The Vertex Linear Arboricity of Integer Distance Graph $G(D_{m,1,4})$

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Abstract: An integer distance graph is a graph $G(D)$ with the set Z of all integers as vertex set and two vertices $u, v \in Z$ are adjacent if and only if $|u-v| \in D$, where the distance set D is a subset of positive integers. A k-vertex coloring of a graph G is a mapping f from $V(G)$ to $[0, k-1]$. A path k-vertex coloring of a graph G is a k-vertex coloring such that every connected component is a path in the induced subgraph of $V_i (1 \leq i \leq k)$, where the vertex set V_i is the subset of vertices assigned color i. The vertex linear arboricity of a graph G is the minimum positive integer k such that G has a path k-vertex coloring. In this paper, we studied the vertex linear arboricity of the integer distance graph $G(D_{m,1,4})$, where $D_{m,1,4} = [1,m] \setminus [1,4]$, and proved that $vla(G(D_{m,1,4})) = \lceil \frac{m}{7} \rceil + 1$ for every integer $m \geq 6$.

Key–Words: Integer distance graph; Vertex linear arboricity; Path coloring

1 Introduction

In this paper, R and Z denote the sets of all real numbers and all integers, respectively. For $x \in R$, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and $\lceil x \rceil$ denote the least integer not less than x. Let $[m, n] = \{m, \cdots, n\}$ denote the set of all integers from m to n where $m \leq n$ and $[m, n] = \emptyset$ if $m > n$. $|S|$ denotes the cardinality of a set S and $|S| = +\infty$ means that S is an infinite set.

In recent years, many parameters and graph classes were studied. For examples, He et al. in [7] obtained the linear k-arboricity of the Mycielski graph $M(K_n)$, Lai et al.in [9] gave a survey for the more recent developments of the research on supereulerian graphs and the related problems, and Jiang and Zhang in [8] studied Randomly M_t-decomposable multigraphs and M_2-equipackable multigraphs.

Coloring of graphs is one of the most fascinating and well-studied topic in graph theory. The problem can be traced back to the Four Color Conjecture. It was motivated by application problems as the frequency assignment problem (e.g., $L(2, 1)$-labeling and the multi-level distance labeling), the control of traffic signals (e.g., circular coloring) and other problems from wide range of industrial and technology areas. A vertex coloring can be viewed as a function from V to Z. More precisely, a vertex k-coloring of a graph G is a mapping f from $V(G)$ to $[1, k]$. Given a vertex k-coloring, let V_i denote the set of all vertices of G which colored with i, and $\langle V_i \rangle$ denote the subgraph induced by V_i in G. If V_i is an independent set for every $1 \leq i \leq k$, then f is called a *proper k-coloring*. The *chromatic number* $\chi(G)$ of a graph G is the minimum integer k for which G has a proper k-coloring. If V_i induces a subgraph whose connected components are paths, then f is called a *path* k-coloring. The *vertex linear arboricity* of a graph G, denoted by $vla(G)$, is the minimum number k such that G has a path k-coloring. Clearly, $\chi(G) \geq vla(G)$ for any graph G.

Matsumoto [11] proved that for a finite graph G,

$$vla(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil;$$

moreover, if $\Delta(G)$ is even, then

$$vla(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$$

if and only if G is a complete graph of order $\Delta(G) + 1$ or a cycle. Goddard [5] and Poh [12] proved that $vla(G) \leq 3$ for a planar graph G. Akiyama et al. [1] proved that $vla(G) \leq 2$ if G is an outerplanar graph.

Let S be a subset of real numbers and D a set of positive real numbers. Then *distance graph* $G(S, D)$ has the vertex set S and two real numbers x and y are adjacent if and only if $|x-y| \in D$, where the set

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D is called the *distance set*. In particular, if all elements of D are positive integers and $S = Z$, then the graph $G(Z, D)$, or $G(D)$ in short, is called *integer distance graph*. The distance graphs were introduced by Eggleton et al.[3] in 1985 to study the chromatic number. They proved that $\chi(G(R, D)) = n + 2$, where D is an interval between 1 and δ, and n satisfies $1 \le n < \delta \le n+1$. They also partially determined the values of $\chi(G(D_{m,k}))$, where $D_{m,k} = [1, m] \setminus \{k\}$. The complete solution to $\chi(G(D_{m,k}))$ is provided by Chang et al.in [2]. Many peoples discussed the chromatic number of integer distance graph $G(D)$. More results on the chromatic number of integer distance graphs, see [3, 4, 6, 10, 13] and [14]. In [16] and [17], it is considered that vertex linear arboricity of the real distance graphs. In [15], it is studied that the vertex linear arboricity of $G(D_{m,k})$ where $D_{m,k} = [1, m] \setminus \{k\}$. In [18], it is obtained that $vla\left(G\left(D_{m,1,3}\right)\right) = \lceil \frac{m}{6} \rceil + 1$.

Now the integer distance graph is applied widely to gene sequence, sequential series, on-line computing and so on.

Let $D_{m,1,4} = [1, m] \setminus \{1, 2, 3, 4\}$. In this paper, we shall prove that

$$vla\left(G\left(D_{m,1,4}\right)\right) = \lceil \tfrac{m}{7} \rceil + 1$$

for $m \ge 6$.

2 Main results

For $m = 5$, $D_{5,1,4} = \{5\}$, so we have

$$vla(G(D_{5,1,4})) = 1.$$

For $6 \le m \le 7$, let $n = 14l + j$, $f(n) = 0$ if $0 \le j < 7$, and $f(n) = 1$ if $7 \le j < 14$. Then f is a path coloring, and thus

$$vla\left(G\left(D_{m,1,4}\right)\right) \le 2.$$

Since vertices 0, 5, 10, 15, 20, 25, 30, 24, 18, 12, 6, 0 in $G\left(D_{m,1,4}\right)$ induce a cycle, we obtained that

$$vla\left(G\left(D_{m,1,4}\right)\right) = 2.$$

Theorem 1. *For any integer $m \ge 8$, we have*

$$vla\left(G\left(D_{m,1,4}\right)\right) = \lceil \tfrac{m}{7} \rceil + 1.$$

Proof. At first we give a path coloring of $G\left(D_{m,1,4}\right)$.

Let $f(n) = i$ for $n = 7i + j, 0 \le j \le 6, 0 \le i \le \lceil \frac{m}{7} \rceil$, and for any integer t, let

$$f\left(7t\left(\lceil \tfrac{m}{7} \rceil + 1\right) + n\right) = f(n).$$

Then f is a path coloring, and

$$vla\left(G\left(D_{m,1,4}\right)\right) \le \lceil \tfrac{m}{7} \rceil + 1.$$

In the following, we shall show that

$$vla\left(G\left(D_{m,1,4}\right)\right) \ge \lceil \tfrac{m}{7} \rceil + 1$$

by contradiction approach.

Assume that the result is not right, that is,

$$vla\left(G\left(D_{m,1,4}\right)\right) \le \lceil \tfrac{m}{7} \rceil = q,$$

then $G\left(D_{m,1,4}\right)$ has a path q-coloring f. Clearly, f is also a path q-coloring of the subgraph H induced by vertex subset $[0, 7q]$ of $G\left(D_{m,1,4}\right)$. Note that $|V(H)| = 7q + 1$. Hence there are at least eight vertices

$$(0 \le)\, a_0 < a_1 < \cdots < a_7\, (\le 7q)$$

with the same color α.

Claim 1: If $a_6 - a_0 \le m$, then

$$\begin{aligned} a_6 &= a_5 + 1 = a_4 + 2 = a_3 + 3 \\ &= a_2 + 4 = a_1 + 5 = a_0 + 6. \end{aligned}$$

Otherwise, there is some $0 \le i \le 5$ such that

$$a_{i+1} - a_i > 1,$$

then a_0a_6, a_0a_5, $a_0a_4 \in E(H)$ or a_0a_6, a_1a_6, $a_2a_6 \in E(H)$, i. e., a_0, a_4, a_5, a_6 form a $K_{1,3}$, or a_0, a_1, a_2, a_6 form a $K_{1,3}$, a contradiction. Hence Claim 1 holds.

Claim 2: $\min\{a_6 - a_0, a_7 - a_1\} > m$.

Assume that $a_6 - a_0 \le m$, then by Claim 1, we can obtain that

$$\begin{aligned} a_6 &= a_5 + 1 = a_4 + 2 = a_3 + 3 \\ &= a_2 + 4 = a_1 + 5 = a_0 + 6, \end{aligned}$$

so $a_0a_6, a_0a_5, a_1a_6 \in E(H)$, thus $a_0a_7, a_6a_7 \notin E(H)$. Therefore $a_7 - a_0 > m$, and $a_7 - a_6 = t \le 4$ or $a_7 - a_6 > m$. If $a_7 - a_6 = t \le 4$, then $a_2a_7, a_3a_7, a_4a_7 \in E(H)$ when $3 \le a_7 - a_6 \le 4$, and $a_0a_7, a_1a_7, a_2a_7 \in E(H)$ when $1 \le a_7 - a_6 \le 2$, a contradiction. Hence $a_7 - a_6 > m$, so $a_7 \ge a_6 + m + 1 \ge m + 7 > 7q$, a contradiction, too.

Therefore $a_6 - a_0 > m$. Similarly, $a_7 - a_1 > m$. Thus Claim 2 is proved.

By Claim 2, we have $m \le 7q - 2$.

Claim 3: If $a_ia_{i+j} \in E(H)$ for some $j \ge 3$, then

$$a_ia_{i+j-2}, a_{i+2}a_{i+j}, a_{i+1}a_{i+j-1} \notin E(H).$$

Otherwise, if $a_ia_{i+j-2} \in E(H)$, then

$$5 \le a_{i+j-2} - a_i < a_{i+j-1} - a_i < a_{i+j} - a_i \le m,$$

so $a_i, a_{i+j-2}, a_{i+j-1}, a_{i+j}$ form a $K_{1,3}$, a contradiction. Thus $a_ia_{i+j-2} \notin E(H)$. Similarly, $a_{i+2}a_{i+j} \notin E(H)$. If $a_{i+1}a_{i+j-1} \in E(H)$, then a_i, a_{i+j-1}, a_{i+1}, a_{i+j} form a 4-cycle, a contradiction, too.

Claim 4: There are at most eight vertices in H with the same color.

Otherwise, assume that there are nine vertices

$$(0 \le)\, a_0 < a_1 < \cdots < a_8\, (\le 7q)$$

with the same color α, then $a_{i+6}-a_i > m$ by Claim 2, so $a_i \in [i, i+3]$, and $a_{i+6} \in [m+i+1, m+i+4]$ where $i \in [0, 2]$.

(1)If $a_2 = 5$, then $a_8 = m+6, m = 7(q-1)+1$, and $a_2a_7 \in E(H)$. Moreover, by Claim 3, $a_2a_5, a_4a_7 \notin E(H)$, i. e., $3 \le a_5 - a_2 \le 4$, and $3 \le a_7 - a_4 \le 4$, so $8 \le a_5 \le 9$, and $a_4 \ge a_7 - 4 \ge m-2$, thus $m = 8$, $7 \le a_4 \le 8$ and $10 \le a_7 \le 12$. Hence a_3, a_4, a_5, a_8 form a $K_{1,3}$, a contradiction.

(2) If $a_2 = 4$, then we have $a_8 \ge m+5$, $m = 7(q-1)+j, 1 \le j \le 2$, and $a_2a_6 \in E(H)$. By Claim 3, $a_2a_4, a_4a_6 \notin E(H)$, so

$$m - 3 \le a_6 - a_2 \le 8,$$

and $6 \le a_4 \le 8$, thus $m \in [8, 9]$, $9 \le a_6 \le 12$, and $m+5 \le a_8 \le m+6$. If $a_2a_7 \in E(H)$, then $a_2a_5 \notin E(H)$, i. e., $a_5 - a_2 \le 4$, and $7 \le a_5 \le 8$, so $a_5a_8, a_4a_8 \in E(H)$, thus $a_3a_8 \notin E(H)$, hence $a_3 = 5$, $a_8 = m+6$, and $m = 8$. Moreover, $a_3a_7 \in E(H)$, and then $a_3a_6, a_4a_7 \notin E(H)$ (otherwise, a_2, a_6, a_3, a_7 form a cycle, or a_2, a_3, a_4, a_7 form a $K_{1,3}$), that is, $a_6 = 9, a_7 \le a_4 + 4 \le 11$, thus a_1, a_2, a_6, a_8 form a $K_{1,3}$, a contradiction. Therefore $a_2a_7 \notin E(H)$, i. e., $a_7 = m+5, a_8 = m+6$, then $m = 8$ since $j = 1$, and $a_4a_7, a_4a_8 \in E(H)$, so $a_3 = 5$ (otherwise, a_3, a_7, a_4, a_8 form a 4-cycle), and then $a_5 \ge 9$ (otherwise, a_4, a_7, a_5, a_8 form a 4-cycle), thus $a_2a_5, a_3a_6, a_3a_7 \in E(H)$, and $a_3a_5 \notin E(H)$,that is, $a_5 = 9$, but $a_2, a_5, a_8, a_4, a_7, a_3, a_6$ form a 7-cycle in this case, a contradiction, too.

(3)Assume that $a_2 = 3$. By Claim 2, it is easy to know that

$$m = 7(q-1) + j$$

with $1 \le j \le 3$.

Suppose that $a_2a_6 \in E(H)$, then, by Claim 3, $a_2a_4, a_4a_6, a_3a_5 \notin E(H)$, so $m-2 \le a_6 - a_2 \le 8$, $a_5 - a_3 \le 4$, and $5 \le a_4 \le 7$, thus $m \in [8, 10]$, $9 \le a_6 \le 11$, and $m+4 \le a_8 \le m+6$ by Claim 2. If $a_2a_7 \in E(H)$, then $a_2a_5 \notin E(H)$, i. e., $a_5 - a_2 \le 4$ and $6 \le a_5 \le 7$, so $a_0a_5, a_5a_8 \in E(H)$, thus $a_5 - a_1 = 4$, $a_5 = 6$, and $a_7 \le a_5 + 4 = 10$, hence $a_7 = a_6 + 1 = 10$, that is, $m = 8$, and a_2, a_3, a_4, a_7 form a $K_{1,3}$, a contradiction. Therefore, $a_2a_7 \notin E(H)$, i. e., $a_7 \ge m+4, a_8 \ge m+5$, thus $j \le 2$, and $a_4a_7 \in E(H)$, so $a_3a_7 \notin E(H)$ or $a_5a_7 \notin E(H)$, i. e., $a_3 = 4$ and $a_7 = m+5$, or $a_7 - a_5 \le 4$. In the former case, $a_3a_6 \in E(H)$, $6 \le a_5 \le 8$, so a_0, a_5, a_7, a_8 form a $K_{1,3}$, a contradiction. In the latter case, we may suppose that $a_3a_7 \in E(H)$, then $a_2a_5 \in E(H)$ (otherwise, $5 \le a_5 \le 7$, then $a_7 \le a_5 + 4 \le 11$, which contradicts $a_7 \ge m+4$), so $a_5a_8, a_1a_6, a_0a_5 \notin E(H)$ (otherwise, a_1, a_2, a_5, a_8 form a $K_{1,3}$, or a_1, a_5, a_2, a_6 form a cycle, or a_0, a_1, a_2, a_5 form a $K_{1,3}$), then $a_4a_8 \in E(H)$ (otherwise, $a_4 = 5$ and $a_8 = m+6$, so $a_3 = 4$, and $a_5 \le 8$ which contradicts $a_5a_8 \notin E(H)$), thus $a_0a_4 \notin E(H)$ (otherwise, a_0, a_4, a_7, a_8 form a $K_{1,3}$), that is, $a_4 = a_0 + 4$, hence $a_4a_6 \in E(H)$, and a_4, a_6, a_7, a_8 form a $K_{1,3}$, a contradiction, too.

Suppose that $a_2a_6 \notin E(H)$, then $a_6 = m + 4, a_7 = m+5$, $a_8 = m+6$, and $m = 7(q-1)+1$, so $a_2a_5 \in E(H)$ (otherwise, a_5, a_6, a_7, a_8 form a $K_{1,3}$), and $a_2a_3, a_3a_4, a_4a_5 \notin E(H)$ by Claim 3, thus $a_3 \le 7$, and $a_5a_7 \notin E(H)$ (otherwise, a_2, a_5, a_7, a_8 form a $K_{1,3}$), i. e., $a_5 \ge m+1$. If $a_2a_4 \notin E(H)$, then $a_4 = 5$ (otherwise, $6 \le a_4 \le 7$ and a_4, a_6, a_7, a_8 form a $K_{1,3}$), thus, $a_5 = 9$ and $m = 8$, hence a_1, a_2, a_5, a_8 form a $K_{1,3}$, a contradiction. Therefore, $a_2a_4 \in E(H)$, so $a_1a_5, a_0a_4 \notin E(H)$ (otherwise, a_1, a_4, a_2, a_5 form a 4-cycle, or a_0, a_1, a_2, a_4 form a $K_{1,3}$), thus $a_4 \ge m+1$, $a_5 - a_1 \ge m+1$, and $a_3a_8 \notin E(H)$ (otherwise, a_3, a_6, a_7, a_8 form a $K_{1,3}$), i. e., $4 \le a_3 \le 5$, and $m = 8$. Moreover, $a_4 = 9$ and $a_3 = 5$, then a_3, a_5, a_6, a_7 form a $K_{1,3}$, a contradiction, too.

(4) Suppose that $a_2 = 2$. Then $a_1 = 1$, and $a_0 = 0$.

If $a_2a_6 \in E(H)$, then $m+1 \le a_6 \le m+2$, $a_5 - a_3 \le 4, m-3 \le a_6 - 4 \le a_4 \le 6$, $a_8 \ge m+3$ by Claims 2-3, thus $a_0a_4 \in E(H)$, and $m \in [8, 9]$. Moreover, $a_1a_4 \notin E(H)$ or $a_4a_7 \notin E(H)$, i. e., $a_4 = 5$, or $a_4 = 6$ and $a_7 = m+2 = 10$. In the former case, it is obvious that a_0, a_1, a_5, a_8 induce a $K_{1,3}$ if $a_5 = 6$, a_0, a_1, a_2, a_5 induce a $K_{1,3}$ if $6 < a_5 \le m$, and a_1, a_2, a_3, a_5 induce a $K_{1,3}$ if $a_5 = m+1$, a contradiction. In the latter case, a_0, a_1, a_4, a_8 induce a $K_{1,3}$, a contradiction.

Suppose that $a_2a_6 \notin E(H)$, i. e., $a_6 \ge m+3$. If $a_2a_5 \in E(H)$, then $a_5 \ge m+1$ (otherwise, a_0, a_1, a_2, a_5 induce a $K_{1,3}$), so $a_3 \le 6$ and $a_4 \ge a_5 - 4$ by Claim 3, thus $a_3 \le 4$ (otherwise, a_0, a_3, a_6, a_7 induce a $K_{1,3}$ if $a_3 = 5$, and a_0, a_1, a_3, a_8 induce a $K_{1,3}$ if $a_3 = 6$), hence $a_3a_5 \in E(H)$, $a_5 \ge m+2$ and $a_4 \ge a_5 - 4 \ge m-2$ by $a_1a_5, a_4a_5 \notin E(H)$, therefore a_0, a_1, a_4, a_8 induce a $K_{1,3}$ if $a_4 = 6$, a_0, a_1, a_2, a_4 induce a $K_{1,3}$ if $6 < a_4 \le m$, and a_2, a_4, a_3, a_5 induce a cycle if $a_4 > m$, a contradiction. If $a_2a_5 \notin E(H)$, then $a_5 \le 6$ or $a_5 \ge m+3$. In the former case, $a_0a_5 \in E(H)$, and $a_5a_7 \notin E(H)$

or $a_5a_8 \notin E(H)$, so $a_5 = 6, a_7 = 10$ and $m = 8$, or $a_5 = 5$ and $a_8 = m + 6$, hence a_0, a_1, a_5, a_8 induce a $K_{1,3}$, or a_0, a_5, a_6, a_7 induce a $K_{1,3}$, a contradiction. In the latter case, $a_5 = m + 3 = a_6 - 1 = a_7 - 2 = a_8 - 3$, then $a_4 = 4$ or $a_4 = m + 2$ (otherwise, a_4, a_5, a_6, a_7 induce a $K_{1,3}$ when $a_4 = 5$, a_0, a_1, a_4, a_8 induce a $K_{1,3}$ when $6 \leq a_4 \leq m$, and a_1, a_2, a_4, a_8 induce a $K_{1,3}$ when $a_4 = m + 1$). If $a_4 = 4$, then every color colored seven consecutive vertices except α and some color β that colored

$$b_5 > b_4 > b_3 > b_2 > b_1 > b_0 (\geq 5)$$

in H, so vertices $m + 8, m + 9, m + 10$ receive color β and are all adjacent to b_5, a contradiction. If $a_4 = m + 2$, then $a_3 = 3$ (otherwise, a_3, a_4, a_5, a_6 induce a $K_{1,3}$ when $4 \leq a_3 \leq m - 3$, a_0, a_1, a_3, a_8 induce a $K_{1,3}$ when $m-2 \leq a_3 \leq m$, and a_1, a_2, a_3, a_8 induce a $K_{1,3}$ when $a_3 = m + 1$), thus every color colored seven consecutive vertices except α and some color β that colored

$$(m + 1 \geq) b_5 > b_4 > b_3 > b_2 > b_1 > b_0 (\geq 4)$$

in H, so vertices $-4, -3, -2$ receive color β and are all adjacent to b_0, a contradiction, too.

Hence Claim 4 holds.

Claim 5: In H, if

$$b - a = 7(q - 1) - 2, \quad t \geq m + 1,$$
$$a + t - b > 1 \qquad (i.e., b - t < a - 1)$$

and $7(q - 1)$ vertices in $[a, b] \cup \{a + t\}$ (or $\{b - t\} \cup [a, b]$) colored $q - 1$ colors, then a and $a + t$ (or a and $b - t$) have the same color.

Assume that a and $a + t$ have different colors, then, by Claim 1, each color colored consecutive seven vertices, which is impossible. Similarly, a and $b-t$ have the same color.

Claim 6: $m = 7(q - 1) + 1$.

Suppose that

$$m \geq 7(q - 1) + 2 \geq 9.$$

By Claim 2, $a_6 - a_0 > m$ and $a_7 - a_1 > m$. Then there is $1 \leq h \leq 5$ such that $a_h - a_0 \leq m$, and $a_{h+1} - a_0 > m$.

Case 1. $h = 1$.

Then $a_1 - a_0 \leq m$, and $a_2 - a_0 > m$, so

$$a_7 - a_2 \leq 7q - (m + 1) \leq (m + 5) - (m + 1) \leq 4,$$

which contradicts $a_7 - a_2 \geq 5$.

Case 2. $h = 2$.

We have $a_2 - a_0 \leq m, a_3 - a_0 > m$, and

$$a_7 - a_3 \leq 7q - (m + 1) \leq 4$$

in this case, so $a_7 - a_3 = 4$, i. e., $a_3 = m + 1$, $a_7 = a_6 + 1 = a_5 + 2 = a_4 + 3 = a_3 + 4 = 7q$, $m = 7(q - 1) + 2$, and $a_0 = 0$. Since $a_7 - a_1 > m$,

$$a_1 \leq a_7 - (m + 1) \leq (m + 5) - (m + 1) \leq 4,$$

that is, $1 \leq a_1 \leq 4$, then $a_1a_3 \in E(H)$.

If $a_1 = 1$, then the remainder $7(q - 1)$ vertices $[2, m] \setminus \{a_2\}$ in H colored $q - 1$ colors by Claim 1, such that each color colored seven vertices as

$$u(\geq 2), u + 1, u + 2, u + 3, u + 4, u + 5, u + 6,$$

by Claim 2, so $m + 6$ and $m + 7$ would color α, but they form a $K_{1,3}$ with a_3, a_1, a contradiction.

If $2 \leq a_1 \leq 4$, then the remainder $7(q - 1)$ vertices $[1, m] \setminus \{a_1, a_2\}$ in H colored $q - 1$ colors, by Claim 1, each color would color consecutive seven vertices, which is impossible.

Case 3. $h = 3$.

We have $a_3 - a_0 \leq m, a_4 - a_0 > m$, so

$$m + 1 \leq a_4 \leq m + 2,$$

and $0 \leq a_0 \leq 1$. By $a_7 - a_1 > m$, we can obtain that $1 \leq a_1 \leq 4$.

(1) If $a_4 = m + 2$, then

$$a_7 = a_6 + 1 = a_5 + 2 = a_4 + 3 = 7q,$$

and $m = 7(q - 1) + 2$.

(1.1) Assume that $2 \leq a_1 \leq 4$, then $a_1a_4 \in E(H)$, so $a_1a_2, a_3a_4 \notin E(H)$ by Claim 3, i. e., $a_2 - a_1 \leq 4$, and $a_4 - a_3 \leq 4$, thus m-2 $\leq a_3 \leq m + 1$. If

$$m - 2 \leq a_3 \leq m,$$

then $a_0a_3, a_3a_7 \in E(H)$, so $a_3a_6 \notin E(H)$, i. e., $a_6 - a_3 \leq 4$, thus $a_3 \geq a_6 - 4 \geq m$, moreover $a_3 = m$, and $a_1a_3 \in E(H)$, hence a_0, a_1, a_3, a_7 form a $K_{1,3}$, a contradiction. Therefore,$a_3 = m + 1$, and $a_0 = 1$, then $a_0a_3, a_1a_3 \in E(H)$, so $a_1a_5 \notin E(H)$, i. e., $a_5 - a_1 > m$, and $a_1 \leq a_5 - (m + 1) \leq 2$, thus $a_1 = 2$. Hence the remainder $7(q - 1)$ vertices

$$\{0\} \cup [3, m] \setminus \{a_2\}$$

in H colored $q - 1$ colors, by Claim 1, each color colored seven consecutive vertices, which is impossible.

(1.2) Assume that $a_1 = 1$, then $a_0 = 0$, and $3 \leq a_3 \leq m$. If $6 \leq a_3 \leq m$, then $a_0a_3, a_1a_3, a_3a_7 \in E(H)$, a contradiction. If $4 \leq a_3 \leq 5$, then $a_3a_4, a_3a_5, a_3a_6 \in E(H)$, a contradiction. Hence

$a_3 = 3$, and $a_2 = 2$, so the remainder $7(q-1)$ vertices $[4, m+1]$ in H colored $q-1$ colors such that each color colored seven vertices as

$$u(\geq 4), u+1, u+2, u+3, u+4, u+5, u+6$$

by Claim 1, then by Claim 2, $m+9$ would color α and form a $K_{1,3}$ with a_4, a_5, a_6, a contradiction.

(2) Suppose that $a_4 = m+1$. Then $a_0 = 0$, and $a_1a_4 \in E(H)$, so a_1a_2, $a_3a_4 \notin E(H)$ by Claim 3, i. e., $a_2 - a_1 \leq 4$, and $a_4 - a_3 \leq 4$, thus $m - 3 \leq a_3 \leq m$. If $m-3 \leq a_3 \leq m-1$, then a_0a_3, $a_3a_7 \in E(H)$, so a_1a_3, $a_3a_6 \notin E(H)$, i. e., $a_3 - a_1 \leq 4$, and $a_6 - a_3 \leq 4$, thus $a_3 \geq a_6 - 4 \geq m-1$, and $a_3 = m-1$, $a_1 \geq a_3 - 4 \geq m-5 \geq 4$, hence $a_1 = 4$, and a_1, a_4, a_5, a_6 induce a $K_{1,3}$, a contradiction. Therefore, $a_3 = m$, and $a_0a_3, a_1a_3 \in E(H)$, so a_2a_3,$a_3a_7 \notin E(H)$, that is, $a_7 - a_3 = 4$, and then $a_7 = m+4$, $a_6 = m+3, a_5 = m+2$, and $m-4 \leq a_2 \leq m-1$. Moreover, a_0a_2, $a_2a_7 \in E(H)$, so a_1a_2, $a_2a_6 \notin E(H)$, i. e., $a_6 - a_2 = 4$, and $a_2 - a_1 \leq 4$, hence $a_2 = m-1$, and $a_1 = 4$, thus $a_7 - a_1 = m$ which contradicts Claim 2.

Case 4. $h = 4$.

We have $a_4 - a_0 \leq m$, and $a_5 - a_0 > m$, so $m+1 \leq a_5 \leq m+3$, and $0 \leq a_0 \leq 2$.

Assume that $a_4 - a_0 = 4$. If $a_0 = 2$, then $a_1 = 3$, $a_2 = 4$, $a_3 = 5$ and $a_4 = 6$, so $a_1a_5, a_2a_5, a_3a_5 \in E(H)$, a contradiction. If $a_0 = 1$, then $a_1 = 2$, $a_2 = 3$, $a_3 = 4$ and $a_4 = 5$, so $a_2a_5, a_3a_5, a_4a_5 \in E(H)$, a contradiction. If $a_0 = 0$, then $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$,thus $a_2a_5, a_3a_5, a_4a_5 \in E(H)$ when $m+1 \leq a_5 \leq m+2$, so $a_5 = m+3$, $a_6 = m+4$, and $a_7 = m+5$. The remainder at least $7(q-1)$ vertices $[5, m+2]$ in H are colored $q-1$ colors such that each color colored seven vertices as

$$u(\geq 5), u+1, u+2, u+3, u+4, u+5, u+6,$$

by Claim 1, hence $m+10$ would color α by Claim 2 and be adjacent to a_5, a_6, a_7, a contradiction.

Therefore, $5 \leq a_4 - a_0 \leq m$, so $a_0a_4 \in E(H)$, then, by Claim 3, $a_0a_2, a_2a_4 \notin E(H)$, i. e., $a_2 - a_0 \leq 4$, and $a_4 - a_2 \leq 4$, thus $5 \leq a_4 - a_0 \leq 8$.

(1) If $a_4 - a_0 = 5$, then $5 \leq a_4 \leq 7$, so $a_4a_7 \in E(H)$, thus $a_4a_6 \notin E(H)$, i. e., $a_6 - a_4 \leq 4$, and

$$11 \leq m+2 \leq a_6 \leq 11.$$

Hence $m = 9$, $a_6 = 11$, $a_5 = 10$, $a_4 = 7$, and $a_0 = 2$, so $a_0a_6 \in E(H)$ which contradicts Claim 2.

(2) If $a_4 - a_0 = 6$, then $a_6 - a_0 \geq m+2$ by $a_5 - a_0 > m$, so

$$\begin{aligned} 5 \leq m-4 &\leq (a_6 - a_0) - (a_4 - a_0) \\ &= a_6 - a_4 < a_7 - a_4 \leq m, \end{aligned}$$

Thus a_0, a_4, a_6, a_7 induce a $K_{1,3}$, a contradiction.

(3) If $a_4 - a_0 = 7$, then $3 \leq a_4 - a_2 \leq 4$, $a_7 - a_0 \geq m+3$ by $a_5 - a_0 > m$, thus

$$\begin{aligned} 5 \leq m-4 &\leq (a_7 - a_0) - (a_4 - a_0) \\ &= a_7 - a_4 \leq m. \end{aligned}$$

Hence $a_4a_7 \in E(H)$, so $a_1a_4, a_4a_6 \notin E(H)$, that is, $a_6 - a_4 \leq 4$, and $3 \leq a_4 - a_1 \leq 4$, then

$$\begin{aligned} 11 \leq m+2 &\leq a_6 - a_0 \\ &= (a_6 - a_4) + (a_4 - a_0) \leq 11. \end{aligned}$$

Therefore,$a_6 - a_0 = 11$, $m = 9$, $a_5 - a_0 = 10$, $a_4 - a_1 = 4$, and $a_4 - a_2 = 3$, so $a_1a_5, a_1a_6, a_2a_5, a_2a_6 \in E(H)$, i. e., a_1, a_2, a_5,a_6 form a 4-cycle, a contradiction.

(4) If $a_4 - a_0 = 8$, then $a_4 - a_2 = a_2 - a_0 = 4$, so $a_1a_4 \in E(H)$, thus $a_4a_7 \notin E(H)$, i. e., $a_7 - a_4 \leq 4$, hence $a_7 - a_2 \leq 8$, and $a_2a_5, a_2a_6, a_2a_7 \in E(H)$, a contradiction, too.

Case 5. $h = 5$.

We have $a_5 - a_0 \leq m$, and $a_6 - a_0 > m$, then $a_0a_5 \in E(H)$, so $a_0a_3, a_1a_4, a_2a_5 \notin E(H)$ by Claim 3, i. e., $3 \leq a_3 - a_0 \leq 4$, $a_4 - a_1 \leq 4$, and $3 \leq a_5 - a_2 \leq 4$, hence $2 \leq a_2 - a_0 \leq 3$, moreover, $5 \leq a_5 - a_0 \leq 7$, $4 \leq a_4 - a_0 \leq 6$, $1 \leq a_1 - a_0 \leq 2$, and

$$m+1 \leq a_6 \leq m+4.$$

(1) If $a_6 = m+1$, then $a_0 = 0$. Hence $1 \leq a_1 \leq 2$, $2 \leq a_2 \leq 3$, and $3 \leq a_3 \leq 4$, so a_1a_6,$a_2a_6, a_3a_6 \in E(H)$, a contradiction.

(2) Assume that $a_6 = m+2$, then $0 \leq a_0 \leq 1$. If $a_0 = 0$, then $3 \leq a_3 \leq 4$, $4 \leq a_4 \leq 6$, and $2 \leq a_2 \leq 3$, so $a_2a_6, a_3a_6, a_4a_6 \in E(H)$, a contradiction. If $a_0 = 1$, then $2 \leq a_1 \leq 3$, $3 \leq a_2 \leq 4$, and $4 \leq a_3 \leq 5$, so a_1a_6, a_2a_6, $a_3a_6 \in E(H)$, a contradiction.

(3) Assume that $a_6 = m+3$, then $0 \leq a_0 \leq 2$. If $a_0 = 0$, then $5 \leq a_5 \leq 7$, so a_5a_6, $a_5a_7 \in E(H)$, thus a_0, a_5, a_6, a_7 induce a $K_{1,3}$, a contradiction. If $a_0 = 1$, then $4 \leq a_3 \leq 5$, $3 \leq a_2 \leq 4$, and $5 \leq a_4 \leq 7$, then $a_2a_6, a_3a_6, a_4a_6 \in E(H)$; if $a_0 = 2$, then $3 \leq a_1 \leq 4$, $4 \leq a_2 \leq 5$, and $5 \leq a_3 \leq 6$, so $a_1a_6, a_2a_6, a_3a_6 \in E(H)$, a contradiction, too.

(4) Assume that $a_6 = m+4$, then $a_7 = m+5$, and $0 \leq a_0 \leq 3$. If $0 \leq a_0 \leq 1$, then $5 \leq a_5 \leq 8$, and a_5a_6, $a_5a_7 \in E(H)$, so a_0, a_5, a_6, a_7 form a $K_{1,3}$, a contradiction. If $a_0 = 2$, then $4 \leq a_2 \leq 5$, $5 \leq a_3 \leq 6$, and $6 \leq a_4 \leq 8$, so $a_2a_6, a_3a_6, a_4a_6 \in E(H)$, a contradiction. Hence $a_0 = 3$, $5 \leq a_2 < a_3 \leq 7$, and $7 \leq a_4 \leq 9$, then $a_2a_7, a_3a_7, a_4a_7 \in E(H)$, a contradiction, too.

Therefore, we have

$$m = 7(q-1) + 1,$$

and thus Claim 6 holds.

Claim 7: $a_4 \le 4$, that is, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_4 = 4$.

Subclaim 7.1 $a_4 - a_0 = 4$.

Otherwise, $a_4 - a_0 > 4$. We shall get a contradiction according to the related positions of a_4 and a_0.

Case 1. $a_4 - a_0 > m$.

There is $1 \le h < 4$, such that $a_h - a_0 \le m$, and $a_{h+1} - a_0 > m$. By Claim 2, it is easy to see that $2 \le h \le 3$.

(1) Suppose that $h = 2$, then $a_2 - a_0 \le m$, and $a_3 - a_0 > m$, so $m + 1 \le a_3 \le m + 2$. By $a_7 - a_1 > m$, we have $1 \le a_1 \le 5$.

(1.1) Assume that $a_3 = m + 1$, then $a_0 = 0$. If $1 \le a_1 \le 4$, then $a_1 a_3 \in E(H)$, and thus $a_2 a_3 \notin E(H)$ or $a_3 a_7 \notin E(H)$. If $a_2 a_3 \notin E(H)$, then $a_3 - a_2 \le 4$, i. e., $m - 3 \le a_2 \le m$, so $a_0 a_2$, $a_2 a_6$, $a_2 a_7 \in E(H)$ when $m - 2 \le a_2 \le m - 1$, and $a_0 a_2, a_2 a_4, a_2 a_5 \in E(H)$ when $a_2 = m - 3$, a contradiction. Hence $a_2 = m$, so $a_0 a_2$, $a_2 a_7 \in E(H)$, thus $a_1 a_2, a_2 a_6 \notin E(H)$, i. e., $a_6 - a_2 = 4$ and $4 \le m - 4 \le a_2 - 4 \le a_1 \le 4$. Moreover, $m = 8$, $a_1 = 4$, $a_6 = m + 4$, $a_5 = m + 3$, $a_4 = m+2$, and $a_1 a_4, a_1 a_5, a_1 a_6 \in E(H)$, a contradiction. Therefore, $a_2 a_3 \in E(H)$, then $a_3 a_7 \notin E(H)$, i. e., $a_7 - a_3 = 4$, and $a_3 - a_2 \ge 5$, so $2 \le a_2 \le m - 4$, $a_7 = m + 5, a_6 = m + 4, a_5 = m + 3$, and $a_4 = m + 2$. If $3 \le a_2 \le m - 4$, then a_2, a_3, a_4, a_5 induce a $K_{1,3}$, a contradiction. Hence $a_2 = 2$, $a_1 = 1$, and $a_2 a_4 \in E(H)$, so the remainder $7(q - 1)$ vertices $[3, m] \cup \{m + 6\}$ in H colored $q - 1$ colors. By Claim 5, there is some color β colored seven vertices

$$3 = h_0 < h_1 < h_2 < h_3 < h_4 < h_5 < h_6 = m + 6,$$

but h_3, h_4, h_5 are all adjacent to $m+6$ since $6 \le h_3 < h_4 < h_5 \le m$, a contradiction.

Therefore, $a_1 = 5$, and $a_7 = m + 6$, then $a_0 a_1, a_1 a_5, a_1 a_6 \in E(H)$, a contradiction, too.

(1.2) Assume that $a_3 = m + 2$, then $a_4 = m + 3$, $a_5 = m + 4$, $a_6 = m + 5$, and $a_7 = m + 6$.

(1.2.1)If $4 \le a_1 \le 5$, then $a_1 a_3, a_1 a_4, a_1 a_5 \in E(H)$, a contradiction.

(1.2.2) If $a_1 = 3$, then $a_1 a_3, a_1 a_4 \in E(H)$, then $a_1 a_2 \notin E(H)$, i. e., $4 \le a_2 \le 7$. We have $a_2 a_3, a_2 a_4, a_2 a_5 \in E(H)$ when $4 \le a_2 \le 5$, and $a_2 a_5, a_2 a_6, a_2 a_7 \in E(H)$ when $6 \le a_2 \le 7$, a contradiction.

(1.2.3) If $a_1 = 2$, then $a_1 a_3 \in E(H)$. For $a_0 = 0$, the remainder $7(q - 1)$ vertices $\{1\} \cup [3, m + 1] \setminus \{a_2\}$ in H colored $q - 1$ colors such that each color colored seven consecutive vertices by Claim 1, which is impossible. For $a_0 = 1$, we have $3 \le a_2 \le m + 1$, and $a_2 a_3, a_2 a_4, a_2 a_5 \in E(H)$ when $4 \le a_2 \le m - 3$, $a_0 a_2, a_2 a_6, a_2 a_7 \in E(H)$ when $m - 2 \le a_2 \le m$, and $a_0 a_2, a_1 a_2, a_2 a_7 \in E(H)$ when $a_2 = m + 1$, hence $a_2 = 3$, then $a_1 a_3, a_2 a_3 \in E(H)$, and the remainder $7(q - 1)$ vertices $\{0\} \cup [4, m + 1]$ in H colored $q - 1$ colors, such that each color colored seven consecutive vertices except β colored seven vertices

$$0 < h_1 < h_2 < h_3 < h_4 < h_5 < m + 1$$

by Claim 5, but h_2, h_3, h_4 are all adjacent to 0 since

$$5 \le h_2 < h_3 < h_4 \le m - 1,$$

a contradiction.

(1.2.4) If $a_1 = 1$, then $a_0 = 0$, thus the remainder $7(q - 1)$ vertices $[2, m + 1] \setminus \{a_2\}$ in H colored $q - 1$ colors such that each color colored seven vertices as

$$u\,(\ge 2)\,, u + 1, u + 2, u + 3, u + 4, u + 5, u + 6,$$

by Claim 1, so $m + 7$ and $m + 8$ would color α and be adjacent to a_3, so $a_2 a_3 \notin E(H)$, i. e., $m - 2 \le a_2 \le m + 1$. If $m - 2 \le a_2 \le m$, then $a_0 a_2, a_1 a_2, a_2 a_6 \in E(H)$. Hence $a_2 = m + 1$, so $m + 8$ is adjacent to a_2, a_3, a_4, a contradiction, too.

(2) Suppose that $h = 3$, then $a_3 - a_0 \le m$, and $a_4 - a_0 > m$, so $m + 1 \le a_4 \le m + 3$.

If $a_4 a_7 \in E(H)$, then $a_7 - a_4 \ge 5$, so $a_7 = m + 6$, $a_4 = m + 1$, and $a_0 = 0$. By $a_7 - a_1 > m$, we have $1 \le a_1 \le 5$. If $a_1 = 5$, then $a_0 a_1, a_1 a_5, a_1 a_6 \in E(H)$, a contradiction. Hence $1 \le a_1 \le 4$, so $a_1 a_4 \in E(H)$, and then $a_2 a_4 \notin E(H)$, i. e., $a_4 - a_2 \le 4$, thus $m - 3 \le a_2 \le m - 1$. For $m - 2 \le a_2 \le m - 1$, $a_0 a_2, a_0 a_3, a_2 a_7, a_3 a_7 \in E(H)$, i. e., a_0, a_2, a_7, a_3 form a 4-cycle, a contradiction. For $a_2 = m - 3$, $a_0 a_2, a_2 a_5, a_2 a_6 \in E(H)$, a contradiction. Therefore, $a_4 a_7 \notin E(H)$, that is, $a_7 - a_4 \le 4$.

(2.1) Assume that $a_4 = m + 1$, then $a_0 = 0$, $m + 4 \le a_7 \le m + 5$, and $1 \le a_1 \le 4$, so $a_1 a_4 \in E(H)$, and thus $a_3 a_4 \notin E(H)$ by Claim 3, i. e., $a_4 - a_3 \le 4$, hence $m - 3 \le a_3 \le m$.

(2.1.1) If $m - 3 \le a_3 \le m - 2$, then $a_0 a_3, a_3 a_6, a_3 a_7 \in E(H)$, a contradiction.

(2.1.2) If $a_3 = m - 1$, then $a_0 a_3, a_3 a_7 \in E(H)$, so $a_1 a_3, a_3 a_6 \notin E(H)$, i. e., $a_3 - a_1 \le 4$, and $a_6 - a_3 \le 4$, hence $a_6 = m + 3$, $a_5 = m + 2$, $3 \le m - 5 \le a_1 \le 4$, and a_1, a_4, a_5, a_6 induce a $K_{1,3}$, a contradiction.

(2.1.3) If $a_3 = m$, then $a_0 a_3 \in E(H)$, so $a_2 a_3 \notin E(H)$, i. e., $a_3 - a_2 \le 4$. If $a_3 a_7 \in E(H)$, then $a_7 = m + 5$, and $a_1 a_3 \notin E(H)$, i. e., $a_1 \ge m - 4$, so $a_2 \ge m - 3$, and a_0, a_2, a_7, a_3 induce a 4-cycle, a contradiction. Therefore, $a_3 a_7 \notin E(H)$, $a_7 = m+4$, $a_6 = m + 3$, $a_5 = m + 2$, and $1 \le a_1 \le 3$, thus, $a_1 a_3, a_1 a_4, a_1 a_5 \in E(H)$ when $2 \le a_1 \le 3$, then

$a_1 = 1$, so $a_1a_3, a_1a_4 \in E(H)$, and $a_1a_2 \notin E(H)$, that is, $4 \le m-4 \le a_2 \le a_1+4 \le 5$, hence $a_2a_5, a_2a_6, a_2a_7 \in E(H)$, a contradiction.

(2.2) Suppose that $a_4 = m+2$, then $m+5 \le a_7 \le m+6$, $0 \le a_0 \le 1$, and $1 \le a_1 \le 5$.

(2.2.1) Assume that $2 \le a_1 \le 5$, then $a_1a_4 \in E(H)$, so $a_3a_4 \notin E(H)$, i. e., $a_4 - a_3 \le 4$, and $m-2 \le a_3 \le m+1$.

(2.2.1.1) If $m-2 \le a_3 \le m-1$, then a_0a_3, a_3a_6, $a_3a_7 \in E(H)$, a contradiction.

(2.2.1.2) If $a_3 = m$, then $a_0a_3, a_3a_7 \in E(H)$, so $a_1a_3, a_3a_6 \notin E(H)$, i. e., $a_6-a_3 \le 4$, and $a_3-a_1 \le 4$, thus $a_6 = m+4$, $a_5 = m+3$, $4 \le m-4 \le a_1 \le 5$, and a_1, a_4, a_5, a_6 induce a $K_{1,3}$, a contradiction.

(2.2.1.3) If $a_3 = m+1$, then $a_0 = 1$, and $a_0a_3 \in E(H)$, so $a_2a_3 \notin E(H)$, i. e., $m-3 \le a_2 \le m$, thus, $a_0a_2, a_2a_6, a_2a_7 \in E(H)$ when $m-2 \le a_2 \le m-1$, and $a_2a_4, a_2a_5, a_2a_6 \in E(H)$ when $a_2 = m-3$, a contradiction. Hence $a_2 = m$, so $a_0a_2, a_2a_7 \in E(H)$, then $a_1a_2, a_2a_6 \notin E(H)$, i. e., $a_6 - a_2 = 4$, and $a_2 - a_1 \le 4$, thus $a_6 = m+4$, $a_5 = m+3$, $4 \le m-4 \le a_1 \le 5$, and a_1, a_4, a_5, a_6 induce a $K_{1,3}$, a contradiction, too.

(2.2.2) Assume that $a_1 = 1$, then $a_0 = 0$, and $3 \le a_3 \le m$.

(2.2.2.1) If $6 \le a_3 \le m$, then $a_0a_3, a_1a_3, a_3a_7 \in E(H)$, a contradiction.

(2.2.2.2) If $a_3 = 5$, then a_0a_3, a_3a_4, $a_3a_5 \in E(H)$, a contradiction.

(2.2.2.3) If $a_3 = 4$, then $a_3a_4, a_3a_5 \in E(H)$, so $a_3a_6 \notin E(H)$, thus $a_7 = m+6$, $a_6 = m+5$, and $2 \le a_2 \le 3$, hence $a_2 = 3$, $a_5 = m+3$, and a_2, a_3, a_4, a_5 form a 4-cycle when $a_2a_5 \in E(H)$, a contradiction. Therefore, $a_2a_5 \notin E(H)$. If $a_2 = 3$, then $a_5 = m+4$, and the remainder $7(q-1)$ vertices $[5, m+1] \cup \{2, m+3\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices, except some color β colored vertices

$$2 < h_1 < h_2 < h_3 < h_4 < h_5 < m+3$$

by Claim 5, so β would color $m+8$ which is adjacent to $m+3$, h_4, and h_5, a contradiction. Hence $a_2 = 2$, and $m+3 \le a_5 \le m+4$. If $a_5 = m+4$, then the remainder $7(q-1)$ vertices $[5, m+1] \cup \{3, m+3\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices by Claim 1, which is impossible. Hence $a_5 = m+3$, then the remainder $7(q-1)$ vertices $[5, m+1] \cup \{3, m+4\}$ in H colored $q-1$ colors such that each color colored seven consecutive vertices, except some color β colored vertices

$$3 < h_1 < h_2 < h_3 < h_4 < h_5 < m+4,$$

but h_1, h_2, h_3 are all adjacent to $m+4$ since $5 \le h_1 < h_2 < h_3 \le m-1$, a contradiction.

(2.2.2.4) Assume that $a_3 = 3$, then $a_2 = 2$, so $a_2a_4, a_3a_4 \in E(H)$.If $a_7 = m+5$, then $a_6 = m+4$, and $a_5 = m+3$, so the remainder $7(q-1)$ vertices $[4, m+1] \cup \{m+6\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices except some color β colored vertices

$$4 < h_1 < h_2 < h_3 < h_4 < h_5 < m+6$$

by Claim 5, but h_3, h_4, h_5 are all adjacent to $m+6$ since

$$7 \le h_3 < h_4 < h_5 \le m+1,$$

a contradiction. Hence $a_7 = m+6$. If $a_6 = m+5$, then the remainder $7(q-1)$ vertices $[4, m+1] \cup \{m+3\}$ or $[4, m+1] \cup \{m+4\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices by Claim 1, which is impossible. Therefore, $a_6 = m+4$, and $a_5 = m+3$, then the remainder $7(q-1)$ vertices $[4, m+1] \cup \{m+5\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices except some color β colored vertices

$$4 < h_1 < h_2 < h_3 < h_4 < h_5 < m+5$$

by Claim 5, but h_1, h_2, h_3 are all adjacent to $m+5$ since

$$5 \le h_1 < h_2 < h_3 \le m-1,$$

a contradiction, too.

(2.3) Assume that $a_4 = m+3$, then $a_5 = m+4$, $a_6 = m+5$, $a_7 = m+6$, and $0 \le a_0 \le 2$.

(2.3.1) If $6 \le a_3 \le m-1$, then a_3a_5, a_3a_6, $a_3a_7 \in E(H)$, a contradiction.

(2.3.2) If $a_3 = m$, then a_0a_3, a_3a_6, $a_3a_7 \in E(H)$, a contradiction.

(2.3.3) If $a_3 = m+1$, then $1 \le a_0 \le 2$, so $a_0a_3, a_3a_7 \in E(H)$, thus $a_1a_3, a_3a_6 \notin E(H)$, so $a_3 - a_1 \le 4$ and $a_7 - a_3 = 5$, hence $9 \le m+1 \le a_7 - a_1 \le 9$, thus $a_7 - a_1 = 9$, i. e., $a_1 = 5$, and $a_1a_4, a_1a_5, a_1a_6 \in E(H)$, a contradiction.

(2.3.4) Assume that $a_3 = m+2$, then $a_0 = 2$, and $3 \le a_1 \le 5$, so $a_0a_3, a_1a_3 \in E(H)$, hence $a_2a_3 \notin E(H)$, i. e., $m-2 \le a_2 \le m+1$. If $a_2 = m-2$, then $a_2a_4, a_2a_5, a_2a_6 \in E(H)$, a contradiction. If $m-1 \le a_2 \le m+1$, then a_0a_2, $a_2a_7 \in E(H)$, so $a_2a_6 \notin E(H)$, i. e., $a_6 - a_2 = 4$, and $a_2 = m+1$, thus, $a_1a_2 \in E(H)$ and a_0, a_2, a_1, a_3 induce a 4-cycle when $3 \le a_1 \le 4$, and a_1, a_3, a_4, a_5 induce a $K_{1,3}$ when $a_1 = 5$, a contradiction.

(2.3.5) If $a_3 = 5$, then $a_3a_4, a_3a_5, a_3a_6 \in E(H)$, a contradiction.

(2.3.6) If $a_3 = 4$ and $a_0 = 1$, then $a_1 = 2$, $a_2 = 3$, and $a_3a_4, a_3a_5, a_2a_4 \in E(H)$, so the remainder $7(q-1)$ vertices $\{0\} \cup [5, m+2]$ in H colored $q-1$

colors, such that each color colored seven consecutive vertices, except some color β colored vertices

$$0 < h_1 < h_2 < h_3 < h_4 < h_5 < m+2$$

by Claim 5, but h_1, h_2, h_3 are all adjacent to 0 since $5 \le h_1 < h_2 < h_3 \le m-1$, a contradiction. If $a_3 = 4$ and $a_0 = 0$, then a_2 is 2 or 3 when $a_1 = 1$, and in this case the remainder $7(q-1)$ vertices $\{2\} \cup [5, m+2]$ or $\{3\} \cup [5, m+2]$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices which is impossible, hence $a_1 = 2$, $a_2 = 3$, $a_2a_4, a_3a_4 \in E(H)$, and in this case the remainder $7(q-1)$ vertices $\{1\} \cup [5, m+2]$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices, except some color β colored vertices

$$1 < h_1 < h_2 < h_3 < h_4 < h_5 < m+2$$

by Claim 5, but h_2, h_3, h_4 are all adjacent to 1 since

$$6 \le h_2 < h_3 < h_4 \le m,$$

a contradiction.

(2.3.7) If $a_3 = 3$, then $a_1 = 1$, $a_2 = 2$,and $a_0 = 0$, so the remainder $7(q-1)$ vertices $[4, m+2]$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices as

$$u\,(\ge 4)\,, u+1, u+2, u+3, u+4, u+5, u+6,$$

thus $m+10$ would color α and be adjacent to a_4, a_5, a_6, a contradiction, too.

Case 2. $5 \le a_4 - a_0 \le m$.

Since $a_0a_4 \in E(H)$, we have $a_0a_2, a_2a_4, a_1a_3 \notin E(H)$ by Claim 3, i. e., $a_2 - a_0 \le 4$, $a_4 - a_2 \le 4$, and $a_3 - a_1 \le 4$, then $5 \le a_4 - a_0 \le 8$. By $a_7 - a_1 \ge m+1$, we have $a_7 - a_0 \ge m+2$. In the following we shall get a contradiction according to the related positions of a_0 and a_4.

It is obvious that

$$\begin{aligned} a_6 - a_4 &= (a_6 - a_0) - (a_4 - a_0) \\ &\ge m + 1 - 5 \ge m - 4. \end{aligned}$$

(1) Suppose that $a_4 - a_0 = 5$.

(1.1) Assume that $a_4 = 5$, then $a_0 = 0$. If $m+2 \le a_7 \le m+5$, then $a_4a_7 \in E(H)$, so $a_4a_6 \notin E(H)$, i. e., $4 \le m-4 \le a_6 - a_4 \le 4$, then $m = 8$, $a_6 = 9$, and $a_1a_6, a_2a_6, a_3a_6 \in E(H)$, a contradiction. Hence $a_7 = m+6$, so $a_1a_6, a_2a_6, a_3a_6 \in E(H)$ if $a_6 = m+1$, a contradiction. Therefore, $m+2 \le a_6 \le m+5$, then $a_4a_6 \in E(H)$, so $a_4a_5 \notin E(H)$, i. e., $6 \le a_5 \le 9$, we have $a_1a_5, a_2a_5, a_3a_5 \in E(H)$ when $a_5 = 9$, $a_0a_5, a_1a_5, a_5a_7 \in E(H)$ when $7 \le a_5 \le 8$, thus we have $a_5 = 6$, and $a_0a_5, a_5a_7 \in E(H)$, which induces $a_5a_6 \notin E(H)$, i. e., $10 \le m+2 \le a_6 \le 10$, that is, $a_6 = 10$, $m = 8$,and $a_2a_6, a_3a_6, a_4a_6 \in E(H)$, a contradiction.

(1.2) If $a_4 \ge 6$, then $a_4a_6 \notin E(H)$ (otherwise, a_0, a_4, a_6, a_7 induce a $K_{1,3}$), i. e., $4 \le m-4 \le a_6 - a_4 \le 4$, thus $m = 8$, $a_6 - a_4 = 4$, so $a_4a_7 \in E(H)$, and $a_4 - a_1 \le 4$, hence $a_1a_6, a_2a_6, a_3a_6 \in E(H)$, a contradiction, too.

(2) Suppose that $a_4 - a_0 = 6$, then $a_7 - a_4 = (a_7 - a_0) - (a_4 - a_0) \ge m - 4$.

(2.1) Assume that $a_7 - a_4 \ge m-3$, then $a_4a_7 \in E(H)$, so $a_1a_4, a_4a_6 \notin E(H)$, i. e., $a_4 - a_1 \le 4$, $a_6 - a_4 \le 4$, and $a_1 - a_0 = (a_4 - a_0) - (a_4 - a_1) \ge 2$, thus $3 \le a_2 - a_0 \le 4$. Hence $5 \le a_6 - a_1 \le 8$, and $m - 3 \le (a_6 - a_0) - (a_2 - a_0) = a_6 - a_2 = (a_6 - a_4) + (a_4 - a_2) \le 8$, so $a_1a_6, a_2a_6 \in E(H)$, then $a_3a_6, a_6a_7 \notin E(H)$, i. e., $a_6 - a_3 \le 4$, $a_7 - a_6 \le 4$, thus $4 \le m - 4 \le a_6 - a_2 - 1 \le a_6 - a_3 \le 4$. Therefore, $m = 8$, and $a_6 - a_3 = 4$. Moreover,

$$\begin{aligned} 5 &\le a_7 - a_4 < a_7 - a_3 \\ &= (a_7 - a_6) + (a_6 - a_3) \le 8, \end{aligned}$$

and

$$\begin{aligned} m - 3 &\le (a_6 - a_0) - (a_6 - a_3) \\ &= a_3 - a_0 < a_4 - a_0 \le 8, \end{aligned}$$

so $a_0a_3, a_0a_4, a_3a_7, a_4a_7 \in E(H)$, i. e., a_0, a_3, a_4, a_7 form a 4-cycle, a contradiction.

(2.2) Assume that $a_7 - a_4 = m - 4$. Then

$$a_7 - a_0 = (a_7 - a_4) + (a_4 - a_0) = m + 2,$$

and $a_6 - a_0 = m+1$, so $a_1 - a_0 = 1$, $a_7 - a_1 = m+1$,

$$a_4 - a_1 = (a_7 - a_1) - (a_7 - a_4) = 5,$$

and

$$a_6 - a_1 = (a_6 - a_0) - (a_1 - a_0) = m,$$

thus $a_1a_4, a_1a_5, a_1a_6 \in E(H)$, a contradiction.

(3) Suppose that $a_4 - a_0 = 7$, then

$$a_7 - a_4 = (a_7 - a_0) - (a_4 - a_0) \ge m - 5.$$

(3.1) Assume that

$$a_7 - a_4 \ge m - 3.$$

Then $a_4a_7 \in E(H)$, so $a_1a_4, a_4a_6 \notin E(H)$, i. e., $a_4 - a_1 \le 4$, and $a_6 - a_4 \le 4$. Hence

$$a_1 - a_0 = (a_4 - a_0) - (a_4 - a_1) \ge 3,$$

and then $a_2 - a_0 = 4$, and $a_4 - a_2 = 3$. Thus

$$5 \le a_6 - a_2 < a_6 - a_1 \le 8,$$

so $a_1a_6, a_2a_6 \in E(H)$, and then $a_3a_6, a_6a_7 \notin E(H)$, i. e., $a_6 - a_3 \leq 4$, and $a_7 - a_6 \leq 4$, hence

$$m - 3 \leq (a_6 - a_0) - (a_6 - a_3) \\ = a_3 - a_0 < a_4 - a_0 = 7,$$

which induces $m = 8$, and

$$5 \leq a_7 - a_4 < a_7 - a_3 = a_7 - a_6 + a_6 - a_3 \leq 8,$$

therefore a_0, a_3, a_4, a_7 form a 4-cycle, a contradiction.

(3.2) Assume that

$$a_7 - a_4 = m - 4,$$

then

$$a_7 - a_0 = (a_7 - a_4) + (a_4 - a_0) = m + 3,$$

and

$$m + 1 \leq a_7 - a_1 \leq m + 2,$$

so

$$5 \leq (a_7 - a_1) - (a_7 - a_4) = a_4 - a_1 \leq 6,$$

and $a_1a_4 \in E(H)$, thus $a_4a_7 \notin E(H)$, i. e.,

$$a_7 - a_4 = m - 4 \leq 4,$$

that is, $a_7 - a_4 = 4$, and $m = 8$. Clearly, $2 \leq a_6 - a_4 \leq 3$, and $1 \leq a_5 - a_4 \leq 2$ in this case.

(3.2.1) If $a_4 - a_1 = 5$, then $a_1a_4, a_1a_5, a_1a_6 \in E(H)$, a contradiction.

(3.2.2) If $a_4 - a_1 = 6$, then $a_1 - a_0 = 1$, so we have $a_1a_5 \in E(H)$, thus $a_5 - a_4 = 2$ (otherwise, $a_5 - a_4 = 1$, and a_0, a_4, a_1, a_5 induce a 4-cycle), and $a_7 - a_6 = a_6 - a_5 = 1$, hence a_0, a_2, a_3, a_4 induce a $K_{1,3}$ when $a_4 - a_2 = 2$, and a_2, a_5, a_6, a_7 induce a $K_{1,3}$ when $3 \leq a_4 - a_2 \leq 4$, a contradiction.

(3.3) Assume that

$$a_7 - a_4 = m - 5.$$

Then

$$a_7 - a_0 = (a_7 - a_4) + (a_4 - a_0) = m + 2,$$

$a_6 - a_0 = m + 1$, and $a_1 - a_0 = 1$, so

$$a_4 - a_1 = (a_4 - a_0) - (a_1 - a_0) = 6, \\ a_6 - a_1 = (a_6 - a_0) - (a_1 - a_0) = m,$$

and $a_1a_4, a_1a_5, a_1a_6 \in E(H)$, a contradiction,too.

(4) Suppose that $a_4 - a_0 = 8$, then $a_0a_4 \in E(H)$, and $a_4 - a_2 = 4$ by Claim 3, so $a_1a_4 \in E(H)$, and thus $a_4a_7 \notin E(H)$, i. e., $a_7 - a_4 \leq 4$, hence $a_2a_5, a_2a_6, a_2a_7 \in E(H)$, a contradiction.

By two cases above, we have $a_4 - a_0 \leq 4$, i. e., $a_4 - a_0 = 4$. Hence Subclaim 7. 1 holds.

Subclaim 7.2 $a_0 = 0$.

Otherwise, we have $a_0 \geq 1$, and $2 \leq a_1 \leq 5$ by $a_7 - a_1 > m$.

(1) If $a_1 = 5$, then $a_0 = 4$, $a_2 = 6, a_3 = 7, a_4 = 8, a_7 = m+6$, and $a_6 = m+5$, so $a_1a_6, a_2a_6, a_3a_6 \in E(H)$, a contradiction.

(2) If $a_1 = 4$, then $a_0 = 3$, $a_2 = 5$, $a_3 = 6$, $a_4 = 7$, and $m + 4 \leq a_6 \leq m + 5$,so $a_2a_6, a_3a_6, a_4a_6 \in E(H)$, a contradiction.

(3) If $a_1 = 3$, $a_0 = 2$, $a_2 = 4$, $a_3 = 5$, $a_4 = 6$, then $m + 3 \leq a_6 \leq m + 5$, so $a_3a_6, a_4a_6 \in E(H)$, thus $a_2a_6, a_5a_6 \notin E(H)$, i. e., $a_6 - a_2 \geq m + 1$, and $a_6 - a_5 \leq 4$, hence $a_6 = m+5$, $a_7 = m+6$, and $a_5 \geq m + 1$. Clearly, $a_4a_7 \in E(H)$, so $a_4a_5 \notin E(H)$, i. e., $9 \leq m + 1 \leq a_5 \leq 10$, thus $a_0a_5, a_1a_5, a_2a_5 \in E(H)$, a contradiction.

(4) Suppose that $a_1 = 2$, then $a_0 = 1$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$,

$$m + 2 \leq a_6 \leq m + 5,$$

and

$$m + 3 \leq a_7 \leq m + 6,$$

so $a_4a_6 \in E(H)$, and thus $a_4a_5 \notin E(H)$ or $a_5a_6 \notin E(H)$.

(4.1) Assume that $a_4a_5 \notin E(H)$, then $a_5 - a_4 \leq 4$, i. e., $6 \leq a_5 \leq 9$.

(4.1.1) If $8 \leq a_5 \leq 9$, then $a_0a_5, a_1a_5, a_2a_5 \in E(H)$, a contradiction.

(4.1.2) If $a_5 = 7$, then $a_0a_5, a_1a_5 \in E(H)$, so $a_5a_7 \notin E(H)$, i. e.,

$$11 \leq m + 3 \leq a_7 \leq 11,$$

hence $a_7 = 11, m = 8$ and $a_2a_7, a_3a_7, a_4a_7 \in E(H)$, a contradiction.

(4.1.3) If $a_5 = 6$,then $a_0a_5, a_5a_7 \in E(H)$, so $a_5a_6 \notin E(H)$, i. e.,

$$10 \leq m + 2 \leq a_6 \leq 10,$$

hence $a_6 = 10$, $m = 8$, and $a_1a_6, a_2a_6, a_3a_6 \in E(H)$, a contradiction, too.

(4.2) Assume that $a_4a_5 \in E(H)$, and $a_5a_6 \notin E(H)$, then $a_3a_5 \in E(H)$, $a_6 - a_5 \leq 4$, so $a_2a_5 \notin E(H)$, i. e., $a_5 \geq m + 4$, hence $a_5 = m + 4 = a_6 - 1 = a_7 - 2$. Therefore, the remainder $7(q-1)$ vertices $\{0\} \cup [6, m+3]$ in H colored $q - 1$ colors, such that each color colored seven consecutive vertices, except some color β colored vertices

$$0 < h_1 < h_2 < h_3 < h_4 < h_5 < m + 3,$$

but h_1, h_2, h_3 are all adjacent to 0 since

$$6 \leq h_1 < h_2 < h_3 \leq m,$$

a contradiction, too.

Therefore, $a_0 = 0$, and then Subclaim 7.2 is proved.

In a word, we have $a_4 \leq 4$, and Claim 7 holds.

Claim 8: $a_6 \geq 7(q-1)+6$, that is, $a_6 = 7q-1$, and $a_7 = 7q$.

If $a_6 \leq 7q-2$, i. e., $a_6 \leq m+4$, then $a_4a_6 \in E(H)$, so $a_4a_5 \notin E(H)$ or $a_5a_6 \notin E(H)$.

(1) Suppose that $a_4a_5 \notin E(H)$, then $5 \leq a_5 \leq 8$.

(1.1) If $7 \leq a_5 \leq 8$, then $a_0a_5, a_1a_5, a_2a_5 \in E(H)$, a contradiction.

(1.2) If $a_5 = 6$, then $a_0a_5, a_1a_5 \in E(H)$, so $a_5a_7 \notin E(H)$, and $10 \leq m+2 \leq a_7 \leq 10$, thus $a_7 = 10$, $m = 8$, $a_6 = 9$, and $a_2a_7, a_3a_7, a_4a_7 \in E(H)$, a contradiction.

(1.3) Assume that $a_5 = 5$, then $a_0a_5 \in E(H)$. If $m+2 \leq a_7 \leq m+5$, then $a_5a_7 \in E(H)$, so $a_5a_6 \notin E(H)$, i. e., $9 \leq m+1 \leq a_6 \leq 9$, thus $a_6 = 9$, $m = 8$, and $a_1a_6, a_2a_6, a_3a_6 \in E(H)$, a contradiction. Hence $a_7 = m+6$, and then the remainder $7(q-1)$ vertices $[6, m+5] \setminus \{a_6\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices by Claim 1, which is impossible since $m+1 \leq a_6 \leq m+4$.

(2) Suppose that $a_4a_5 \in E(H)$, and $a_5a_6 \notin E(H)$. Then $a_3a_5 \in E(H)$, so $a_7 - a_5 \leq 4$, $a_6 - a_3 \geq m+1$, and $a_5 - a_2 \geq m+1$, hence $a_6 = m+4$ and $a_5 = m+3$. If $a_7 = m+6$, then the remainder $7(q-1)$ vertices $[5, m+2] \cup \{m+5\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices by Claim 1, which is impossible. Hence $a_7 = m+5$, and the remainder $7(q-1)$ vertices $[5, m+2] \cup \{m+6\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices, except some color β colored vertices

$$5 < h_1 < h_2 < h_3 < h_4 < h_5 < m+6,$$

but h_1, h_2, h_3 are all adjacent to $m+6$ since

$$6 \leq h_1 < h_2 < h_3 \leq m,$$

a contradiction, too.

Therefore, we have $a_6 = 7q-1$, and $a_7 = 7q$.

Claim 9: $a_5 = 5$ or $a_5 = m+4$.

Assume that

$$6 \leq a_5 \leq m+3.$$

If

$$6 \leq a_5 \leq m,$$

then $a_0a_5, a_1a_5, a_5a_7 \in E(H)$, a contradiction. If

$$m+1 \leq a_5 \leq m+2,$$

then $a_2a_5, a_3a_5, a_4a_5 \in E(H)$, a contradiction. If $a_5 = m+3$, then the remainder $7(q-1)$ vertices $[5, m+2] \cup \{m+4\}$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices by Claim 1, which is impossible. Hence Claim 9 holds.

Without loss of generality, suppose that $a_5 = 5$. Then the remainder $7(q-1)$ vertices $[6, m+4]$ in H colored $q-1$ colors, such that each color colored seven consecutive vertices as $(6 \leq)u, u+1, \cdots, u+6$ by Claim 1, hence $m+11, m+12$ would color α and induce a 4-cycle along with a_6, a_7, a contradiction, too.

In a word, we have shown that

$$vla\left(G\left(D_{m,1,4}\right)\right) \geq \lceil \tfrac{m}{7} \rceil + 1.$$

Therefore, we obtain that

$$vla\left(G\left(D_{m,1,4}\right)\right) = \lceil \tfrac{m}{7} \rceil + 1.$$

□

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