y_{1,t}(1,t), \\ T_2 y_{2,x}(1,t) - T_4 y_{4,x}(0,t) - T_6 y_{6,x}(0,t) \\ = -\alpha_2 y_{2,t}(1,t), \\ T_3 y_{3,x}(1,t) - T_5 y_{5,x}(0,t) - T_7 y_{7,x}(0,t) \\ = -\alpha_3 y_{3,t}(1,t), \\ T_j y_{j,x}(1,t) = -\alpha_j y_{j,t}(1,t), j = 4,5,8 \\ T_6 y_{6,x}(1,t) - T_8 y_{8,x}(0,t) = -\alpha_6 y_{6,t}(1,t), \\ T_7 y_{7,x}(1,t) - T_8 y_{8,x}(0,t) = -\alpha_7 y_{7,t}(1,t), \\ y_j(x,0) = y_{j,0}(x), y_{j,t}(x,0) = y_{j,1}(x). \end{cases} \tag{1.1}$$

The contents of this paper is organized as follows. In section 2, we shall discuss the well-posedness and the the asymptotic stability of the system (1.1). In section 3, we shall carry out a complete asymptotic analysis for the spectrum of the system operator. We shall prove that the operator has a compact resolvent whose spectrum is located in a strip, parallel to the imaginary axis under certain conditions. In section 4, we prove that the generalized eigenvectors of the system operator are complete, and there is a sequence of generalized eigenvectors that form a Riesz basis with parentheses. We show that the system satisfies the spectrum determined growth condition. Therefore, if the imaginary axis is not an asymptote of spectrum, then the system decays exponentially.

## 2 Well-posedness of the system

In this section we shall study the well-posedness of the closed loop system (1.1). To this aim, we begin with formulating this system into an appropriate Hilbert state space.

Set

$$Y(x,t) = (y_1(x,t), y_2(x,t), \cdots, y_8(x,t)),$$

We define $n \times n$ matrices by

$$T = \text{diag}\{T_1, T_2, \cdots, T_8\},$$

$$M = \text{diag}\{m_1, m_2, \cdots, m_8\}$$

$$\Gamma = \text{diag}\{\alpha_1, \alpha_2, \cdots, \alpha_8\},$$

where $\alpha_i > 0, i = 1, \cdots, 8$.

Then equation (1.1) can be rewritten in the following form

$$\begin{cases} TY_{xx}(x,t) = MY_{tt}(x,t) \\ Y(0,t) = CY(1,t) \\ TY_x(1,t) - C^\tau TY_x(0,t) = -\Gamma Y_t(1,t) \\ y_j(x,0) = y_{j,0}(x), y_{j,t}(x,0) = y_{j,1}(x). \end{cases} \tag{2.2}$$

where

$$C = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0 \end{pmatrix}$$

Let the state space be $\mathcal{H} = \{(f,g)^\tau \in H^1((0,1),\mathbb{C}^8) \times L^2([0,1],\mathbb{C}^8) | f(0) = Cf(1)\}$ equipped with an inner product, for $\forall (f,g), (\hat{f},\hat{g}) \in \mathcal{H}$, via $\langle (f,g),(\hat{f},\hat{g})\rangle_{\mathcal{H}} = \int_0^1 (Tf'(x), \hat{f}'(x))dx + \int_0^1 (Mg(x),\hat{g}(x))dx$, where$(\cdot,\cdot)$ denotes the inner product in $\mathbb{C}^8$, a direct verification shows that $||(f,g)||^2 = ((f,g),(f,g))_{\mathcal{H}}$ induces a norm on $\mathcal{H}$ and $\mathcal{H}$ is a Hilbert space.

We define an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ (f,g) \in \mathcal{H} \mid \begin{array}{l} f \in H^2((0,1),\mathbb{C}^8), \\ g \in H^1((0,1),\mathbb{C}^8) \\ Tf'(1) - C^\tau Tf'(0) \\ = -\Gamma g(1) \end{array} \right\} \tag{2.3}$$

$$\mathcal{A}(f,g) = (g(x), M^{-1}Tf''(x)) \tag{2.4}$$

Now we rewrite (2.2) as an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t), t > 0 \\ U(0) = U_0 \end{cases} \tag{2.5}$$

where $U(t) = (Y, Y_t)^\tau$ and $U_0 = (Y_0, Y_1)$ is given.

**Theorem 1** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Then $\mathcal{A}$ is dissipative, $\mathcal{A}^{-1}$ is compact, and hence $\mathcal{A}$ generates a $C_0$ semigroup of contractions on $\mathcal{H}$.*

**Proof** First, we prove that $\mathcal{A}$ is a dissipative operator. For any $(f,g) \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} & \langle \mathcal{A}(f,g),(f,g)\rangle_{\mathcal{H}} \\ = & \int_0^1 (Tg'(x), f'(x))dx \\ + & \int_0^1 (M(M^{-1}T)f''(x), g(x))dx \\ = & (Tg(x), f'(x))|_0^1 \\ + & \int_0^1 [(f''(x), Tg(x)) - (Tg(x), f''(x))]dx \end{aligned}$$

and hence,

$$\begin{aligned} & \Re\langle \mathcal{A}(f,g),(f,g)\rangle_{\mathcal{H}} \\ = & \Re(Tg(x), f'(x))|_0^1 \\ = & \Re(Tf'(1), g(1)) - \Re(Tf'(0), g(0)) \\ = & \Re(Tf'(1) - C^\tau Tf'(0), g(1)) \\ = & \Re(-(\Gamma g(1), g(1)) \le 0 \end{aligned}$$

So, $\mathcal{A}$ is a dissipative operator in $\mathcal{H}$.

Next, we shall prove that $\mathcal{A}^{-1}$ is compact. Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. For any fixed $(\mu,\nu) \in \mathcal{H}$, we consider the solvability of the equation $\mathcal{A}(f,g) = (\mu,\nu), (f,g) \in \mathcal{D}(\mathcal{A})$, i.e.,

$$\begin{cases} g(x) = \mu(x), x \in (0,1), \\ M^{-1}Tf''(x) = \nu(x), x \in (0,1), \\ f(0) = Cf(1), \\ Tf'(1) - C^\tau Tf'(0) = -\Gamma g(1). \end{cases} \tag{2.6}$$

Integrating the second equation in (2.6) from $x$ to $1$ leads to

$$Tf'(1) - Tf'(x) = \int_x^1 M\nu(s)ds \tag{2.7}$$

and

$$(1-x)Tf'(1) - Tf(1) + Tf(x) = \int_x^1 dr \int_r^1 M\nu(s)ds. \tag{2.8}$$

From (2.7) and (2.8), we have

$$Tf'(1) - Tf'(0) = \int_0^1 M\nu(s)ds \tag{2.9}$$

and

$$Tf'(1) - Tf(1) + Tf(0) = \int_0^1 dr \int_r^1 M\nu(s)ds. \tag{2.10}$$

Using the boundary conditions in (2.6) we get

$$(I - C^\tau)Tf'(1) = -\Gamma\mu(1) - \int_0^1 C^\tau M\nu(s)ds,$$

and

$$Tf'(1) = -(I - C^\tau)^{-1}[\Gamma\mu(1) + \int_0^1 C^\tau M\nu(s)ds].$$

Using the condition $f(0) = Cf(1)$ leads to

$$Tf'(1) - Tf(1) + TCf(1) = \int_0^1 dr \int_r^1 M\nu(s)ds,$$

Further we have

$$\begin{aligned} f(1) = & -(I-C)^{-1}T^{-1}(I-C^\tau)^{-1} \\ & [\Gamma\mu(1) + \int_0^1 C^\tau M\nu(s)ds] \\ & -(I-C)^{-1}T^{-1}\int_0^1 dr \int_r^1 M\nu(s)ds. \end{aligned}$$

Thus, we get

$$\begin{aligned} & f(x) \\ = & -T^{-1}(I-C^\tau)^{-1}[\Gamma\mu(1) + \int_0^1 C^\tau M\nu(s)ds] \\ & (x-1) - (I-C)^{-1}T^{-1}(I-C^\tau)^{-1} \\ & [\Gamma\mu(1) + \int_0^1 C^\tau M\nu(s)ds] - (I-C)^{-1}T^{-1} \\ & \int_0^1 dr \int_r^1 M\nu(s)ds + T^{-1}\int_x^1 dr \int_r^1 M\nu(s)ds \end{aligned}$$

From discussion above we see that for each $(\mu,\nu) \in \mathcal{H}$, there exists unique a solution $(f,g) \in D(\mathcal{A})$. So $\mathcal{A}^{-1}$ exists and $\mathcal{A}^{-1}(\mu,\nu) = (f,g)$, the Sobolev Embedding Theorem asserts that $\mathcal{A}^{-1}$ is compact. Thus according to the Lumer-Phillips theorem (see, [27]), $\mathcal{A}$ generates a $C_0$ semigroup of contractions.

As a consequence of Theorem 1, we have the following result.

**Corollary 2** *The spectrum $\sigma(\mathcal{A})$ consists of isolated eigenvalues of $\mathcal{A}$ of finite multiplicity, i.e., $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.*

**Corollary 3** *Let $\mathcal{A}$ be defined as before, $S(t)$ be the $C_0$ semigroup generated by $\mathcal{A}$. Then it holds that $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid \Re\lambda < 0\}$ and hence $S(t)$ is asymptotically stable.*

**Proof** For any $\lambda \in \sigma(\mathcal{A})$, we will prove $\Re\lambda < 0$. If it is not true, there exists at least one $\lambda_0 \in \sigma(\mathcal{A})$ with $\Re\lambda_0 = 0$. Clearly, $\lambda_0 \neq 0$. For this $\lambda_0$, let $(f,g) \in \mathcal{D}(\mathcal{A})$ be a corresponding eigenvector. Then from $A(f,g) = \lambda_0(f,g)$ we get that $g(x) = \lambda_0 f(x)$ and

$$\begin{aligned} 0 = \Re\lambda_0 \|(f,g)\|^2_{\mathcal{H}} & = \Re\lambda_0\langle (f,g),(f,g)\rangle_{\mathcal{H}} \\ & = \Re\langle \mathcal{A}(f,g),(f,g)\rangle_{\mathcal{H}} \\ & = -(\Gamma g(1), g(1)) \end{aligned}$$

Since $\Gamma = \text{diag}\{\alpha_1, \alpha_2, \cdots, \alpha_8\}$,where $\alpha_i > 0, i = 1, \cdots, 8$, we can obtain $g(1) = 0$ and, so $f(1) = 0$. From $f(0) = Cf(1)$, we have $f(0) = 0$. Thus $f(x)$ satisfies the following differential equations

$$\begin{cases} \lambda_0^2 Mf(x) = Tf''(x), x \in (0,1), \\ f(0) = 0 = f(1), \\ Tf'(1) - C^\tau Tf'(0) = 0. \end{cases} \tag{2.11}$$

For the sake of convenience, we set

$$B^2 = T^{-1}M = \operatorname{diag}\{\rho_1^2, \rho_2^2, \cdots, \rho_8^2\},$$

$$\rho_j = \sqrt{\frac{m_j}{T_j}}, j = 1, 2, \cdots, 8.$$

Since $M$ and $T$ are positive definite matrices, so $B$ is a positive definite matrices, too. Therefore, the general solution of the equation (2.11) is of the form

$$f(x) = e^{x\lambda_0 B}u + e^{-x\lambda_0 B}v, \quad u, v \in \mathbb{C}^8.$$

From (2.11) and $\lambda_0 \neq 0$, we have

$$\begin{cases} u + v = e^{\lambda_0 B}u + e^{-\lambda_0 B}v = 0, \\ T\lambda_0 B(e^{\lambda_0 B}u - e^{-\lambda_0 B}v) = C^\tau T\lambda_0 B(u - v). \end{cases} \tag{2.12}$$

From the first equation of (2.12) we get that $u = -v$ and $\sinh \lambda_0 Bu = 0$; the second equation becomes

$$TM \cosh \lambda_0 Bu = C^\tau TBu.$$

Note that $TM$, $\sinh \lambda_0 B$ and $\cosh \lambda_0 B$ are diagonal matrices, so we have $TM \cosh \lambda_0 B = \cosh \lambda_0 BTM$, $TM \sinh \lambda_0 B = \sinh \lambda_0 BTB$, which will lead to

$$\cosh \lambda_0 BTMu = C^\tau TMu, \quad \sinh \lambda_0 BTBu = 0.$$

Hence, $(e^{\lambda_0 B} - C^\tau)TMu = 0$. Since $\det(e^{\lambda_0 B} - C^\tau)) = e^{\lambda_0 \sum_{k=1}^{8} \rho_k} \neq 0$, we get that $u = 0$. Therefore, $f(x) = 0$, and $(f, g) = 0$. This contradicts to $(f, g)$ being an eigenvector of $\mathcal{A}$. Thus, for any $\lambda \in \sigma(\mathcal{A}), \Re\lambda < 0$. The stability Theorem of semigroup (see, [28]) asserts that $S(t)$ is asymptotically stable.

# 3 Asymptotic analysis of spectrum of $\mathcal{A}$

In this section, we will discuss the asymptotic distribution of the spectrum of $\mathcal{A}$. Thanks to Corollary 3, we need only to discuss the eigenvalue problem of $\mathcal{A}$.

Let $\lambda \in \mathbb{C}$ and $(f, g) \in \mathcal{D}(\mathcal{A})$ be a non-zero vector such that $(\lambda I - \mathcal{A})(f, g) = 0$. It is equivalent to the following equations

$$\begin{cases} g(x) = \lambda f(x), x \in (0, 1), \\ \lambda B^2 f(x) = f''(x), x \in (0, 1), \\ f(0) = Cf(1), \\ Tf'(1) - C^\tau Tf'(0) = -\lambda \Gamma f(1). \end{cases} \tag{3.13}$$

So the general solution to the differential equation in (3.13) is of the form

$$f(x) = e^{x\lambda B}u + e^{-x\lambda B}v, \quad u, v \in \mathbb{C}^8.$$

Inserting above into the boundary condition in (3.13) leads to a system of algebraic equations

$$\begin{cases} (I - Ce^{\lambda B})u + (I - Ce^{-\lambda B})v = 0, \\ ((TB + \Gamma)e^{\lambda B} - C^\tau TB)u \\ +((\Gamma - TB)e^{-\lambda B} + C^\tau TB)v = 0. \end{cases} \tag{3.14}$$

Clearly, the algebraic equations have non-zero solution if and only if the determinant of the coefficient matrix vanishes, i.e., $D(\lambda) = 0$ where

$$D(\lambda) = \det\begin{pmatrix} I - Ce^{\lambda B} \\ (TB + \Gamma)e^{\lambda B} - C^\tau TB \end{pmatrix.$$

$$\begin{matrix} I - Ce^{-\lambda B} \\ (\Gamma - TB)e^{-\lambda B} + C^\tau TB \end{matrix}\Bigg). \tag{3.15}$$

Conversely, if $\lambda \in \mathbb{C}$ such that $D(\lambda) = 0$, the equation (3.15) has at least a non-zero solution, then we can see that $\lambda$ also is an eigenvalue of $\mathcal{A}$.

Note that

$$\begin{aligned} & D(\lambda) \\ = \ & \det\begin{pmatrix} e^{-\lambda B} - C & I - Ce^{-\lambda B} \\ (TB + \Gamma) - C^\tau TBe^{-\lambda B} & (\Gamma - TB)e^{-\lambda B} + C^\tau TB \end{pmatrix} \\ & \det\begin{pmatrix} e^{\lambda B} & O \\ O & I \end{pmatrix} \\ = \ & \det\begin{pmatrix} I - Ce^{\lambda B} & e^{\lambda B} - C \\ (TB + \Gamma)e^{\lambda B} - C^\tau TB & (\Gamma - TB) + C^\tau TBe^{\lambda B} \end{pmatrix} \\ & \det\begin{pmatrix} I & O \\ O & e^{-\lambda B} \end{pmatrix}. \end{aligned}$$

So, we have

$$\begin{aligned} \Delta_+ &= \lim_{\Re\lambda \to +\infty} \tfrac{D(\lambda)}{\det(e^{\lambda B})} \\ &= \det\begin{pmatrix} -C & I \\ TB + \Gamma & C^\tau TB \end{pmatrix} \\ &= (\alpha_1 + T_1\rho_1 + T_2\rho_2 + T_3\rho_3) \\ & (\alpha_2 + T_2\rho_2 + T_4\rho_4 + T_6\rho_6) \\ & (\alpha_3 + T_3\rho_3 + T_5\rho_5 + T_7\rho_7) \\ & (\alpha_4 + T_4\rho_4)(\alpha_5 + T_5\rho_5) \\ & [(\alpha_6 + T_6\rho_6)(\alpha_7 + T_7\rho_7) + (\alpha_6 \\ & +T_6\rho_6 + \alpha_7 + T_7\rho_7)T_8\rho_8] \\ & (\alpha_8 + T_8\rho_8) \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
\Delta_- &= \lim_{\Re\lambda\to-\infty} \frac{D(\lambda)}{\det(e^{-\lambda B})} \\
&= \det\begin{pmatrix} I & -C \\ -C^\tau TB & \Gamma - TB \end{pmatrix} \\
&= (\alpha_1 - T_1\rho_1 - T_2\rho_2 - T_3\rho_3) \\
&(\alpha_2 - T_2\rho_2 - T_4\rho_4 - T_6\rho_6) \\
&(\alpha_3 - T_3\rho_3 - T_5\rho_5 - T_7\rho_7) \\
&(\alpha_4 - T_4\rho_4)(\alpha_5 - T_5\rho_5)[(\alpha_6 - T_6\rho_6 - T_8\rho_8) \\
&(\alpha_7 - T_7\rho_7 - T_8\rho_8) - T_8^2\rho_8^2](\alpha_8 - T_8\rho_8).
\end{aligned} \tag{3.17}
$$

So, if $\lim_{\Re\lambda\to-\infty} \frac{D(\lambda)}{\det(e^{-\lambda B})} \neq 0$, i.e.,

$$
\begin{cases}
\alpha_1 - T_1\rho_1 - T_2\rho_2 - T_3\rho_3 \neq 0, \\
\alpha_2 - T_2\rho_2 - T_4\rho_4 - T_6\rho_6 \neq 0, \\
\alpha_3 - T_3\rho_3 - T_5\rho_5 - T_7\rho_7 \neq 0, \\
\alpha_4 - T_4\rho_4 \neq 0, \\
\alpha_5 - T_5\rho_5 \neq 0, \\
(\alpha_6 - T_6\rho_6 - T_8\rho_8)(\alpha_7 - T_7\rho_7 - T_8\rho_8) \\
-T_8^2\rho_8^2 \neq 0, \\
\alpha_8 - T_8\rho_8 \neq 0,
\end{cases} \tag{3.18}
$$

there exist positive constants $c_1, c_2$ and $\delta$ such that when $|\Re\lambda| > \delta$, we have

$$
c_1 \det(e^{|\lambda|B}) \leq |D(\lambda)| \leq c_2 \det(e^{|\lambda|B}). \tag{3.19}
$$

Hence,

$$
\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid D(\lambda) = 0\} \subset \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq \delta\}. \tag{3.20}
$$

and $D(\lambda)$ is a sine-type function. Levin's theorem asserts that the zero sets of $D(\lambda)$ is a union of finitely many separable sets. So, $\sigma(\mathcal{A})$ is a union of finitely many separable sets too. From Corollary 3, we can obtain

$$
\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} | -\delta \leq \Re(\lambda) < 0\}. \tag{3.21}
$$

Therefore, we can deduce the following result.

**Theorem 4** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, and let $D(\lambda)$ be defined as (3.15). Then $\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid D(\lambda) = 0\}$ and when the condition (3.18) holds, $\sigma(\mathcal{A})$ distributes in a strip parallel to imaginary axis and is a union of finitely many separable sets.*

# 4 Completeness and Riesz basis property of eigenvectors of A

In this section, we shall discuss the completeness and Riesz basis property of the root vectors of $\mathcal{A}$. Firstly, we establish the completeness of the root vectors of $\mathcal{A}$ and then use the spectral distribution of $\mathcal{A}$ to obtain the Riesz basis property.

Let us define a new operator by

$$
\begin{aligned}
&\mathcal{A}_0(f, g) = (g, M^{-1}Tf'') \\
&\mathcal{D}(\mathcal{A}_0) \\
&= \left\{ (f, g) \in \mathcal{H} \;\middle|\; \begin{array}{l} f \in H^2((0,1), \mathbb{C}^8), \\ g \in H^1((0,1), \mathbb{C}^8) \\ |Tf'(1) - C^\tau Tf'(0) = 0 \end{array} \right\}
\end{aligned} \tag{4.22}
$$

**Theorem 5** *$\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ and for $\forall(\mu, \nu) \in \mathcal{H}, \lambda \in \mathbb{R}$, the solution $(f_\lambda, g_\lambda)$ of the equation $\lambda(f, g) - \mathcal{A}_0(f, , g) = (\mu, \nu)$ satisfy*

$$
\|g_\lambda(1)\| \leq K\|(\mu, \nu)\|_{\mathcal{H}}. \tag{4.23}
$$

*where $K$ is a positive constant.*

**Proof.** It is easy to check that $D(\mathcal{A}_0^*) = D(\mathcal{A}_0)$ and $\mathcal{A}_0^* = -\mathcal{A}_0$. In what follows we mainly prove the inequality (4.23).

For $\forall(\mu, \nu) \in \mathcal{H}, \lambda \in \mathbb{R}$. Suppose that $(f_\lambda, g_\lambda)$ satisfy the equation

$$
(\lambda I - \mathcal{A}_0)(f, g) = (\mu, \nu), (f, g, ) \in \mathcal{D}(\mathcal{A}_0).
$$

we have

$$
\lambda f_\lambda(x) - g_\lambda(x) = \mu(x), \lambda g_\lambda(x) - M^{-1}Tf_\lambda''(x) = \nu(x),
$$
$$
f_\lambda(0) = Cf_\lambda(1), Tf_\lambda'(1) - C^\tau Tf_\lambda'(0) = 0
$$

Since

$$
f_\lambda(1) = \int_0^1 f_\lambda'(x)dx + f_\lambda(0) = \int_0^1 f_\lambda'(x)dx + Cf_\lambda(1),
$$

we have

$$
f_\lambda(1) = (I - C)^{-1}(\int_0^1 f_\lambda'(x)dx),
$$

Similarly, we have

$$
\mu(1) = (I - C)^{-1}(\int_0^1 \mu'(x)dx),
$$

Hence,

$$
\begin{aligned}
g_\lambda(1) &= \lambda f_\lambda(1) - \mu(1) \\
&= (I - C)^{-1}T^{-\frac{1}{2}} \\
&\quad [\lambda \textstyle\int_0^1 T^{\frac{1}{2}} f_\lambda'(x)dx - \int_0^1 T^{\frac{1}{2}}\mu'(x)dx]
\end{aligned}
$$

So, we have

$$
\begin{aligned}
& \|g_\lambda(1)\| \\
\leq\ & \|(I - C)^{-1}T^{-\frac{1}{2}}\|[|\lambda|(\textstyle\int_0^1 (Tf_\lambda'(x), f_\lambda'(x))dx)^{\frac{1}{2}} \\
+\ & (\textstyle\int_0^1 (T\mu'(x), \mu'(x))dx)^{\frac{1}{2}}] \\
\leq\ & \|(I - C)^{-1}T^{-\frac{1}{2}}\|(|\lambda|\|\mathcal{R}(\lambda, \mathcal{A}_0)(\mu, \nu)\|_{\mathcal{H}} \\
+\ & \|(\mu, \nu)\|).
\end{aligned}
$$

Since $\mathcal{A}_0$ is a skew-adjoint operator, $\|\lambda\mathcal{R}(\lambda,\mathcal{A}_0)\|\le 1,\lambda\in\mathbb{R}$, we have

$$\|g_\lambda(1)\|\le 2\|(I-C)^{-1}T^{-\frac{1}{2}}\|\|(\mu,\nu)\|_{\mathcal{H}},\forall\lambda\in\mathbb{R}.$$

So,

$$\|g_\lambda(1)\|\le K\|(\mu,\nu)\|_{\mathcal{H}}$$

where $K=2\|(I-C)^{-1}T^{-\frac{1}{2}}\|$.

**Theorem 6** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before and $\mathcal{A}_0$ be defined as (4.22). If the conditions in (3.18) hold, then the system of the root vectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

**Proof.** The completeness of the root vectors of $\mathcal{A}$ is just

$$\mathrm{Sp}(\mathcal{A})=\overline{\{\sum y_k,y_k\in E(\lambda_k,\mathcal{A})\mathcal{H},\forall\lambda_k\in\sigma(\mathcal{A})\}},$$

where $E(\lambda_k,\mathcal{A})$ is the Riesz projection corresponding to $\lambda_k$.

Assuming that $(\mu_0,\nu_0)\in\mathcal{H}$ and $(\mu_0,\nu_0)\perp Sp(\mathcal{A})$, then the resolvent $R^*(\lambda,\mathcal{A})(\mu_0,\nu_0)$ is a $\mathcal{H}$-valued entire function for $\lambda\in\mathbb{C}$. Thus for any $(\mu,\nu)\in\mathcal{H}$, the function

$$F(\lambda)=\langle(\mu,\nu),R^*(\lambda,\mathcal{A})(\mu_0,\nu_0)\rangle_{\mathcal{H}}$$

is an entire function that satisfies

$$|F(\lambda)|\le(\Re\lambda)^{-1}\|(\mu,\nu)\|_{\mathcal{H}}\|(\mu_0,\nu_0,)\|_{\mathcal{H}},\ \Re\lambda>0$$

due to $A$ generates a $C_0$ semigroup of contraction. Thus, $\lim_{\Re\lambda\to+\infty}F(\lambda)=0$.

Now let us consider the resolvent problems for $\lambda\in\rho(\mathcal{A})\cap\rho(\mathcal{A}_0)\cap\mathbb{R}$

$$(\lambda I-\mathcal{A})(f_{1\lambda},g_{1\lambda})=(\mu,\nu),$$

$$(\lambda I-\mathcal{A}_0)(f_{2\lambda},g_{2\lambda})=(\mu,\nu,).$$

Set

$$\widetilde{f}(x)=f_{1\lambda}(x)-f_{2\lambda}(x),\ \widetilde{g}(x)=g_{1\lambda}(x)-g_{2\lambda}(x).$$

Then

$$R(\lambda,\mathcal{A})(\mu,\nu)=R(\lambda,\mathcal{A}_0)(\mu,\nu)+(\widetilde{f},\widetilde{g}),$$

which implies that $\widetilde{g}(x)=\lambda\widetilde{f}(x)$, and $\widetilde{f}$ satisfy the following equation

$$\begin{cases}\lambda^2B^2\widetilde{f}(x)=\widetilde{f}''(x),x\in(0,1),\\ \widetilde{f}(0)=C\widetilde{f}(1),\\ T\widetilde{f}'(1)-C^\tau T\widetilde{f}'(0)+\lambda\Gamma\widetilde{f}(1)=-\Gamma g_{2\lambda}(1).\end{cases}$$

So, we have

$$\widetilde{f}(x)=e^{x\lambda B}y+e^{-x\lambda B}z,\quad y,z\in\mathbb{C}^8,$$

where $y$ and $z$ satisfy the following algebraic equations

$$\begin{cases}(I-Ce^{\lambda B})y+(I-Ce^{-\lambda B})z=0,\\ ((\Gamma+TB)e^{\lambda B}-C^\tau TB)y\\ +((\Gamma-TB)e^{-\lambda B}+C^\tau TB)z=-\lambda^{-1}\Gamma g_{2\lambda}(1).\end{cases}$$

Therefore,

$$\begin{cases}y=(I-Ce^{\lambda B})^{-1}(C-e^{\lambda B})e^{-\lambda B}z\\ =(C-o(\lambda^{-1}))e^{-\lambda B}z,(\lambda\to-\infty);\\ ((\Gamma+TB)e^{\lambda B}-C^\tau TB)y+((\Gamma-TB)e^{-\lambda B}\\ +C^\tau TB)z=-\lambda^{-1}\Gamma g_{2\lambda}(1).\end{cases}$$

Hence, when $\lambda\to-\infty$

$$\begin{aligned}e^{-\lambda B}z&=-\lambda^{-1}(\Gamma-TB-C^\tau TBC\\ &+o(\lambda^{-1}))^{-1}\Gamma g_{2\lambda}(1)\end{aligned}$$

When $|\lambda|$ is large enough, we have

$$\begin{aligned}\widetilde{f}(1)&=e^{\lambda B}y+e^{-\lambda B}z\\ &=e^{\lambda B}(C-o(\lambda^{-1}))e^{-\lambda B}z+e^{-\lambda B}z\\ &=-\lambda^{-1}(I+o(\lambda^{-1}))(\Gamma-TB\\ &-C^\tau TBC+o(\lambda^{-1}))^{-1}\Gamma g_{2\lambda}(1)\end{aligned}$$

Therefore, there exist a positive constants $M_1$ such that

$$\|\widetilde{f}(1)\|\le M_1|\lambda^{-1}|\|g_{2\lambda}(1)\|,$$

and

$$\begin{aligned}&\lambda^{-1}T\widetilde{f}'(0)\\ =&\lambda^{-1}T(C+o(\lambda))\Gamma(\Gamma-TB-C^\tau TBC\\ &+o(\lambda^{-1}))^{-1}g_{2\lambda}(1)\end{aligned}$$

Hence,

$$\begin{aligned}&\|(\widetilde{f},\widetilde{g})\|^2_{\mathcal{H}}\\ =&\int_0^1(T\widetilde{f}'(x),\widetilde{f}'(x))dx+\int_0^1(T\widetilde{g}(x),\widetilde{g}(x))dx\\ =&(T\widetilde{f}'(1),\widetilde{f}'(1))-(T\widetilde{f}'(0),\widetilde{f}'(0))\\ =&-\lambda(\Gamma\widetilde{f}(1),\widetilde{f}(1))-(\Gamma g_{2\lambda}(1),\widetilde{f}(1))\\ \le&|\lambda|\|\Gamma\|\|\widetilde{f}(1)\|^2+\|\Gamma\|\|g_{2\lambda}(1)\|\|\widetilde{f}(1)\|\\ \le&M_2|\lambda^{-1}|\|g_{2\lambda}(1)\|^2\end{aligned}$$

where $M_2$ is a positive constant.

From theorem 4.1, we have

$$\|g_{2\lambda}(1)\|\le K\|(\mu,\nu)\|_{\mathcal{H}}$$

Thus, there exist a positive constant $M_3$,

$$\|(\widetilde{f},\widetilde{g})\|_{\mathcal{H}}\le M_3\sqrt{|\lambda^{-1}|}\|(\mu,\nu)\|_{\mathcal{H}}.$$

For $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_I) \cap \mathbb{R}_-$ and $|\lambda|$ sufficiently large, we have

$$\begin{aligned} & |F(\lambda)| \\ = & |\langle R(\lambda,\mathcal{A})(\mu,\nu),(\mu_0,\nu_0)\rangle_{\mathcal{H}}| \\ = & |\langle R(\lambda,\mathcal{A}_0)(\mu,\nu),(\mu_0,\nu_0)\rangle_{\mathcal{H}} \\ + & \langle(\widetilde{f},\widetilde{g}),(\mu_0,\nu_0)\rangle_{\mathcal{H}}| \\ \leq & |\lambda^{-1}|\|(\mu,\nu)\|_{\mathcal{H}}\|(\mu_0,\nu_0)\|_{\mathcal{H}} \\ + & \sqrt{|\lambda^{-1}|}M_3\|(\mu,\nu)\|_{\mathcal{H}} \end{aligned}$$

Hence, we can get

$$\lim_{\Re\lambda\to-\infty} F(\lambda) = 0.$$

Since $F(\lambda)$ is an entire function of finite exponential type, the Phragmén-Linderöf Theorem (see, [29]) says that

$$|F(\lambda)| \leq M, \forall\lambda \in \mathbb{C}.$$

So Liouvills Theorem asserts that $F(\lambda) \equiv 0$. This means that $(\mu_0,\nu_0) = 0$. Therefore, $Sp(\mathcal{A}) = \mathcal{H}$.

In order to obtain the Riesz basis generation of the root vectors of $\mathcal{A}$, we need the following Lemma (see, [30] and [25]).

**Lemma 7** *Let $\mathcal{A}$ be the generator of a $C_0$ semigroup $T(t)$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied*

*(1) The spectrum of $\mathcal{A}$ has a decomposition*

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$$

*where $\sigma_2(\mathcal{A})$ consists of the isolated eigenvalues of $\mathcal{A}$ of finite multiplicity (repeated many times according to its algebraic multiplicity).*

*(2) There exists a real number $\alpha \in \mathbb{R}$ such that*

$$\sup\{\Re\lambda, \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re\lambda, \lambda \in \sigma_2(\mathcal{A})\}$$

*(3) The set $\sigma_2(\mathcal{A})$ is a union of finite many separated sets.*

*Then the following statements are true:*

*(a) There exist two $T(t)-$invariant closed subspaces $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ such that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$;*

*(b) there exists a finite combination $E(\Omega_k,\mathcal{A})$ of some $\{E(\lambda_k,\mathcal{A})\}_{k=1}^{\infty}$ :*

$$E(\Omega_k,\mathcal{A}) = \sum_{\lambda\in\Omega_k\cap\sigma_2(\mathcal{A})} E(\lambda,\mathcal{A})$$

*such that $\{E(\Omega_k,\mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$. Furthermore,*

$$\mathcal{H} = \overline{\mathcal{H}_1 \bigoplus \mathcal{H}_2}.$$

*(c) If $\sup_{k\geq1}\|E(\lambda_k,\mathcal{A})\| < \infty$, then*

$$\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \bigoplus \mathcal{H}_2 \subset \mathcal{H}.$$

*(d) $\mathcal{H}$ has a decomposition of the topological direct sum, $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$, if and only if*

$$\sup_{n\geq1}\|\sum_{k=1}^{n} E(\Omega_k,\mathcal{A})\| < \infty.$$

Now applying Lemma 7 to our model, combining Theorem 1, Theorem 4 and Theorem 6, we have the following result.

**Theorem 8** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. If the conditions in Theorem 4 are fulfilled, then there is a sequence of eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$. Indeed, in this case, $\mathcal{A}$ generates a $C_0$ group on $\mathcal{H}$. In particular, the system associated with $\mathcal{A}$ will satisfy the spectrum determined growth condition.*

**Proof** Set $\sigma_1(\mathcal{A}) = \{-\infty\}, \sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Theorem 4 shows that all hypotheses in Lemma 7 are fulfilled. So the results of Lemma 7 are true. Hence there is a sequence of eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}_2$. Theorem 6 says that the eigenvectors is complete in $\mathcal{H}$, that is $\mathcal{H}_2 = \mathcal{H}$. Therefore the sequence is also a Riesz basis with parentheses for $\mathcal{H}$. The Riesz basis property of the eigenvectors together with distribution of spectrum of $\mathcal{A}$ implies that $\mathcal{A}$ generates a $\mathcal{C}_0$ group on $\mathcal{H}$. At the same time, the Riesz basis property together with the uniform boundedness of multiplicities of eigenvalues of $\mathcal{A}$ ensure that the system associated with $\mathcal{A}$ satisfies the spectrum determined growth condition. The proof is then complete.

Set $\sigma(\mathcal{A}) = \{\lambda_n, n \in \mathbb{N}\}$, and $\lambda_n = \alpha_n + i\beta_n$. According to Theorem 6, we have $D(\lambda_n) \equiv 0$ for $n \in \mathbb{N}$. Consider the difference

$$\begin{aligned} & D(\alpha_n + i\beta_n) - D(i\beta_n) \\ & = \alpha_n D'(i\beta_n) + \frac{(\alpha_n)^2}{2} D''(\eta_n + i\beta_n) \end{aligned}$$

where $\eta_n \in (0,\alpha_n)$. From above we see that $D(i\beta_n) \to 0$ if and only if $\alpha_n \to 0$. Therefore, as a consequence of Theorem 8, we have the following result.

**Corollary 9** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Suppose that conditions in (3.18) hold. Then the following statements are true:*

(1) *If $\inf_{\lambda\in i\mathbb{R}}|D(\lambda)| \neq 0$, then the system (2.5) is exponentially stable;*

(2) *If $\inf_{\lambda\in i\mathbb{R}}|D(\lambda)| = 0$, then the system (2.5) is asymptotically stable and but not exponentially stable.*

From corollary 9 we see that in order to assert the exponentially stability of the system, we must judge whether or not $\inf_{\lambda\in i\mathbb{R}} |D(\lambda)|$ is zero. In general, it is very difficult to verify it. From known stability result of the 1-d wave networks we know that the stability of irrational ratio of $\rho_j/\rho_k$ is better than the rational ratio of $\rho_j/\rho_k$. Here, we only give a conclusion for a special situation.

Assume that $T_1 = T_2 = \cdots = T_8$ and $m_1 = m_2 = \cdots = m_8$. Let $\rho = \sqrt{m_i/T_i}, i = 1, 2\cdots, 8$ and

$$\begin{cases} w_i(\lambda) = \cosh\lambda\rho + \beta_i \sinh\lambda\rho, \\ v_i(\lambda) = \sinh\lambda\rho + \beta_i \cosh\lambda\rho. \end{cases}$$

$$\begin{cases} F(\lambda) = w_2w_4w_6 + v_4v_6 \sinh\lambda\rho + w_4v_6 \sinh\lambda\rho, \\ F'(\lambda) = v_2w_4w_6 + v_4w_6 \cosh\lambda\rho + w_4v_6 \cosh\lambda\rho. \end{cases}$$

$$\begin{cases} G(\lambda) = w_3w_5w_7 + v_7w_5 \sinh\lambda\rho + w_7v_5 \sinh\lambda\rho, \\ G'(\lambda) = v_3w_5w_7 + v_7w_5 \cosh\lambda\rho + w_7v_5 \cosh\lambda\rho. \end{cases}$$

where $\beta_i = \frac{\alpha_i}{T_i\rho} > 0, i = 1, 2, \cdots, 8$.

After complex calculation, we get

$$\begin{aligned} D(\lambda) &= (w_1F(\lambda)G(\lambda) + \sinh\lambda\rho F(\lambda)G'(\lambda) \\ &+ \sinh\lambda\rho F'(\lambda)G(\lambda))w_8 + v_8 \sinh\lambda\rho \\ &\quad [\omega_1(w_3w_5 + v_5 \sinh\lambda\rho + w_5 \cosh\lambda\rho) \\ &+ \sinh\lambda\rho v_3w_5 + \sinh\lambda\rho \cosh\lambda\rho v_5 \\ &+ \cosh^2\lambda\rho w_5)F(\lambda) + (\sinh\lambda\rho w_3w_5 \\ &+ \sinh^2\lambda\rho v_5 + \sinh\lambda\rho\cosh\lambda\rho w_5)F'(\lambda) \\ &+ w_1(w_2w_4 + v_4 \sinh\lambda\rho + w_4 \cosh\lambda\rho) \\ &+ \sinh\lambda\rho v_2\omega_4 + \sinh\lambda\rho \cosh\lambda\rho v_4 \\ &+ \cosh^2\lambda\rho w_4)G(\lambda) + (\sinh\lambda\rho w_2w_4 \\ &+ \sinh^2\lambda\rho v_4 + \sinh\lambda\rho\cosh\lambda\rho w_4)G'(\lambda) \\ &+ 2w_4w_5] \end{aligned}$$

and $D(\lambda)$ has the following form:

$$D(\lambda) = a_1e^{16\lambda\rho} + a_2e^{14\lambda\rho} + \cdots\cdots + a_8e^{2\lambda\rho} + a_9$$

where each $a_i$ is real constant. Let $z = e^{2\lambda\rho}$. Then $D(\lambda) = 0$ is equivalent to

$$a_1z^8 + a_2z^7 + a_3z^6 + \cdots\cdots + a_8z + a_9 = 0$$

Let $z_j, j = 1, 2, \cdots, 8$ are the zeros of above algebraic equation, we have

$$D(\lambda) = a_1 \prod_{j=1}^{8} (e^{2\lambda\rho} - z_j).$$

Since there is no zero of $D(\lambda)$ on the imaginary axis(see corollary 3), so $|z_j| \neq 1$, Thus, $\inf_{\lambda\in i\mathbb{R}} |D(\lambda)| \neq 0$, then the system (2.5) is exponentially stable.

**Remark 10** *From above calculation we see that if there exists an $\rho$ such that $\rho_j = k_j\rho$ for some $k_j \in \mathbb{N}$, then $D(\lambda)$ also is a polynomial of $z = e^{2\lambda\rho}$. So the system also is exponentially stable provided there is no zeros of $D(\lambda)$.*

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