Optimal investment and risk management for open-ended funds under Heston's SV model

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Abstract: This paper considers the optimal investment and risk management strategies for a manager of open-ended funds under Heston's stochastic volatility model. The manager is allowed to invest the fund in a financial market, which consists of one risk-free asset and one risky asset whose price process satisfies Heston's SV model. The objective of the fund manager is to maximize the expected exponential utility of the terminal wealth of the fund assets. We obtain the optimal strategies and value function via stochastic optimal control approach explicitly. Moreover, a verification theorem is provided and the properties of the optimal strategies are discussed. Finally, sensitivity analysis is presented to illustrate the influences of parameters on the optimal investment strategy and redemption limit.

Key–Words: Open-ended funds, Subscribe and redeem, Compound Poisson process, Stochastic volatility, Hamilton-Jacobi-Bellman equations, Stochastic optimal control

1 Introduction

As a type of professionally managed collective investment scheme, the first mutual fund known as the Foreign & Colonial Government Trust, was established in London in 1868, and then the mutual funds were introduced into the United States in 1890s. Benefiting from the scale advantage brought by the property of collective investment, they gradually emerged from other financial instruments and became popular during the 1920s, soon after that came the funds-oriented investment boom throughout the world. Nowadays, they play an important role in household finances, most notably in retirement planning. At the end of 2011, funds accounted for 23% of household financial assets. Their role in retirement planning is particularly significant. Roughly half of assets in 401(k) plans and individual retirement accounts were invested in mutual funds [1]. Compared to direct investment in individual securities, mutual funds have some significant advantages, such as increased diversification, daily liquidity (this concept applies only to open-ended funds), professional investment management and ability to participate in investments that may be available only to larger investors.

According to the organizational forms, mode of operation and investment objective, mutual funds are classified to different types. In terms of the operation mode, mutual funds are usually divided into closed-ended and open-ended. The original design of mutual funds is the closed-ended type with a fix number of shares which are irredeemable from the fund and can only be purchased and sold in the market. This means an investor of close-ended funds can only turn to the market once he/she urgently needs to realise part of his/her fund shares, which would not be timely sometimes and always costly when encountered with a market downturn.

An excellent solution to such kind of liquidity shortage is attributed to the innovation in operation mode brought by the open-ended fund, which can issue and redeem shares at any time. Unlike closed-ended funds, new shares/units in an open-ended fund can be created by managers to meet demand from investors and an investor will generally purchase shares in the fund directly from the fund itself rather than from the existing shareholders. That is to say, anytime when the investor wants to sell his/her shares, the redemption requests will be available, which has long been a great attraction to millions of investors, especially to those in favor of short-term investment. As a much more flexible investment option, open-ended funds soon became the most common type of mutual funds though it once accounted for only 5% of the industry's $27 billion in total assets by 1929. At the end of 2011, in contrast to 634 closed-end funds amounted to $239 billion, there were 7,581 open-end mutual funds in the United States with combined assets of \$11.6 trillion [1].

Admittedly, with such an aggressive operating mode, allowing the investors to buy back their shares at any time, the managers of open-ended funds are required to be capable of managing the liquidity risk, especially for the Quantity of Large Redemption, which often occurs along with a market slump and may cause bankruptcy of the funds. Therefore, how to deal with the Quantity of Large Redemption is the key to liquidity management for open-ended funds. The research on open-ended funds' operation mode and trigger mechanism of the Quantity of Large Redemption will greatly help the fund managers to make proper and rational decisions. Scholars have achieved some fundamental results about the open-ended funds, though not that much but some are really interesting.

Edelen [2] took the lead in proposing the concept of investor's flows and used a separate regression of both inflow and outflow of funds to analyze the gross flow. He confirmed a statistically significant indirect cost in the form of a negative relation between a fund's abnormal return and investor flows, which was attributed to the costs of liquidity-motivated trading. Stein [3] explained why most funds are open-ended in the perspective of competition and the limits of arbitrage, he suggested that an over-marketing strategy to attract investors may bring about cutthroat competition, and the fund management companies would have to choose the open-ended type passively. He points out that economically large mispricings can coexist with rational, competitive arbitrageurs, even without short-sales constraints or other frictions. By using the theory of Random process and the theory of Sequential Decision, Cheng et al [4] considered the optimal investment decision of open-ended funds, which is based on the benefit of investors and the cost of transaction. They introduced a method of random discounting factor, by which investors can choose optimal investment strategy, and concluded that when the fund management company operated only one type of fund in long-term which has the largest profit, it also should strengthen management for other types of funds in order to improve yield rate.

However, although aforementioned works have provided a wide range of ideas and prospectives, their results are fragmented and a systematic and completed research method is still needed. It is difficult but significant to establish the open-ended funds' typical redemption mechanism, meaning that when and how many the investors choose to redeem their shares should be quantified or modeled, especially for the case of Quantity of Large Redemption. Notably, some research on investment and insurance may lead to a solution to the problem. Some mature methods applied to studying reinsurance and investment for insurers will provide a cogent and thoughtful guidance to our work.

For example, Hipp and Taksar [5] used the compound Poisson process to describe the insurer's surplus process and studied the optimal investment problem in new business to minimize the ruin probability. Yang and Zhang [6] then extended the study of Hipp and Taksar [5] to the case of an insurer with jump-diffusion risk model. With the deepening of the insurance and reinsurance research, academe subdivides the insurance problems into different circumstances, then follows the proportional reinsurance, excess-of-loss reinsurance and other categories. Subsequently, Zhao et al [7] considered the optimal investment and excess-of-loss reinsurance problem for an insurer with jump-diffusion risk process. The application of jump-diffusion risk model in insurance greatly inspires the ideas and methods to study the open-ended funds, the stochastic subscription and redemption of open-ended funds can be seen as the random claims in insurance and reinsurance problems, and the redemption fee is similar to premium.

In this paper, we will try to apply the compound Poisson process to simulate the stochastic processes for investors to subscribe and redeem their shares. Furthermore, we assume that the price process of risky asset satisfies the Heston's stochastic volatility (SV) model. Heston model is proposed by Heston [8] and is a common used SV model. Compared to the Stein-Stein model [9] under which the risky asset' price process and its stochastic varying volatility parameter are driven by two independent Wiener processes, Heston's SV model holds a similar hypothesis with assuming that the two Wiener processes are correlated and the risky asset's appreciation rate is also stochastic. It has already gained its popularity in option pricing and is used in investment problems recently.

2 Model and assumptions

To meet the daily requirement of redemption from fund shareholders, a fund manager has to reserve some liquid assets, such as cash, sight deposits and short-term bonds, as part of the whole fund assets. The price process of such assets (risk-free assets) is denoted by $B(t)$ which follows

$$dB(t) = r_0 B(t)dt,$$

where $r_0 > 0$ represents the risk-free interest rate. The price process $S(t)$ of the risky asset is assumed to follow the Heston's SV model,

$$dS(t) = S(t)[(r_0 + \lambda L(t))dt + \sqrt{L(t)}dW_1(t)]$$

$$S(0) = s_0,$$
$$dL(t) = k(\theta - L(t))dt + \sigma\sqrt{L(t)}dW_2(t),$$
$$L(0) = l_0,$$

where k,θ,σ andλ are all positive constants. $\{W_1(t)\}$ and $\{W_2(t)\}$ are two correlated standard Brownian motions with $E[W_1(t)W_2(t)] = \rho t$. In addition, we also assume $2k\theta \geq \sigma^2$ to insure that $L(t)$ is almost surely non-negative as mentioned in Cox et al [10]. Here $L(t)$ is related to the appreciation rate and volatility of risky asset and $\{W_2(t)\}$ represents the corresponding random fluctuation. $\lambda L(t)$ can be regarded as the risk premium (see $R_m - R_f$ in William Sharpe's CAPM) for taking risk $\sqrt{L(t)}$. Thus, the unit risk premium can be written as $\lambda\sqrt{L(t)}$. Considering that the financial market is supposed to include only one risky asset (and a risk-free asset), then $\lambda\sqrt{L(t)}$ can be seen as a criterion to measure the financial market's performance. We suppose the Quantity of Large Redemption will only occurs when the unit risk premium tumbles below a particular level ζ. Let $\chi(t)$ denotes the indicator function for the potential large redemption,

$$\chi(t) = \begin{cases} 1 & , L(t) < (\frac{\zeta}{\lambda})^2 \doteq \zeta_0, \\ 0 & , L(t) \geq \zeta_0, \end{cases}$$

apparently, $L(t)$ is an almost everywhere continuous function on a given duration $[0, T]$, then $\chi(t) \equiv 1$ holds over some intervals, in which the unit risk premium is less than ζ, i.e., $L(t) < \zeta_0$.

The surplus process of open-ended funds is described by a jump-diffusion risk model, which we denote by $\{R(t) : t \geq 0\}$ satisfies

$$dR(t) = \chi(t)c(t)dt + \beta dW_0(t) + dI(t) - \chi(t)dO(t) \tag{1}$$

where $\beta > 0$ is a constant, and $\{W_0(t)\}$ with β standing for the uncertainty associated with the surplus of open-ended funds at time t, is another standard Brownian motion which is independent with $\{W_1(t)\}$ and $\{W_2(t)\}$. $I(t) = \sum_{i=1}^{N_1(t)} I_i$ is a compound Poisson process denoting the cumulative net buying by time t, where $N_1(t)$, the number of subscription occurring in the time interval $[0, t]$, is a homogeneous Poisson process with intensity α_1, and the subscription sizes $\{I_i|i \geq 1\}$ are independent and identically distributed positive random variables with common distribution $F_1(x)$. $O(t) = \sum_{i=1}^{N_2(t)} O_i$ as a representation of cumulative Quantity of Large Redemption, is another compound Poisson process, where $N_2(t)$ with intensity α_2 denotes the number of large redemption by time t, and $\{O_i : i \geq 1\}$ have the similar properties as $\{I_i : i \geq 1\}$ with common distribution $F_2(x)$.

Remark 1 *For the convenience of mathematical calculation, we assume $N_1(t) = N_2(t) = N(t)$ and $\alpha_1 = \alpha_2 = \alpha$. It should be noted this assumption can not be extended to the more general cases. Actually, the net buying process $I(t)$ is relatively independent of the Quantity of Large Redemption process $O(t)$. The statistical relativity between them should be specified by the practical data, which is a direction of our future efforts.*

Besides that, other random variables are all supposed to be independent with each other.

When Quantity of Large Redemption appears, we assume that O_i stands for the total redemption each time (like every business day) from all the fund shareholders, in other words, a single shareholder will participate in the joint redemption with others. Hence, the entire fund can theoretically be redeemed at any time. Apparently, a fund manager will try to shrink this sort of liquidity risk to avoid bankruptcy, then comes the redemption limit $a(t)$, the maximum amount of a single large redemption. The part of redemption at time t exceeding $a(t)$ is supposed to be invalid or get into the next redemption request automatically.

In consideration of the impact cost (extra cost for a trader due to market slippage) for massive sale of securities (Chang et al [13] points out that the trading turnover has a significant effect on the short-run abnormal returns and long-run cumulative returns of portfolios) to meet the shareholders' redemption needs, and opportunity cost for holding some cash assets with low yields, redemption fees are charged from traders as a stiff penalty used to discourage short-term, in-and-out trading of mutual fund shares. We denote the redemption fees (subtracting the corresponding impact cost and opportunity cost) by $c(t)$ in (1), which is supposed to be a certain percentage $\eta > 0$ of the expected quantity of large redemption and credited to the fund's assets. Thus,

$$\begin{aligned} c(t) &= \eta\alpha\left(\int_0^{a(t)} y dF_2(y) + \int_{a(t)}^{\infty} a(t) dF_2(y)\right) \\ &= \eta\alpha \int_0^{a(t)} \overline{F_2(y)} dy \end{aligned} \tag{2}$$

where $\overline{F_2(x)} = 1 - F_2(x)$.

Let $\pi(t)$ denotes the money amount invested in the risky asset at time t, hence a trading strategy for a fund manager can be described by $(\pi(t), a(t))$. Associated with the trading strategy $(\pi(t), a(t))$, open-ended fund's wealth process $X(t)$ then evolves as

$$\begin{aligned} dX(t) &= \Big(\pi(t)\lambda L(t) + \chi(t)\eta\alpha\int_0^{a(t)}\overline{F_2(y)}dy \\ &\quad + r_0X(t)\Big)dt + \pi(t)\sqrt{L(t)}dW_1(t) \\ &\quad + \beta dW_0(t) + d\sum_{i=1}^{N(t)} I_i - \chi(t)d\sum_{i=1}^{N(t)} O_i \end{aligned} \tag{3}$$

Remark 2 *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ *be a complete probability space satisfying the usual condition, where* $\{\mathscr{F}_t\}$ *is generated by process* $\{X(t)\}$*. All stochastic processes introduced in this paper are supposed to be adapted processes in this space. An admissible trading strategy means that* $(\pi(t), a(t))$ *is* $\mathscr{F}_t$*-progressively measurable,* $0 \leqslant a(t) \leqslant X(t)$*,* $E[\int_0^T \pi^2(t)L(t)dt] < \infty$ *for all* $T < \infty$*, and* (3) *has a unique strong solution.*

Π denotes the set of all admissible strategies. And for simplicity, an admissible strategy $(\pi(t), a(t))$ *and the indicator function for the large redemption* $\chi(t)$ *will be written as* (π, a) *and* χ *accordingly in the next sections.*

Here we suppose that the fund manager has an exponential utility function $U(\cdot)$, which is strictly concave and continuously differentiable on $(-\infty, \infty)$, and aims to maximize the expected utility of the fund's terminal wealth, i.e.,

$$\max_{(\pi,a)\in\Pi} E[U(X(T))] \tag{4}$$

with $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$ and coefficient of absolute risk aversion $\gamma > 0$.

Remark 3 *Of course, we can choose power utility and logarithm utility as candidates for the utility function, some related examples can be seen in the work of Chang et al [11] and Li et al [12]. But exponential utility is the only utility function to make the principle of "zero utility" give a fair premium, which leads to an extensive application of exponential utility in insurance mathematics.*

3 Optimal strategies and verification theorem

For an admissible strategy (π, a), the value function H from state (x, l) at time t is defined as

$$H(t, x, l) = \sup_{(\pi,a)\in\Pi} E[U(X(T)|X(t) = x, L(t) = l], \tag{5}$$

where $0 \le t < T$, and H satisfies the boundary condition $H(T, x, l) = -\frac{1}{\gamma}e^{-\gamma x}$.

For any $H(t, x, l) \in C^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, we define an operator $\mathcal{A}^{\pi,a}$, satisfying

$$\begin{aligned} \mathcal{A}^{\pi,a}H(t,x,l) &= H_t + \Big(r_0x + \chi\eta\alpha\int_0^a \overline{F_2(y)}dy \\ &\quad + \pi l\lambda\Big)H_x + k(\theta - l)H_l + \frac{1}{2}(\beta^2 + \pi^2 l)H_{xx} \\ &\quad + \frac{1}{2}\sigma^2 lH_{ll} + \rho\sigma\pi lH_{xl} + \alpha E\big[H\big(t, x + I_i \\ &\quad - \chi\min(O_i, a), l\big) - H(t, x, l)\big] \end{aligned} \tag{6}$$

where H_x, H_l, H_{xx}, H_{ll} and H_{xl} denote the first and second-order partial derivatives with respect to corresponding variables x and l.

According to Fleming and Soner [14], the optimal value function $V(t, x, l)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\sup_{(\pi,a)\in\Pi} \mathcal{A}^{\pi,a}V(t, x, l) = 0, \quad t < T \tag{7}$$

with boundary condition $V(t, x, l) = -\frac{1}{\gamma}e^{-\gamma x}$.

Inspired by Browne [15], we solve the HJB equation (7) by making use of *ansatz* with the form below:

$$V(t, x, l) = -\frac{1}{\gamma}\exp\Big\{-\gamma\big[xe^{r_0(T-t)} + f(t) + g(t, l)\big]\Big\} \tag{8}$$

with $f(T) = 0$ and $g(T, l) = 0$. Then we have

$$\begin{aligned} &V_t = \gamma\big(r_0xe^{r_0(T-t)} - f'(t) - g_t\big)V, \\ &V_x = -\gamma e^{r_0(T-t)}V, \quad V_{xx} = \gamma^2e^{2r_0(T-t)}V, \\ &V_l = -\gamma g_lV, \quad V_{ll} = (-\gamma g_{ll} + \gamma^2 g_l^2)V, \\ &V_{xl} = \gamma^2e^{r_0(T-t)}g_lV. \end{aligned}$$

Suppose that $F_1(x)$ is a uniform distribution on $[0, D]$, then we can derive

$$\begin{aligned} &E\big[V\big(t, x + I_i - \chi\min(O_i, a), l\big) - V\big(t, x, l\big)\big] \\ &= V\Big\{\frac{1 - e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}\big[\chi\gamma e^{r_0(T-t)}\int_0^a \overline{F_2(y)} \\ &\quad \cdot e^{\chi\gamma ye^{r_0(T-t)}}dy + 1\big] - 1\Big\} \end{aligned} \tag{9}$$

Substituting the above derivatives into (7) yields

$$\gamma(f'(t)+g_t)-\frac{1}{2}\beta^2\gamma^2e^{2r_0(T-t)}+k(\theta-l)\gamma g_l$$
$$-\frac{1}{2}\sigma^2 l(-\gamma g_{ll}+\gamma^2 g_l^2)+\max_{\pi\in\mathbb{R}}\Big\{(\lambda-\rho\sigma\gamma g_l)$$
$$\cdot\gamma l e^{r_0(T-t)}\pi-\frac{1}{2}\gamma^2 l e^{2r_0(T-t)}\pi^2\Big\}+\alpha\max_{a\in(0,x)}\Big\{$$
$$\chi\eta\gamma e^{r_0(T-t)}\int_0^a\overline{F_2(y)}dy+1-\frac{1-e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}\big[$$
$$\chi\gamma e^{r_0(T-t)}\int_0^a e^{\chi\gamma y e^{r_0(T-t)}}\overline{F_2(y)}dy+1\big]\Big\}=0 \tag{10}$$

If $\chi=0$, Quantity of Large Redemption will not happen according to our assumption above, then we only need to consider the case of $\chi=1$. So all the following discussions will assume $\chi=1$.

The optimal investment policy $\pi^*(t)$ is obtained by taking derivative of (10) with respect to π,

$$\pi^*=\frac{\lambda}{\gamma}e^{-r_0(T-t)}-\rho\sigma e^{-r_0(T-t)}g_l. \tag{11}$$

Similarly, by taking derivative of (10) with respect to a, we figure out the minimizer a_0, which satisfies

$$\Big(\eta-\frac{1-e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}e^{\gamma a_0 e^{r_0(T-t)}}\Big)\overline{F_2(a_0)}=0. \tag{12}$$

We let $x_0=\sup\{x: F_2(x)<1\}$ denote the investors' maximum redemptions in theory. In consideration of that the investors can redeem all their shares at any time, $x_0\geq x(t)$ should hold for any $t\in[0,T]$. Then we have the following deduction.

If $a_0\in(0,x_0)$, then $0<\overline{F_2(a_0)}\leqslant 1$ and

$$a_0=\frac{\ln\Big(\frac{\eta D\gamma e^{r_0(T-t)}}{1-e^{-\gamma e^{r_0(T-t)}D}}\Big)}{\gamma e^{r_0(T-t)}}. \tag{13}$$

Actually, D (the maximum amount for net buying) is always large enough to make $\eta D\gamma>1$, thus $a_0>0$ will hold in most cases when $\chi=1$. Here we assume $\eta D\gamma>1$ in all cases.

We next prove the optimal strategy for redemption $a^*=\min(a_0,x_0)$.

Theorem 4 *If $\chi=1$ and $\eta D\gamma>1$, then the optimal strategy for redemption $a^*=\min(a_0,x_0)$.*

Proof: If $a_0\in(0,x_0)$, apparently, we have $a^*=a_0$. Hence, we only consider the case when $a_0\geqslant x_0$. Since $\frac{1-e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}>0$, then $\kappa(a)\triangleq\eta-\frac{1-e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}e^{\gamma a e^{r_0(T-t)}}$ as part of (12) is a decreasing function with respect to a. For $\eta D\gamma>1$, we have $\kappa(0)>0$. If a_0 in (13) is not less than x_0, then $\kappa(a)>0$ holds for all $a\in(0,x_0)$ and (12) is satisfied when $a=x_0$ for $\overline{F_2(x_0)}=0$.

So we have proved that $a^*=\min(a_0,x_0)$ is the optimal strategy for redemption. □

Remark 5 *We need to point out that the restriction $a^*\leqslant x_0$ is not necessary. Indeed, the shareholders can only redeem the fund's whole wealth $x\leq x_0$ at most, which means $a^*=\min(a_0,x)$ is more practical. Therefore, we will denote a^* as $\min(a_0,x)$ in the rest of this paper.*

Inserting (11) and (13) into (10), we obtain

$$\gamma(f'(t)+g_t)-\frac{1}{2}\beta^2\gamma^2e^{2r_0(T-t)}+k(\theta-l)\gamma g_l$$
$$-\frac{1}{2}\sigma^2\gamma^2 l(1-\rho^2)g_l^2+\frac{1}{2}\sigma^2\gamma l g_{ll}+\frac{1}{2}l\lambda^2$$
$$-\lambda\rho\sigma\gamma l g_l+\alpha\Big\{\chi\eta\gamma e^{r_0(T-t)}\int_0^{a^*(t)}\overline{F_2(y)}dy+1$$
$$-\frac{1-e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}\big[\chi\gamma e^{r_0(T-t)}\int_0^{a^*(t)}e^{\chi\gamma y e^{r_0(T-t)}}$$
$$\cdot\overline{F_2(y)}dy+1\big]\Big\}=0. \tag{14}$$

Given the independency between $f(t)$ and $g(t,l)$, (14) is decomposed into two equations:

$$\gamma f'(t)-\frac{1}{2}\beta^2\gamma^2e^{2r_0(T-t)}+h(t)=0, \tag{15}$$

$$g_t-\frac{1}{2}\sigma^2\gamma l(1-\rho^2)g_l^2+\frac{1}{2}\sigma^2 l g_{ll}+\Big(k(\theta-l)-\lambda\rho\sigma l\Big)g_l+\frac{l\lambda^2}{2\gamma}=0, \tag{16}$$

where

$$h(t)=\alpha\Big\{-\frac{1-e^{-\gamma e^{r_0(T-t)}D}}{D\gamma e^{r_0(T-t)}}\big[\chi\gamma e^{r_0(T-t)}\cdot\int_0^{a^*(t)}e^{\chi\gamma y e^{r_0(T-t)}}\overline{F_2(y)}dy+1\big]+\chi\eta\gamma e^{r_0(T-t)}\int_0^{a^*(t)}\overline{F_2(y)}dy+1\Big\}. \tag{17}$$

On the basis of the boundary condition $f(T)=0$, the solution to (15) is

$$f(t)=\frac{\beta^2\gamma}{4r_0}\Big(1-e^{2r_0(T-t)}\Big)+\frac{1}{\gamma}\int_t^T h(s)ds. \tag{18}$$

To solve (16), we suppose the solution has the following form

$$g(t,l) = A(t)l + B(t) \tag{19}$$

with $A(T) = B(T) = 0$ (for $g(T,l) = 0$). Then (16) can be written as

$$B'(t) + k\theta A(t) + \Big(A'(t) - \frac{1}{2}\sigma^2\gamma(1-\rho^2)A^2(t) - (k+\lambda\rho\sigma)A(t) + \frac{\lambda^2}{2\gamma}\Big)l = 0. \tag{20}$$

In order to eliminate dependence on l, we decompose (20) into

$$A'(t) - \frac{1}{2}\sigma^2\gamma(1-\rho^2)A^2(t) - (k+\lambda\rho\sigma)A(t) + \frac{\lambda^2}{2\gamma} = 0 \tag{21}$$

and

$$B'(t) + k\theta A(t) = 0. \tag{22}$$

Allowing for the boundary conditions, we derive

$$A(t) = \begin{cases} \frac{\lambda^2}{2\gamma(k+\lambda\sigma)}\big(1 - e^{-(k+\lambda\sigma)(T-t)}\big), & \rho = 1, \\ \frac{\lambda^2}{2\gamma(k-\lambda\sigma)}\big(1 - e^{-(k-\lambda\sigma)(T-t)}\big), & \rho = -1 \\ & \text{and } k \neq \lambda\sigma, \\ \frac{\lambda^2}{2\gamma}(T-t), & \rho = -1 \text{ and } k = \lambda\sigma, \\ \frac{v_1v_2 - v_1v_2e^{-\frac{1}{2}\sigma^2\gamma(1-\rho^2)(T-t)(v_1-v_2)}}{v_2 - v_1e^{-\frac{1}{2}\sigma^2\gamma(1-\rho^2)(T-t)(v_1-v_2)}}, & \rho^2 \neq 1, \end{cases}$$
$$B(t) = k\theta\int_t^T A(s)ds, \tag{23}$$

where v_1, v_2 are two different roots of the following equation with respect to x,

$$\frac{1}{2}\sigma^2\gamma(1-\rho^2)x^2 + (k+\lambda\rho\sigma)x - \frac{\lambda^2}{2\gamma} = 0.$$

Thus,

$$v_{1,2} = \frac{-(k+\lambda\rho\sigma) \pm \sqrt{k^2 + 2k\lambda\rho\sigma + \sigma^2\lambda^2}}{\sigma^2\gamma(1-\rho^2)} \tag{24}$$

By now, we have already obtained the solution $V(t,x,l)$ to HJB equation (7) and the optimal strategy (π^*, a^*) for a fund manager with exponential utility under the Heston model,

$$\pi^* = \frac{\lambda}{\gamma}e^{-r_0(T-t)} - \rho\sigma e^{-r_0(T-t)}A(t),$$
$$a^* = \frac{\ln\Big(\frac{\eta D\gamma e^{r_0(T-t)}}{1-e^{-\gamma e^{r_0(T-t)}D}}\Big)}{\chi\gamma e^{r_0(T-t)}}, \quad \chi = 1 \text{ and } \eta D\gamma > 1, \tag{25}$$

$$V(t,x,l) = -\frac{1}{\gamma}\exp\Big\{ -\gamma\big[xe^{r_0(T-t)} + f(t) + A(t)l + B(t)\big]\Big\}, \tag{26}$$

where $A(t)$, $B(t)$ and $f(t)$ are given by (23) and (18), respectively.

Theorem 6 *Suppose $V(t,x,l)$ is a solution to* (7)*, then the value function is $H(t,x,l) = V(t,x,l)$. For the wealth process $X(t)$ associated with an admissible strategy (π, a), we have*

$$E[U(X(T))] \leqslant V(0, x_0, l_0).$$

In particular, for (π^*, a^*) given in (25) *and the corresponding wealth process $X^*(t)$,*

$$E[U(X^*(T))] = V(0, x_0, l_0).$$

According to Theorem 5, the (π^*, a^*) we have obtained is indeed the optimal strategy among all the admissible ones, and the supremum of the expected exponential utility $E[U(X(T))]$ is achievable. The corresponding proof can be found in Appendix of Zhao et al [7].

Furthermore, we find that $A(t)$ is a decreasing function by taking derivative of (23) with respect to time t ($A'(t) < 0$ holds in all cases).

Remark 7 *From* (25)*, we find the optimal amount π^* invested in risky asset is independent of the fund's wealth x. This is due to the fact that the exponential utility function has constant absolute risk aversion ($-U'(x)/U''(x) = \frac{1}{\gamma}$).*

Remark 8 *In addition, the risky asset's appreciation rate $L(t)$ (also to be its volatility) doesn't directly influence the optimal investment strategy π^*. But the correlation coefficient ρ between $L(t)$ and the risky asset's price $S(t)$ has an influence on the investment strategy, so does $L(t)$'s own volatility σ and the drift-related parameter k.*

In a nutshell, the investment strategy mainly depends on the time value $e^{-r_0(T-t)}$, the fund manager's risk aversion γ and the correlation between risky asset and its volatility ρ.

Remark 9 *Unlike the investment strategy, the optimal redemption limit a^* is independent of the risky asset's volatility $L(t)$ and its parameters. When the financial market runs well ($\chi = 0$), then such a redemption limit is not necessary because the Quantity of Large Redemption won't appear according to our assumptions. But it should draw the fund manager's attention when the whole market is deep down*

($\chi = 1$). In this case, the redemption limit has connection with the manager's risk aversion γ, unit redemption fee η and the time value $e^{-r_0(T-t)}$. The specific influences of relevant parameters on the optimal strategy (π^*, a^*) will be demonstrated in the next section.

4 Numerical analysis

In this section, some numerical simulations are presented to illustrate the effects of parameters on the optimal redemption limit and investment strategy. Firstly, we consider the optimal redemption limit α^* in a specific market. Since the emergence of large redemption depends on the market's performance, a simulation of the financial market $L(t)$ and the corresponding trigger line ζ_0 should be given out preferentially. Considering that investors always participate in the collective investment scheme when the market environment is relatively moderate, θ and l_0 are supposed to be similar to each other, otherwise, the financial market will be expected to get a strong trend according to our numerical tests. Here we assume the market-related parameters are given by $k = 0.3$, $\zeta_0 = 0.65$, $\theta = l_0 = 1$ and $\sigma = 0.5$, and other basic parameters are listed as $\rho = 0.4$, $r_0 = 0.15$, $\lambda = 1.5$, $\eta = 0.01$, $D = 1000$, $T = 10$, $x_0 = 1$ and $\gamma = 0.5$. Then the net buying I_i follows uniform distribution $U(0, 1000)$ and Figure 1 shows the simulation of the financial market with trigger line $\zeta_0 = 0.65$. Therefore, Quantity of Large Redemption only occurs when the market $L(t)$ is deep down below the trigger line.

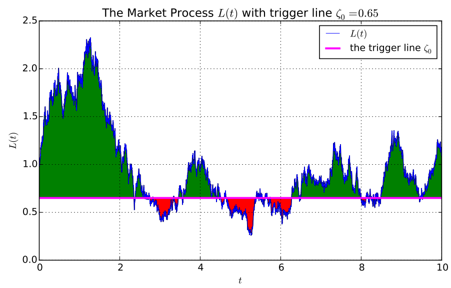

Figure 1: A simulation of the financial market with the trigger line $\zeta_0 = 0.65$.

Figure 2 illustrates the optimal redemption limit a^* corresponding to the financial market in Figure 1. As is shown in Figure 2, the magenta curve denotes the optimal redemption limit with trigger line $\zeta_0 = +\infty$, which means large redemption will appear throughout the whole period $[0, T]$. Similarly, the red part in stem form represents the optimal strategy for redemption in the case $\zeta_0 = 0.65$, and the redemption limit for the situation $\chi = 0$ (i.e., $L(t) \geqslant \zeta_0 = 0.65$)

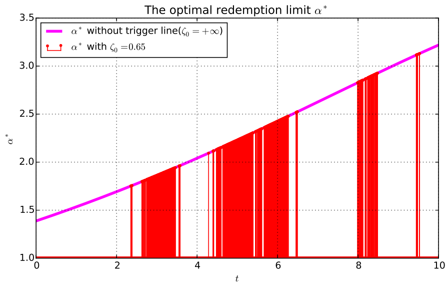

Figure 2: The optimal redemption limit a^* corresponding to Figure 1

is supposed to be the initial wealth $x_0 = 1$, though it is not that necessary. Furthermore, we notice that time t has a positive influence on the redemption limit, this can be explained by the fact that the fund's wealth/asset always increases with time (rising incomes like cumulative net buying from shareholders and investment returns), which leads to greater guarantees for liquidity risk, so higher redemption limit is approved to attract more investors.

Figure 3-4 shows the impacts of parameters on the optimal redemption limit (for ease of comparison, we assume the trigger line $\zeta_0 = +\infty$). Figure 3 indicates that the optimal redemption limit increases initially and then declines with time t when the risk aversion coefficient γ is close to 0.1 (when $\gamma = 0.1$, then $\eta D \gamma = 1$). Moreover, the peak's occurrence delays with the increase of γ. Though there is no rigid distinctions between the redemption limits with different risk aversions, the redemption limit decreases with γ increasing in most cases, which means the fund manager with higher risk aversion coefficient will get smaller tolerance of liquidity risk. From the left part of Figure 4, we find that the optimal redemption limit increases with the unit redemption fee η and the gap between two adjacent curves expends along with time t. Actually, when a fund manager wants to charge more for the same redemption, he should raise the redemption limit as compensation to maintain the old investors and attract the new ones. The right part of Figure 4 illustrates the effect of interest rate r_0 on the optimal redemption limit. As seen, the optimal redemption limit is a decreasing function of r_0 (this is partly because a higher interest rate means higher opportunity costs, then the fund manager has to lower the redemption limit to maintain a relatively stable profit) and all the curves converge to the same point at time T. Thus, the gap between two cases with different interest rates will gradually disappear along with the loss of time value.

Figure 5-8 demonstrate the sensitivity of optimal investment strategy π^* with respect to the relevant pa-

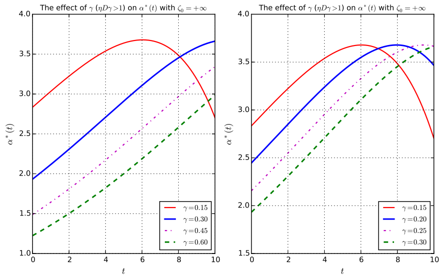

Figure 3: The effect of risk aversion coefficient γ on optimal redemption limit a^*.

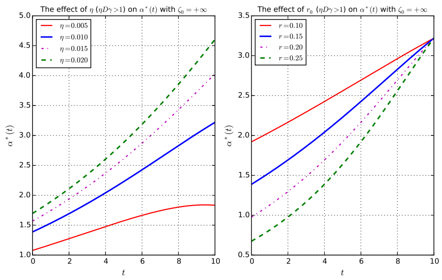

Figure 4: The effect of unit redemption fee η and interest rate r_0 on optimal redemption limit a^*.

rameters. From Figure 5, we can see the optimal investment strategy decreases with ρ when $\rho > 0$ and increases with $|\rho|$ when $\rho < 0$. As a measurement of relevancy, ρ denotes the risky asset's direction of price movement relative to the market trend. In the case of $\rho > 0$, the risky asset's price will move in the same direction of the market trend, otherwise, the two will go in the opposite directions. In consideration of the uncertainty in financial market, for a risk-neutral manager who prefers to maximize the expected monetary value, it will be rational of him/her to invest less money in the risky asset with an increasing correlation coefficient ρ when $\rho > 0$ and invest more in the risky asset as $|\rho|$ increases when $\rho < 0$.

For Heston model, k and σ are parameters of $L(t)$, which denotes the financial market conditions as well as the volatility of risky asset's price. The influences of such parameters on the optimal investment strategy are presented in Figure 6 and Figure 7. Figure 6 illustrates that the optimal investment strategy decreases with respect to σ when $\rho > 0$ and increases with σ when $\rho < 0$. For the case $\rho > 0$, the volatility of risky asset's price will fluctuate a little drastically when σ increases, which leads to a volatile volatility for the risky asset and the manager will reduce his investment in the risky asset. But a negative ρ will help to weaken the sensitivity of the risky asset's price to σ and the volatility of risky asset gets more stable, thus the fund manager prefers to invest more in the risky asset. From Figure 7, we find that k, which reflects the speed of $L(t)$ approaching to θ, has a positive effect on the optimal investment strategy when $\rho > 0$ and a negative effect when $\rho < 0$. That means a larger k brings about a more stable $L(t)$ and the fund manager would like to increase investment in risky asset for the case of $\rho > 0$. While considering the negative correlation between $S(t)$ and $L(t)$ for $\rho < 0$, it is more likely that there will be an expected decline in the risky asset's price as k increases. Therefore, the optimal investment strategy decreases with k when $\rho < 0$.

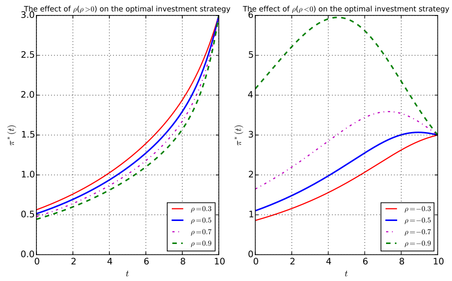

Figure 5: The effect of the correlation coefficient ρ on the optimal investment strategy π^*.

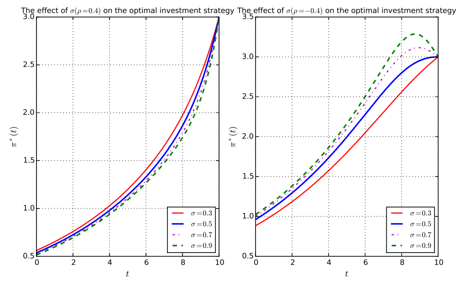

Figure 6: The effect of market-related parameter σ on the optimal investment strategy π^*.

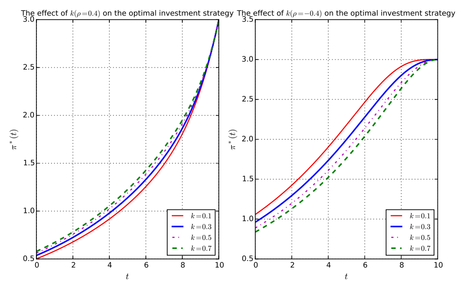

Figure 7: The effect of market-related parameter k on the optimal investment strategy π^*.

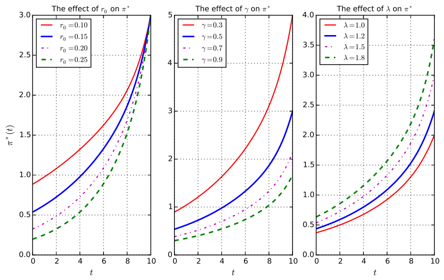

Figure 8: The effect of interest rate r_0, risk aversion coefficient γ and λ on the optimal investment strategy π^*.

The left part of Figure 8 demonstrates that the optimal investment in risky asset decreases with respect to the interest rate r_0, which consists with intuition. The manager would like to invest less in the risky asset to reduce the opportunity cost resulting from a higher interest rate. From the middle part of Figure 8, we find that the risk aversion coefficient γ has a negative influence on the optimal investment strategy. Obviously, a fund manager with larger risk aversion prefers the risk-free asset to avoid risk. Finally, as shown in the right part of Figure 8, the optimal investment strategy increases with the unit risk premium λ. The risky asset with a larger λ gains more popularity to the fact that more expected return will be obtained.

Remark 10 *It would be specially mentioned that the optimal investment strategy π^* will not be paired with the optimal redemption limit α^* once any of the market-related parameters (k, θ and σ) is altered in our numerical analysis.*

5 Conclusion

In this paper, we have considered the optimal investment and risk management strategies for open-ended funds under Heston's SV model. The surplus process of the open-ended fund is assumed to follow a jump-diffusion risk model consisted of two compound Poisson processes. We suppose there are only two assets available for investment in the financial market, a risk-free asset and a risky asset, whose price satisfies the Heston model. By applying the stochastic control approach, we obtain the optimal investment strategy and the optimal redemption limit explicitly. Furthermore, we obtain the value function and present the corresponding verification theorem. Finally, we give a numerical example to illustrate the effects of parameters on the optimal strategies.

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