Dynamics of an epidemic model with saturation recovery and delays

JUAN LIU
Bengbu University
Department of Mathematics and Physics
Caoshan Street 1866, 233030, Bengbu
CHINA
liujuan7216@163.com

CHANGJIN XU
Guizhou University of Finance and Economics
Guizhou Key Laboratory of Economics System Simulation
School of Mathematics and Statistics
Guiyang 550004
CHINA
Corresponding:xcj403@126.com

Abstract: This paper discusses the dynamical behaviors of an SIRS epidemic model with saturation recovery and two delays. The main results are given in terms of local stability and Hopf bifurcation. By choosing the diverse delay as a bifurcation parameter, we show that the complex Hopf bifurcation phenomenon at the positive equilibrium of the model can occur as the diverse delay crosses the corresponding critical value. Particularly, the direction and stability of the local Hopf bifurcation are determined by using the normal form theory and center manifold theorem. Finally, some numerical simulations supporting our theoretical results are presented.

Key–Words: Epidemic model, Delays, Hopf bifurcation, Stability, Periodic solution

1 Introduction

Infectious diseases have ranked with wars and famine as major challenges to hunman and society for centuries [1]. Many epidemic dynamical models have been proposed and used to study the dynamics of epidemics in order to understand the pathogenesis of diseases and to control the diseases [2-8]. In [2], Xiao and Ruan studied an epidemic model with nonmonotonic incidence rate and they found that either the number of infective individuals tends to zero as time evolves or the disease persists by investigating the stability of the disease-free equilibrium and the endemic equilibrium. In [5], Wang et al.proposed a HIV model in order to understand HIV dynamics and disease progression. In [7], Wan and Cui proposed the following SIR model with saturation recovery:

$$\begin{cases} \frac{dS(t)}{dt} = A - dS(t) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (d+v)I(t) - \frac{cI(t)}{b+I(t)}, \\ \frac{dR(t)}{dt} = \frac{cI(t)}{b+I(t)} - dR(t), \end{cases} \quad (1)$$

where $S(t)$, $I(t)$ and $R(t)$ denote the susceptible number, the infected number and the recovered number of individuals at time t, respectively. A is the recruitment rate of the susceptible population. d is the natural death rate of the population and v is the death rate due to the disease. β is the disease transmission coefficient. c is the maximum of treatment per unit of time and b measures how soon saturation occurs. τ is the latent period of the epidemic. All the parameters in system (1) are assumed to be positive. Wan and Cui investigated the stability and bifurcations of the endemic equilibria of system (1) in [7].

It has been recognized for a long time that time delays can have a very complicated impact on the dynamics of a dynamical system and dynamics of a dynamical system with delay have been investigated by many authors [9-15]. In [9], Zhuang and Zhu analyzed the existence of Hopf bifurcation for an improved HIV model with time delay and cure rate. In [12], Kar and Ghorai studied the existence and properties of Hopf bifurcation of delayed predator-prey model with harvesting. In [14], Bianca et al. investigated the existence of the Hopf bifurcation of an economic growth model with two delays by regarding different combination of the two delays as a bifurcation parameter. Further, they obtained the explicit formulas determining the stability, direction, and period of bifurcating periodic solutions by using the normal form theory and center manifold theorem. Motivated by the work above and considering that the recovered individuals may be infected again after a temporary immunity period, we propose the following SIRS model with two delays in this paper:

$$\begin{cases} \frac{dS(t)}{dt} = A - dS(t) - \beta S(t-\tau_1)I(t-\tau_1) \\ \qquad +\eta R(t-\tau_2), \\ \frac{dI(t)}{dt} = \beta S(t-\tau_1)I(t-\tau_1) - (d+v)I(t) \\ \qquad -\frac{cI(t)}{b+I(t)}, \\ \frac{dR(t)}{dt} = \frac{cI(t)}{b+I(t)} - dR(t) - \eta R(t-\tau_2), \end{cases} \quad (2)$$

where τ_1 is the latent period of the epidemic and τ_2 is the temporary immunity period after which a recovered individual may be infected again. η is the state transition rate between the recovered class and the susceptible class.

This paper is organized as follows. In Section 2, existence of Hopf bifurcation of system (2) are analyzed by using characteristic root method. Direction and stability of the Hopf bifurcation are determined in Section 3 by using the normal form theory and the center manifold theorem. In Section 4, computer simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2 Existence of Hopf bifurcation

It is not difficult to verify that if $R_0 = \frac{Ab\beta}{bd(d+v)+cd} > 1$, then system (2) admits a unique endemic equilibrium $E_*(S_*, I_*, R_*)$, where

$$\begin{aligned} S_* &= \frac{1}{\beta}(d+v+\frac{c}{b+I_*}), \\ I_* &= \frac{-B_* + \sqrt{B_*^2 - 4A_*C_*}}{2A_*}, \\ R_* &= \frac{cI_*}{(d+\eta)(b+I_*)}, \end{aligned}$$

with

$$\begin{aligned} A_* &= \beta(d+v)(d+\eta), \\ C_* &= (d+\eta)(bd^2 + bdv - Ab\beta), \\ B_* &= (d+\eta)(c\beta + b\beta(d+v) \\ &\quad +d(d+v) - A\beta) - c\beta\eta, \end{aligned}$$

and R_0 is the basic reproduction number.

It is easy to get the liberalization of system (2) at the positive equilibrium of system

$$\begin{cases} \frac{dS(t)}{dt} = a_1S(t) + a_5S(t-\tau_1) + a_6I(t-\tau_1) \\ \qquad +a_9R(t-\tau_2), \\ \frac{dI(t)}{dt} = a_2I(t) + a_7S(t-\tau_1) + a_8I(t-\tau_1), \\ \frac{dR(t)}{dt} = a_3I(t) + a_4R(t) + a_{10}R(t-\tau_2), \end{cases} \tag{3}$$

where

$$\begin{aligned} a_1 &= -d, a_2 = -(d+v) - \frac{bc}{(b+I_*)^2}, \\ a_3 &= \frac{bc}{(b+I_*)^2}, a_4 = -d, \\ a_5 &= -\beta I_*, a_6 = -\beta S_*, a_7 = \beta I_*, \\ a_8 &= \beta S_*, a_9 = \eta, a_{10} = -\eta. \end{aligned}$$

$$\begin{aligned} \lambda^3 &+ A_2\lambda^2 + A_1\lambda + A_0 \\ &+ (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau_1} \\ &+ (C_2\lambda^2 + C_1\lambda + C_0)e^{-\lambda\tau_2} \\ &+ (D_1\lambda + D_0)e^{-\lambda(\tau_1+\tau_2)} \\ &+ (E_1\lambda + E_0)e^{-\lambda 2\tau_1} \\ &+ F_0e^{-\lambda(2\tau_1+\tau_2)} = 0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_0 &= -a_1a_2a_4, A_1 = a_1a_2 + a_1a_4 + a_2a_4, \\ A_2 &= -(a_1+a_2+a_4), B_0 = -(a_1a_8 + a_2a_5)a_4, \\ B_1 &= a_1a_8 + a_2a_5 + (a_5+a_8)a_4, \\ B_2 &= -(a_5+a_8), C_0 = -a_1a_2a_{10}, \\ C_1 &= (a_1+a_2)a_{10}, C_2 = -a_{10}, \\ D_0 &= -(a_3a_7a_9 + a_1a_8a_{10} + a_2a_5a_{10}), \\ D_1 &= (a_5+a_8)a_{10}, E_0 = (a_6a_7 - a_5a_8)a_4, \\ E_1 &= a_5a_8 - a_6a_7, F_0 = (a_6a_7 - a_5a_8)a_{10}. \end{aligned}$$

From the expressions of a_5, a_6, a_7 and a_8, we know that $a_5a_8 = a_6a_7$. Thus, Eq.(4) becomes the following form

$$\begin{aligned} \lambda^3 &+ A_2\lambda^2 + A_1\lambda + A_0 \\ &+ (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau_1} \\ &+ (C_2\lambda^2 + C_1\lambda + C_0)e^{-\lambda\tau_2} \\ &+ (D_1\lambda + D_0)e^{-\lambda(\tau_1+\tau_2)}. \end{aligned} \tag{5}$$

Case 1. $\tau_1 = \tau_2 = 0$.

When $\tau_1 = \tau_2 = 0$, Eq.(5) becomes

$$\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \tag{6}$$

where

$$\begin{aligned} A_{10} &= A_0 + B_0 + C_0 + D_0, \\ A_{11} &= A_1 + B_1 + C_1 + D_1, \\ A_{12} &= A_2 + B_2 + C_2. \end{aligned}$$

If the condition (H_{11}) $A_{12} > 0$, $A_{12}A_{11} > A_{10}$ holds, all the roots of Eq.(6) must have negative real parts. Therefore, the positive equilibrium E_* is locally asymptotically stable when $\tau_1 = \tau_2 = 0$ if the condition (H_{11}) holds.

Case2. $\tau_1 > 0, \tau_2 = 0$.

For $\tau_1 > 0, \tau_2 = 0$, Eq.(4) can be rewritten as following

$$\begin{aligned} \lambda^3 &+ A_{22}\lambda^2 + A_{21}\lambda + A_{20} \\ &+ (B_{22}\lambda^2 + B_{21}\lambda + B_{20})e^{-\lambda\tau_1} = 0, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_{20} &= A_0 + C_0, A_{21} = A_1 + C_1, \\ A_{22} &= A_2 + C_2, B_{20} = B_0 + D_0, \\ B_{21} &= B_1 + D_1, B_{22} = B_2. \end{aligned}$$

Let $\lambda = i\omega_1 (\omega_1 > 0)$ be the root of Eq.(7). Then we obtain

$$\begin{cases} M_{21}(\omega_1)\sin\tau_1\omega_1 + M_{22}(\omega_1)\cos\tau_1\omega_1 = M_{23}(\omega_1), \\ M_{21}(\omega_1)\cos\tau_1\omega_1 - M_{22}(\omega_1)\sin\tau_1\omega_1 = M_{24}(\omega_1), \end{cases}$$

where

$$\begin{aligned} M_{21}(\omega_1) &= B_{21}\omega_1, \\ M_{22}(\omega_1) &= (B_{20} - B_{22}\omega_1^2), \\ M_{23}(\omega_1) &= A_{22}\omega_1^2 - A_{20}, \\ M_{24}(\omega_1) &= \omega_1^3 - A_{21}\omega_1. \end{aligned}$$

Then, we can obtain

$$\omega_1^6 + m_{22}\omega_1^4 + m_{21}\omega_1^2 + m_{20} = 0, \tag{8}$$

where

$$\begin{aligned} m_{20} &= A_{20}^2 - B_{20}^2, \\ m_{21} &= A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}, \\ m_{22} &= A_{22}^2 - B_{22}^2 - 2A_{21}. \end{aligned}$$

Let $\omega_1^2 = v_1$, then Eq.(8) becomes

$$v_1^3 + m_{22}v_1^2 + m_{21}v_1 + m_{20} = 0. \tag{9}$$

Define

$$f_1(v_1) = v_1^3 + m_{22}v_1^2 + m_{21}v_1 + m_{20}.$$

Discussion about the roots of Eq.(9) is similar to that in [16], so we have the following lemma.

Lemma 1 *For the Eq.(9)*

(i) If $m_{20} < 0$, then Eq.(9) has at least one positive root;

(ii) If $m_{20} \geq 0$ and $\Delta_1 = m_{22}^2 - 3m_{21} \leq 0$, then Eq.(9) has no positive roots;

(iii)If $m_{20} \geq 0$ and $\Delta_1 = m_{22}^2 - 3m_{21} > 0$, then Eq.(9) has positive roots if and only if $v_1^* = \frac{-m_{22}+\sqrt{\Delta_1}}{3}$ and $f_1(v_1^*) \leq 0$.

In what follows, we assume that the coefficients in $f_1(v_1)$ satisfy the condition

(H_{21}) (a): $m_{20} < 0$ or (b): $m_{20} \geq 0$, $\Delta_1 > 0$, $v_1^* > 0$ and $f_1(v_1^*) \leq 0$.

If the condition (H_{21}) holds, then Eq.(9) has at least one positive root. Without loss of generality, we assume that Eq.(9) has three positive roots which are denoted as v_{11}, v_{12} and v_{13}, respectively. Then, Eq.(8) has tree positive roots $\sqrt{\omega_{1k}}$, $k = 1, 2, 3$. And for every fixed ω_{1k},

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} \arccos\frac{g_{21}(\omega_{1k})}{h_{21}(\omega_{1k})} + \frac{2j\pi}{\omega_{1k}},$$

where

$$\begin{aligned} g_{21}(\omega_{1k}) &= (B_{21} - A_{22}B_{22})\omega_{1k}^4 \\ &\quad +(A_{20}B_{22} + A_{22}B_{20} - A_{21}B_{21})\omega_{1k}^2 \\ &\quad -A_{20}B_{20}, \\ h_{21}(\omega_{1k}) &= (B_{22}\omega_{1k}^2 - B_{20})^2 + B_{21}^2\omega_{1k}^2. \end{aligned}$$

$$k = 1, 2, 3. j = 0, 1, 2, \cdots.$$

Define

$$\tau_{10} = \min\{\tau_{1k}^{(0)}\}, \omega_{10} = \omega_{1k}|_{\tau_1=\tau_{10}}, k = 1, 2, 3.$$

Let $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ be a root of Eq.(7) near $\tau_1 = \tau_{10}$ such that $\alpha(\tau_{10}) = 0$, $\omega(\tau_{10}) = \omega_{10}$. Next, we verify the transversality condition. Substituting $\lambda(\tau_1)$ into the left side of Eq.(7) and taking the derivative with respect to τ_1, we get

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1}\right]^{-1} &= -\frac{3\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda(\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20})} \\ &\quad + \frac{2B_{22}\lambda + B_{21}}{\lambda(B_{22}\lambda^2 + B_{21}\lambda + B_{20})} - \frac{\tau_1}{\lambda}. \end{aligned}$$

Thus, we have

$$Re\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} = \frac{f_1'(v_{1*})}{B_{21}^2\omega_{10}^2 + (B_{22}\omega_{10}^2 - B_{20})^2},$$

where $v_{1*} = \omega_{10}^2$.

Thus, if (H_{22}) $f_1'(v_{1*}) \neq 0$, then $Re[\frac{d\lambda}{d\tau_1}]^{-1}_{\tau_1=\tau_{10}} \neq 0$. According to the Hopf bifurcation theorem in [17], we have the following results.

Theorem 2 *If the conditions H_{11}, (H_{21}) and (H_{22}) hold, the positive equilibrium E_* of system (2) is asymptotically stable for $\tau_1 \in [0, \tau_{10})$. System (2) undergoes a Hopf bifurcation at $\tau_1 = \tau_{10}$, which means that a branch of periodic solutions will bifurcate from E_* as τ_1 passes through the critical value τ_{10}.*

The results in Theorem 2 show that the latent period delay plays a complicated role in system (2) and it is responsible for the stability switch of the system when $\tau_2 = 0$.

Case 3. $\tau_1 = 0, \tau_2 > 0$.

When $\tau_1 = 0, \tau_2 > 0$, Eq.(5) can be transformed into the following form

$$\begin{aligned}\lambda^3 &+ A_{32}\lambda^2 + A_{31}\lambda + A_{30}\\ &+ (B_{32}\lambda^2 + B_{31}\lambda + B_{30})e^{-\lambda\tau_2} = 0, \quad (10)\end{aligned}$$

where

$$\begin{aligned}A_{30} &= A_0 + B_0, A_{31} = A_1 + B_1,\\ A_{32} &= A_2 + B_2, B_{30} = C_0 + D_0,\\ B_{31} &= C_1 + D_1, B_{32} = C_2.\end{aligned}$$

Let $\lambda = i\omega_2(\omega_2 > 0)$ be the root of Eq.(10). Then, we get

$$\begin{cases} M_{31}(\omega_2)\cos\tau_2\omega_2 - M_{32}(\omega_2)\sin\tau_2\omega_2 = M_{33},\\ M_{31}(\omega_2)\sin\tau_2\omega_2 + M_{32}(\omega_2)\cos\tau_2\omega_2 = M_{34},\end{cases}$$

where

$$\begin{aligned}M_{31}(\omega_2) &= B_{31}\omega_2,\\ M_{32}(\omega_2) &= B_{30} - B_{32}\omega_2^2,\\ M_{33}(\omega_2) &= \omega_2^3 - A_{31}\omega_2,\\ M_{34}(\omega_2) &= A_{32}\omega_2^2 - A_{30}.\end{aligned}$$

Then, we can obtain

$$\omega_2^6 + m_{32}\omega_2^4 + m_{31}\omega_2^2 + m_{30} = 0, \quad (11)$$

with

$$\begin{aligned}m_{30} &= A_{30}^2 - B_{30}^2,\\ m_{31} &= A_{31}^2 - B_{31}^2 - 2A_{30}A_{32} + 2B_{30}B_{32},\\ m_{32} &= A_{32}^2 - B_{32}^2 - 2A_{31}.\end{aligned}$$

Let $\omega_2^2 = v_2$, then Eq.(11) becomes

$$v_2^3 + m_{32}v_2^2 + m_{31}v_2 + m_{30} = 0. \quad (12)$$

Define

$$f_2(v_2) = v_2^3 + m_{32}v_2^2 + m_{31}v_2 + m_{30}.$$

Similar as in **Case 2**, we assume that the coefficients in $f_2(v_2)$ satisfy the condition (H_{31}) (a'): $m_{30} < 0$ or (b'): $m_{30} \geq 0$, $\Delta_2 > 0$, $v_2^* > 0$ and $f_2(v_2^*) \leq 0$, where $\Delta_2 = m_{32}^2 - 3m_{31}$ and $v_2^* = \frac{-m_{32}+\sqrt{\Delta_2}}{3}$.

If the condition (H_{31}) holds, then Eq.(12) has at least one positive root. Similar as in **Case 2**, we assume that Eq.(12) has three positive roots which are denoted as v_{21}, v_{22} and v_{23}, respectively. Then, Eq.(11) has tree positive roots $\omega_{2k} = \sqrt{v_{2k}}$, $k = 1, 2, 3$. For every fixed ω_{2k},

$$\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}}\arccos\frac{g_{31}(\omega_{2k})}{h_{31}(\omega_{2k})} + \frac{2j\pi}{\omega_{2k}},$$

where

$$\begin{aligned}g_{31}(\omega_{2k}) &= (B_{31} - A_{32}B_{32})\omega_{2k}^4 + (A_{30}B_{32} + A_{32}B_{30}\\ &\quad -A_{31}B_{31},\\ h_{31}(\omega_{2k}) &= (B_{32}\omega_{2k}^2 - B_{30})^2 + B_{31}^2\omega_{1k}^2,\end{aligned}$$

$$k = 1, 2, 3. j = 0, 1, 2, \cdots.$$

Define

$$\tau_{20} = min\{\tau_{2k}^{(0)}\}, \omega_{20} = \omega_{2k}|_{\tau_2=\tau_{20}}, k = 1, 2, 3.$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of Eq.(10) near $\tau_2 = \tau_{20}$ such that $\alpha(\tau_{20}) = 0$, $\omega(\tau_{20}) = \omega_{20}$. Similar as in **Case 2**, we can conclude that if (H_{32}) $f_2'(v_{2*}) \neq 0 (v_{2*} = \omega_{20}^2)$ holds, then $Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\tau_2=\tau_{20}} \neq 0$. Thus, we have the following results.

Theorem 3 *If the conditions (H_{11}), (H_{31}) and (H_{32}) hold, the positive equilibrium E_* of system (2) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$. System (2) undergoes a Hopf bifurcation at $\tau_2 = \tau_{20}$, which means that a branch of periodic solutions will bifurcate from E_* as τ_2 passes through the critical value τ_{20}.*

The results in Theorem 3 show that the time delay due to the temporary immunity period can also play a complicated role in system (2) and it is responsible for the stability switch of the system when $\tau_1 = 0$.

Case 4. $\tau_1 = \tau_2 = \tau > 0$.
When $\tau_1 = \tau_2 = \tau > 0$, Eq.(5) becomes

$$\begin{aligned}\lambda^3 &+ A_{42}\lambda^2 + A_{41}\lambda + A_{40}\\ &+ (B_{42}\lambda^2 + B_{41}\lambda + B_{40})e^{-\lambda\tau}\\ &+ (C_{41}\lambda + C_{40})e^{-2\lambda\tau} = 0, \quad (13)\end{aligned}$$

where

$$\begin{aligned}A_{40} &= A_0, A_{41} = A_1,\\ A_{42} &= A_2, B_{40} = B_0 + C_0,\\ B_{41} &= B_1 + C_1, B_{42} = B_2 + C_2,\\ C_{41} &= D_1, C_{40} = D_0.\end{aligned}$$

Multiplying $e^{\lambda\tau}$ on both sides of Eq.(13), we get

$$\begin{aligned}B_{42}\lambda^2 &+ B_{41}\lambda + B_{40}\\ &+ (\lambda^3 + A_{42}\lambda^2 + A_{41}\lambda + A_{40})e^{\lambda\tau}\\ &+ (C_{41}\lambda + C_{40})e^{-\lambda\tau} = 0. \quad (14)\end{aligned}$$

Let $\lambda = i\omega(\omega > 0)$ be the root of Eq.(14). Then, we can obtain

$$\begin{cases} M_{41}(\omega)\cos\tau\omega - M_{42}(\omega)\sin\tau\omega = M_{43}(\omega),\\ M_{44}(\omega)\sin\tau\omega + M_{45}(\omega)\cos\tau\omega = M_{46}(\omega),\end{cases}$$

where

$$\begin{aligned} M_{41}(\omega) &= A_{40}+C_{40}-A_{42}\omega^2, \\ M_{42}(\omega) &= (A_{41}\omega - C_{41}\omega - \omega^3), \\ M_{43}(\omega) &= B_{42}\omega^2 - B_{40}, \\ M_{44}(\omega) &= (A_{40}-C_{40}-A_{42}\omega^2), \\ M_{45}(\omega) &= (A_{41}\omega + C_{41}\omega - \omega^3), \\ M_{46}(\omega) &= -B_{41}\omega. \end{aligned}$$

It follows that

$$\sin\tau\omega = \frac{p_5\omega^5+p_3\omega^3+p_1\omega}{\omega^6+q_4\omega^4+q_2\omega^2+q_0},$$

$$\cos\tau\omega = \frac{p_4\omega^4+p_2\omega^2+p_0}{\omega^6+q_4\omega^4+q_2\omega^2+q_0},$$

where

$$\begin{aligned} p_0 &= (C_{40}-A_{40})B_{40}, \\ p_1 &= (A_{41}+C_{41})B_{40}-(A_{40}+C_{40})B_{41}, \\ p_2 &= (A_{40}-C_{40})B_{42}+(C_{41}-A_{41})B_{41} \\ &\quad +A_{42}B_{40}, \\ p_3 &= A_{42}B_{41}-B_{40}-(A_{41}+C_{41})B_{42}, \\ p_4 &= B_{41}-A_{42}B_{42}, p_5 = B_{42}, \\ q_0 &= A_{40}^2-C_{40}^2, q_2 = A_{41}^2-C_{41}^2-2A_{40}A_{42}, \\ q_4 &= A_{42}^2-2A_{41}. \end{aligned}$$

By $\sin^2\tau\omega+\cos^2\tau\omega=1$, we get

$$\begin{aligned} \omega^{12} &+ m_{45}\omega^{10}+m_{44}\omega^8+m_{43}\omega^6 \\ &+ m_{42}\omega^4+m_{41}\omega^2+m_{40}=0, \end{aligned} \quad (15)$$

where

$$\begin{aligned} m_{40} &= q_0^2-p_0^2, \\ m_{41} &= 2q_0q_2-2p_0p_2-p_1^2, \\ m_{42} &= q_2^2+2q_0q_4-p_2^2-2p_0p_4-2p_1p_3, \\ m_{43} &= 2q_0+2q_2q_4-2p_1p_5-2p_2p_4-p_3^2, \\ m_{44} &= q_4^2+2q_2-p_4^2-2p_3p_5, \\ m_{45} &= 2q_4-p_5^2. \end{aligned}$$

Let $\omega^2=v_3$, then Eq.(15) becomes

$$\begin{aligned} v_3^6 &+ m_{45}v_3^5+m_{44}v_3^4+m_{43}v_3^3 \\ &+ m_{42}v_3^2+m_{41}v_3+m_{40}=0. \end{aligned} \quad (16)$$

In order to give the main results in this paper, we make the following assumption.

(H_{41}) Eq.(16) has at least one positive real root.

Suppose that the condition (H_{41}) holds. Without loss of generality, we assume that Eq.(16) has six positive real roots, which are denoted as $v_{31}, v_{32}, \cdots, v_{36}$, respectively. Then Eq.(15) has six positive real roots $\omega_k=\sqrt{v_{3k}}$, $k=1,2,\cdots,6$. For every fixed ω_k,

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arccos\frac{p_4\omega_k^4+p_2\omega_k^2+p_0}{\omega_k^6+q_4\omega_k^4+q_2\omega_k^2+q_0}+\frac{2j\pi}{\omega_k},$$

with $k=1,2,\cdots,6. j=0,1,2,\cdots$.

Define

$$\tau_0 = min\{\tau_k^{(0)}\}, \omega_0=\omega_k|_{\tau=\tau_0}, k=1,2,\cdots,6.$$

Let $\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$ be a root of Eq.(14) near $\tau=\tau_0$ such that $\alpha(\tau_0)=0, \omega(\tau_0)=\omega_0$. Substituting $\lambda(\tau)$ into the left side of Eq.(14) and taking the derivative with respect to τ, we get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{p_{41}(\lambda)}{q_{41}(\lambda)}-\frac{\tau}{\lambda},$$

with

$$\begin{aligned} p_{41}(\lambda) &= 2B_{42}\lambda+B_{41} \\ &\quad +(3\lambda^2+2A_{42}\lambda+A_{41})e^{\lambda\tau}+C_{41}e^{-\lambda\tau}, \\ q_{41}(\lambda) &= (C_{41}\lambda^2+C_{40}\lambda)e^{-\lambda\tau} \\ &\quad -(\lambda^4+A_{42}\lambda^3+A_{41}\lambda^2+A_{40}\lambda)e^{\lambda\tau}. \end{aligned}$$

Thus, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{P_{4R}Q_{4R}+P_{4I}Q_{4I}}{Q_{4R}^2+Q_{4I}^2},$$

where

$$\begin{aligned} P_{4R} &= (A_{41}+C_{41}-3\omega_0^2)\cos\tau_0\omega_0 \\ &\quad -2A_{42}\omega_0\sin\tau_0\omega_0+B_{41}, \\ P_{4I} &= (A_{41}-C_{41}-3\omega_0^2)\sin\tau_0\omega_0 \\ &\quad +2A_{42}\omega_0\cos\tau_0\omega_0 2+B_{42}\omega_0, \\ Q_{4R} &= (A41\omega_0^2-C_{41}\omega_0^2-\omega_0^4)\cos\tau_0\omega_0 \\ &\quad -(A_{42}\omega_0^3-A_{40}\omega_0-C_{40}\omega_0)\sin\tau_0\omega_0, \\ Q_{4I} &= (A_{41}\omega_0^2+C_{41}\omega_0^2-\omega_0^4)\sin\tau_0\omega_0 \\ &\quad +(A_{42}\omega_0^3-A_{40}\omega_0+C_{40}\omega_0)\cos\tau_0\omega_0. \end{aligned}$$

Obviously, if (H_{42}) $P_{4R}Q_{4R}+P_{4I}Q_{4I}\neq 0$, then $Re[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau_0}\neq 0$. Thus, we have the following results.

Theorem 4 *If the conditions (H_{11}), (H_{41}) and (H_{42}) hold, the positive equilibrium E_* of system (2) is asymptotically stable for $\tau\in[0,\tau_0)$. System (2) undergoes a Hopf bifurcation at $\tau=\tau_0$, which means that a branch of periodic solutions will bifurcate from E_* as τ passes through the critical value τ_0.*

Theorem 4 shows that the time delay τ is vital to the solutions of system (2) and it can establish the existence of bifurcating periodic solutions.

Case 5. $\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$.

We consider Eq.(5) with τ_1 in its stable interval and τ_2 is considered as a parameter. Let $\lambda = i\omega_{2*}(\omega_{2*} > 0)$ be the root of Eq.(5). Then we have

$$\begin{cases} M_{51}(\omega_{2*})\sin\tau_2\omega_{2*} + M_{52}(\omega_{2*})\cos\tau_2\omega_{2*} \\ \qquad = M_{53}(\omega_{2*}), \\ M_{51}(\omega_{2*})\cos\tau_2\omega_{2*} - M_{52}(\omega_{2*})\sin\tau_2\omega_{2*} \\ \qquad = M_{54}(\omega_{2*}), \end{cases}$$

where

$$\begin{aligned} M_{51}(\omega_{2*}) &= D_1\omega_{2*}\cos\tau_1\omega_{2*} \\ &\quad -D_0\sin\tau_1\omega_{2*} + C_1\omega_{2*}, \\ M_{52}(\omega_{2*}) &= D_1\omega_{2*}\sin\tau_1\omega_{2*} \\ &\quad +D_0\cos\tau_1\omega_{2*} + C_0 - C_2\omega_{2*}^2, \\ M_{53}(\omega_{2*}) &= A_2\omega_{2*}^2 - A_0 - B_1\omega_{2*}\sin\tau_1\omega_{1*} \\ &\quad -(B_0 - B_2\omega_{2*}^2)\cos\tau_1\omega_{2*}, \\ M_{54}(\omega_{2*}) &= \omega_{2*}^3 - A_1\omega_{2*} - B_1\omega_{2*}\cos\tau_1\omega_{2*} \\ &\quad +(B_0 - B_2\omega_{2*}^2)\sin\tau_1\omega_{2*}. \end{aligned}$$

Then, we have

$$\begin{aligned} \omega_{2*}^6 &+ g_{14}\omega_{2*}^4 + g_{12}\omega_{2*}^2 + g_{10} \\ &+ 2(g_{24}\omega_{2*}^4 \\ &+ g_{22}\omega_{2*}^2 + g_{20})\cos\tau_1\omega_{2*} \\ &- 2(g_{25}\omega_{2*}^5 \\ &+ g_{23}\omega_{2*}^3 + g_{21}\omega_{2*})\sin\tau_1\omega_{2*}. \end{aligned} \tag{17}$$

where

$$\begin{aligned} g_{10} &= A_0^2 + B_0^2 - C_0^2 - D_0^2, \\ g_{12} &= A_1^2 + B_1^2 - C_1^2 - D_1^2 \\ &\quad -2A_0A_2 - 2B_0B_2 + 2C_0C_2, \\ g_{14} &= A_2^2 + B_2^2 - C_2^2 - 2A_1, \\ g_{20} &= A_0B_0 - C_0D_0, \\ g_{21} &= A_1B_0 + C_0D_1 - A_0B_1 - C_1D_0, \\ g_{22} &= A_1B_1 - A_0B_2 - A_2B_0 - C_1D_1 \\ &\quad +C_2D_0, \\ g_{23} &= A_2B_1 - A_1B_2 - C_2D_1 - B_0, \\ g_{24} &= A_2B_2 - B_1, g_{25} = B_2. \end{aligned}$$

Next, we suppose that (H_{51}) Eq.(17) has at least finite positive root. We denote the positive roots of Eq.(17) as $\omega_{21*}, \omega_{22*}, \cdots, \omega_{2k*}$. For every ω_{2i*}, $i = 1, 2, \cdots, k$,

$$\tau_{2i*}^{(j)} = \frac{1}{\omega_{2i*}}\arccos\frac{g_{51}(\omega_{2i*})}{g_{52}(\omega_{2i*})} + \frac{2j\pi}{\omega_{2i*}},$$

with

$$\begin{aligned} g_{51}(\omega_{2i*}) &= M_{51}(\omega_{2i*}) \times M_{54}(\omega_{2i*}) \\ &\quad +M_{52}(\omega_{2i*}) \times M_{53}(\omega_{2i*}), \\ g_{52}(\omega_{2i*}) &= M_{51}^2(\omega_{2i*}) + M_{52}^2(\omega_{2i*}). \end{aligned}$$

Define

$$\tau_2^* = min\{\tau_{2i*}^{(0)}\}, \omega_2^* = \omega_{2i*}|_{\tau_2=\tau_2^*}, i = 1, 2, \cdots, k.$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of Eq.(5) near $\tau_2 = \tau_2^*$ such that $\alpha(\tau_2^*) = 0, \omega(\tau_2^*) = \omega_2^*$. Substituting $\lambda(\tau_2)$ into the left side of Eq.(5) and taking the derivative with respect to τ_2, we get

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{p_{51}(\lambda)}{q_{51}(\lambda)} - \frac{\tau_1}{\lambda},$$

where

$$\begin{aligned} p_{51}(\lambda) &= 3\lambda^2 + 2A_2\lambda + A_1 \\ &\quad -(\tau_1B_2\lambda^2 - (2B_2 - \tau_1B_1)\lambda \\ &\quad +\tau_1B_0 - B_1)e^{-\lambda\tau_1} \\ &\quad +(2C_2\lambda + C_1)e^{-\lambda\tau_2} \\ &\quad +(D_1 - \tau_1D_1\lambda - \tau_1D_0)e^{-\lambda(\tau_1+\tau_2)}, \\ q_{51}(\lambda) &= (C_2\lambda^3 + C_1\lambda^2 + C_0\lambda)e^{-\lambda\tau_1} \\ &\quad +(D_1\lambda^2 + D_0\lambda)e^{-\lambda(\tau_1+\tau_2)}. \end{aligned}$$

Therefore,

$$Re\left[\frac{d\lambda}{d\tau_2}\right]_{\tau_2=\tau_2^*}^{-1} = \frac{P_{5R}Q_{5R} + P_{5I}Q_{5I}}{Q_{5R}^2 + Q_{5I}^2},$$

where

$$\begin{aligned} P_{5R} &= (2C_2\omega_2^* - (D_1 - \tau_1D_0)\sin\tau_1\omega_2^* \\ &\quad -\tau_1D_1\omega_2^*\cos\tau_1\omega_2^*)\sin\tau_2^*\omega_2^* \\ &\quad +(C_1 + (D_1 - \tau_1D_0)\cos\tau_1\omega_2^* \\ &\quad -\tau_1D_1\omega_2^*\sin\tau_1\omega_2^*)\cos\tau_2^*\omega_2^*, \\ &\quad +A_1 - 3(\omega_2^*)^2 + (2B_2 - \tau_1B_1)\omega_1^*\sin\tau_1\omega_2^* \\ &\quad +(\tau_1B_2(\omega_1^*)^2 + B_1 - \tau_1B_0)\cos\tau_1\omega_2^* \\ P_{5I} &= (2C_2\omega_2^* - (D_1 - \tau_1D_0)\sin\tau_1\omega_2^* \\ &\quad -\tau_1D_1\omega_2^*\cos\tau_1\omega_2^*)\cos\tau_2^*\omega_2^* \\ &\quad -(C_1 + (D_1 - \tau_1D_0)\cos\tau_1\omega_2^* \\ &\quad -\tau_1D_1\omega_2^*\sin\tau_1\omega_2^*)\sin\tau_2^*\omega_2^*, \\ &\quad +2A_2\omega_2^* + (2B_2 - \tau_1B_1)\cos\tau_1\omega_2^* \\ &\quad -(\tau_1B_2(\omega_2^*)^2 + B_1 - \tau_1B_0)\sin\tau_1\omega_2^*, \\ Q_{5R} &= (C_0\omega_2^* - C_2(\omega_2^*)^3 + D_1(\omega_1^*)^2\sin\tau_1\omega_2^* \\ &\quad +D_0\omega_2^*\cos\tau_1\omega_2^*)\sin\tau_2^*\omega_2^* \\ &\quad -(C_1(\omega_2^*)^2 + D_1(\omega_2^*)^2\cos\tau_1\omega_2^* \end{aligned}$$

$$\begin{aligned}
& \quad -D_0\omega_2^* \sin\tau_1\omega_2^*)\cos\tau_2^*\omega_2^*, \\
Q_{5I} &= (C_0\omega_2^* - C_2(\omega_2^*)^3 + D_1(\omega_1^*)^2 \sin\tau_1\omega_2^* \\
& \quad +D_0\omega_2^* \cos\tau_1\omega_2^*)\cos\tau_2^*\omega_2^* \\
& \quad -(C_1(\omega_2^*)^2 + D_1(\omega_2^*)^2 \cos\tau_1\omega_2^* \\
& \quad -D_0\omega_2^* \sin\tau_1\omega_2^*)\sin\tau_2^*\omega_2^*.
\end{aligned}$$

Thus, if (H_{52}) $P_{5R}Q_{5R}+P_{5I}Q_{5I} \neq 0$ holds, then $Re[\frac{d\lambda}{d\tau_2}]^{-1}_{\tau_2=\tau_2^*} \neq 0$. Therefore, we have the following results.

Theorem 5 *If the conditions (H_{11}), (H_{51}) and (H_{52}) hold and $\tau_1 \in (0,\tau_{10})$, the positive equilibrium E_* of system (2) is asymptotically stable for $\tau_2 \in [0,\tau_2^*)$. System (2) undergoes a Hopf bifurcation at $\tau_2 = \tau_2^*$, which means that a branch of periodic solutions will bifurcate from E_* as τ_2 passes through the critical value τ_2^*.*

The results in Theorem 5 can establish the existence of bifurcating periodic solutions when $\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \in (0,\tau_{10})$. Next, we determine the direction of the Hopf bifurcation and stability of periodic solutions in this case by following the algorithm in [17].

3 Direction and stability of the Hopf bifurcation

In this section, we investigate the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of system (2) with respect to τ_2 for $\tau_1 \in (0,\tau_{10})$ by using the normal form method and center manifold theorem introduced by Hassard et al [17]. It is considered that system (2) undergoes a Hopf bifurcation at $\tau_2 = \tau_2^*$, $\tau_1 \in (0,\tau_{10})$. Without loss of generality, we assume that $\tau_2^* > \tau_1^*$, where $\tau_1^* \in (0,\tau_{10})$.

Let $\tau_2 = \tau_2^* + \mu, \mu \in R$, then $\mu = 0$ is the Hopf bifurcation value of system (2). Rescaling the time delay $t \to (t/\tau_2)$, then system (2) can be transformed into a functional differential equation in $C = C([-1,0],R^3)$ as:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t) \tag{18}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in C = C([-1,0],R^3)$ and $L_\mu : C \to R^3, F : R\times C \to R^3$ are given, respectively, by

$$L_\mu\phi = (\tau_2^*+\mu)(M_1\phi(0)+M_2\phi(-\frac{\tau_1^*}{\tau_2^*})+M_3\phi(-1)),$$

$$F(\mu,\phi) = (\tau_2^*+\mu)(F_1,F_2,F_3)^T,$$

where $\phi(\theta) = (\phi_1(\theta),\phi_2(\theta),\phi_3(\theta))^T \in C$,

$$M_1 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & a_4 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} a_5 & a_6 & 0 \\ a_7 & a_8 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & a_9 \\ 0 & 0 & 0 \\ 0 & 0 & a_{10} \end{pmatrix},$$

$$\begin{aligned}
F_1 &= a_{21}\phi_1(-\frac{\tau_1^*}{\tau_2^*})\phi_2(-\frac{\tau_1^*}{\tau_2^*}), \\
F_2 &= b_{21}\phi_1(-\frac{\tau_1^*}{\tau_2^*})\phi_2(-\frac{\tau_1^*}{\tau_2^*}) + b_{22}\phi_2^2(0) \\
& \quad +b_{23}\phi_2^3(0) + \cdots, \\
F_3 &= c_{21}\phi_2^2(0) + c_{22}\phi_2^3(0) + \cdots,
\end{aligned}$$

with

$$\begin{aligned}
a_{21} &= -\beta, b_{21} = \beta, \\
b_{22} &= \frac{bc}{(b+I_*)^3}, b_{23} = -\frac{bc}{(b+I_*)^4}, \\
c_{21} &= -\frac{bc}{(b+I_*)^3}, c_{22} = \frac{bc}{(b+I_*)^4}.
\end{aligned}$$

By the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu), \theta \in [-1,0]$ whose components are of bounded variation, such that

$$L_\mu\phi = \int_{-1}^0 d\eta(\theta,\mu)\phi(\theta), \phi \in C([-1,0],R^3).$$

In fact, we choose

$$\eta(\theta,\mu) = \begin{cases} (\tau_2^*+\mu)(M_1+M_2+M_3), \theta = 0, \\ (\tau_2^*+\mu)(M_2+M_3), \theta \in [-\frac{\tau_1^*}{\tau_2^*},0), \\ (\tau_2^*+\mu)M_3, \theta \in (-1,-\frac{\tau_1^*}{\tau_2^*}), \\ 0, \theta = -1. \end{cases}$$

For $\phi \in C([-1,0],R^3)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^0 d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (18) can be transformed into the following operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{19}$$

The adjoint operator A^* of A is defined by

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases}$$

associated with a bilinear form

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta) d\eta(\theta)\phi(\xi)d\xi, \tag{20}$$

where $\eta(\theta) = \eta(\theta, 0)$.

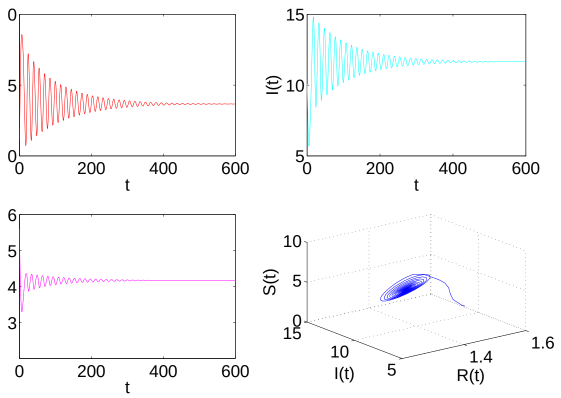

Figure 1: E_* is locally asymptotically stable for $\tau_1 = 3.525 < \tau_{10} = 3.6108$.

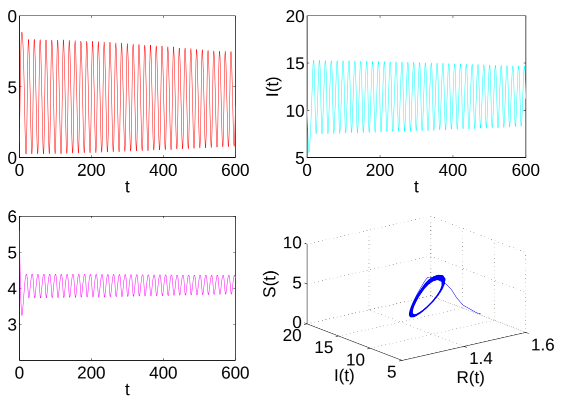

Figure 2: E_* is unstable for $\tau_1 = 3.715 > \tau_{10} = 3.6108$.

From the analysis above, we can conclude that $\pm i\tau_2^*\omega_2^*$ are eigenvalues of $A(0)$ and $A^*(0)$.

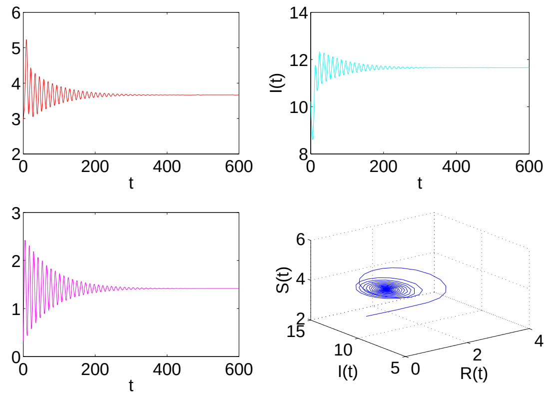

Figure 3: E_* is locally asymptotically stable for $\tau_2 = 3.05 < \tau_{20} = 3.1676$.

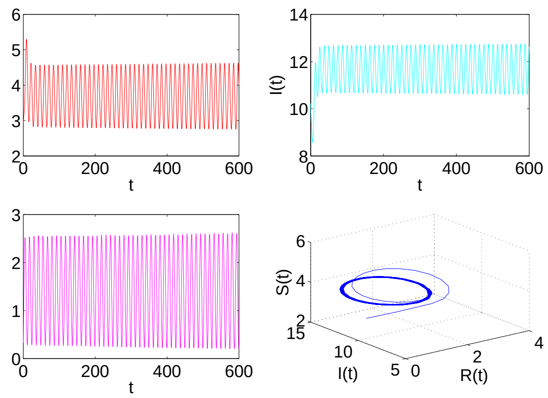

Figure 4: E_* is unstable for $\tau_2 = 3.225 > \tau_{20} = 3.1676$.

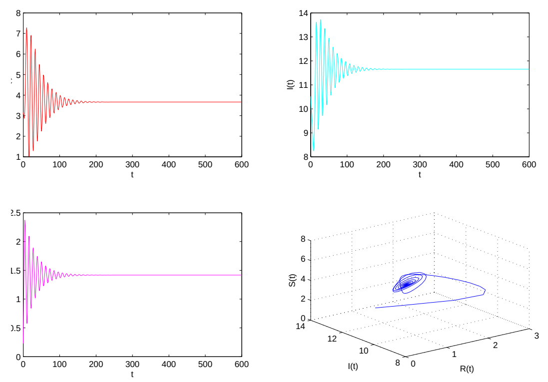

Figure 5: E_* is locally asymptotically stable for $\tau = 2.74 < \tau_0 = 3.0107$.

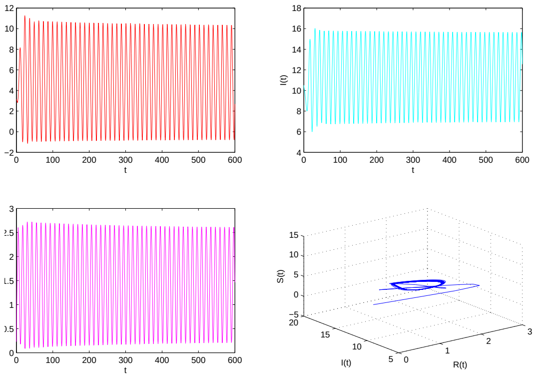

Figure 6: E_* is unstable for $\tau = 3.15 > \tau_0 = 3.0107$.

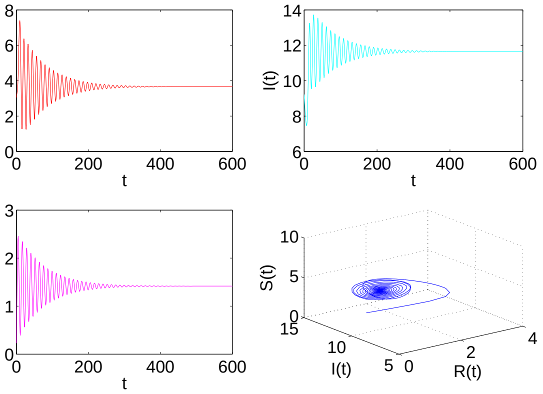

Figure 7: E_* is locally asymptotically stable for $\tau_2 = 2.97 < \tau_2^* = 3.0724$ and $\tau_1 = 2.05$.

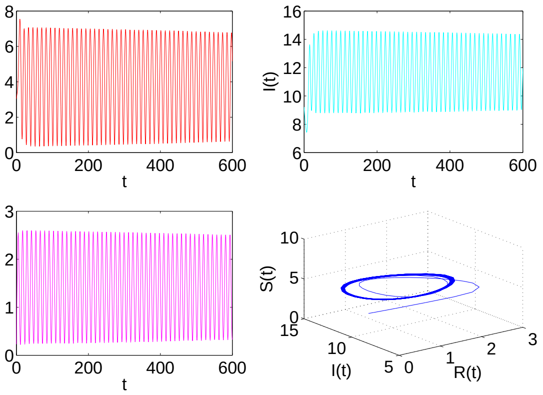

Figure 8: E_* is unstable for $\tau_2 = 3.146 > \tau_2^* = 3.0724$ and $\tau_1 = 2.05$.

Let $\rho(\theta) = (1, \rho_2, \rho_3)^T e^{i\tau_2^*\omega_2^*\theta}$ be the eigenvector of $A(0)$ corresponding to $+i\tau_2^*\omega_2^*$ and $\rho^*(s) = V(1, \rho_2^*, \rho_3^*)^T e^{i\tau_2^*\omega_2^* s}$ be the eigenvector of A^* corresponding to $-i\tau_2^*\omega_2^*$. By some complex computations, we can obtain

$$
\begin{aligned}
\rho_2 &= \frac{a_7}{i\omega_2^* - a_2 - a_8 e^{-i\omega_2^*\tau_1^*}}, \\
\rho_3 &= \frac{a_1 + a_5 e^{-i\omega_2^*\tau_2^*} + a_6\rho_2 e^{-i\omega_2^*\tau_1^*}}{i\omega_2^* - a_9 e^{-i\omega_2^*\tau_2^*}}, \\
\rho_2^* &= -\frac{i\omega_2^* + a_1 + a_5 e^{i\omega_2^*\tau_1^*}}{a_7 e^{i\omega_2^*\tau_1^*}}, \\
\rho_3^* &= -\frac{a_9 e^{i\omega_2^*\tau_2^*}}{i\omega_2^* + a_4 + a_{10} e^{i\omega_2^*\tau_2^*}}.
\end{aligned}
$$

From Eq.(20), we obtain

$$
\begin{aligned}
\bar{V} &= [1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* \\
&\quad + \tau_1^*(a_5 + a_6\rho_2 + \bar{\rho}_2^*(a_7 + a_8\rho_2))e^{-i\omega_2^*\tau_1^*} \\
&\quad + \tau_2^*\rho_3(a_9 + a_{10}\bar{\rho}_3^*)e^{-i\tau_2^*\omega_2^*}]^{-1}
\end{aligned}
$$

such that $\langle \rho^*, \rho \rangle = 1$, $\langle \rho^*, \bar{\rho} \rangle = 0$.

Following the algorithm given in [17] and using the similar computation process in [18], we can get the coefficients used to determine the qualities of the Hopf bifurcation:

$$
\begin{aligned}
g_{20} &= 2\tau_2^*\bar{V}[a_{21}\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*}) \\
&\quad + \bar{\rho}_2^*(b_{21}\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*}) \\
&\quad + 2b_{22}(\rho^{(2)}(0))^2) + 2c_{21}\bar{\rho}_3^*(\rho^{(2)}(0))^2], \\
g_{11} &= \tau_2^*\bar{V}[a_{21}(\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*}) \\
&\quad + \bar{\rho}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*})) \\
&\quad + \bar{\rho}_2^*(b_{21}(\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*}) \\
&\quad + \bar{\rho}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*})) \\
&\quad + 2b_{22}\rho^{(2)}(0)\bar{\rho}^{(2)}(0)) \\
&\quad + 2c_{21}\bar{\rho}_3^*\rho^{(2)}(0)\bar{\rho}^{(2)}(0)], \\
g_{02} &= 2\tau_2^*\bar{V}[a_{21}\bar{\rho}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*}) \\
&\quad + \bar{\rho}_2^*(b_{21}\bar{\rho}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*}) \\
&\quad + 2b_{22}(\bar{\rho}^{(2)}(0))^2) + 2c_{21}\bar{\rho}_3^*(\rho^{(2)}(0))^2], \\
g_{21} &= 2\tau_2^*\bar{V}[a_{21}(W_{11}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}W_{20}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\\
&+W_{11}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\\
&+\frac{1}{2}W_{20}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}(1)(-\frac{\tau_1^*}{\tau_2^*}))\\
&+\bar{\rho}_2^*(b_{21}(W_{11}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\\
&+\frac{1}{2}W_{20}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\\
&+W_{11}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\\
&+\frac{1}{2}W_{20}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}(1)(-\frac{\tau_1^*}{\tau_2^*}))\\
&+b_{22}(2W_{11}^{(2)}(0)\rho^{(2)}(0)\\
&+W_{20}^{(2)}(0)\bar{\rho}^{(2)}(0))+3b_{23}(\rho^{(2)}(0))^2\bar{\rho}^{(2)}(0))\\
&+\bar{\rho}_3^*(c_{21}(2W_{11}^{(2)}(0)\rho^{(2)}(0)\\
&+W_{20}^{(2)}(0)\bar{\rho}^{(2)}(0))+3c_{22}(\rho^{(2)}(0))^2\bar{\rho}^{(2)}(0))],
\end{aligned}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}\rho(0)}{\tau_2^*\omega_2^*}e^{i\tau_2^*\omega_2^*\theta}+\frac{i\bar{g}_{02}\bar{\rho}(0)}{3\tau_2^*\omega_2^*}e^{-i\tau_2^*\omega_2^*\theta}\\
&\quad +E_{20}e^{2i\tau_2^*\omega_2^*\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}\rho(0)}{\tau_2^*\omega_2^*}e^{i\tau_2^*\omega_2^*\theta}+\frac{i\bar{g}_{11}\bar{\rho}(0)}{\tau_2^*\omega_2^*}e^{-i\tau_2^*\omega_2^*\theta}\\
&\quad +E_{11},
\end{aligned}
$$

where E_{20} and E_{11} can be computed by the following equations, respectively

$$
\begin{pmatrix} a'_{11} & a'_{12} & a'_{13}\\ a'_{21} & a'_{22} & 0\\ 0 & -a_3 & a'_{33}\end{pmatrix}E_{20}=2\begin{pmatrix}E_{20}^{(1)}\\ E_{20}^{(2)}\\ 0\end{pmatrix}
$$

$$
\begin{pmatrix} \alpha'_{11} & a_6 & a_9\\ a_7 & \alpha'_{22} & 0\\ 0 & a_3 & \alpha'_{33}\end{pmatrix}E_{11}=-\begin{pmatrix}E_{11}^{(1)}\\ E_{11}^{(2)}\\ 0\end{pmatrix}
$$

with

$$
\begin{aligned}
a'_{11} &= 2i\omega_2^*-a_1-a_5e^{-2i\tau_1^*\omega_2^*},\\
a'_{12} &= -a_6e^{-2i\tau_1^*\omega_2^*},\\
a'_{13} &= -a_9e^{-2i\tau_2^*\omega_2^*},\\
a'_{21} &= -a_7e^{-2i\tau_1^*\omega_2^*},\\
a'_{22} &= 2i\omega_2^*-a_2-a_8e^{-2i\tau_1^*\omega_2^*},\\
a'_{33} &= 2i\omega_2^*-a_4-a_{10}e^{-2i\tau_1^*\omega_2^*},\\
\alpha'_{11} &= a_1+a_5,\\
\alpha'_{22} &= a_2+a_8,\\
\alpha'_{33} &= a_4+a_{10},
\end{aligned}
$$

and with

$$
\begin{aligned}
E_{20}^{(1)} &= a_{21}\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*}),\\
E_{20}^{(2)} &= b_{21}\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\\
&\quad +2b_{22}(\rho^{(2)}(0))^2,\\
E_{11}^{(1)} &= a_{21}(\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\\
&\quad +\bar{\rho}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*})),\\
E_{11}^{(2)} &= b_{21}(\rho^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\bar{\rho}^{(2)}(-\frac{\tau_1^*}{\tau_2^*})\\
&\quad +\bar{\rho}^{(1)}(-\frac{\tau_1^*}{\tau_2^*})\rho^{(2)}(-\frac{\tau_1^*}{\tau_2^*}))\\
&\quad +2b_{22}\rho^{(2)}(0)\bar{\rho}^{(2)}(0).
\end{aligned}
$$

Then, we can get the following coefficients:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\tau_2^*\omega_2^*}(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3})+\frac{g_{21}}{2},\\
\mu_2 &= -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_2^*)\}},\\
\beta_2 &= 2Re\{C_1(0)\},\\
T_2 &= -\frac{Im\{C_1(0)\}+\mu_2 Im\{\lambda'(\tau_2^*)\}}{\tau_2^*\omega_2^*}. \qquad (21)
\end{aligned}
$$

Based on the discussion above, we can obtain the following results.

Theorem 6 *For system (2), if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical); if $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable); if $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solutions increases (decreases).*

The results in Theorem 6 give a description of the Hopf bifurcation and the bifurcating periodic solutions when $\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$. According to the values of μ_2, β_2 and T_2, we can easily determine direction of the Hopf bifurcation and properties of the bifurcating periodic solutions.

4 Numerical simulation

In this section, the interesting dynamical behaviors of system (2) are shown by a numerical example in order to support our theoretical results. We consider system (2) with $A = 1.5$, $d = 0.02$, $\beta = 0.05$, $\eta = 0.5$,

$v = 0.1, c = 0.8, b = 1$, that is

$$\begin{cases} \frac{dS(t)}{dt} = 1.5 - 0.02S(t) - 0.05S(t-\tau_1)I(t-\tau_1) \\ \qquad +0.5R(t-\tau_2), \\ \frac{dI(t)}{dt} = 0.05S(t-\tau_1)I(t-\tau_1) - 0.12I(t) \\ \qquad -\frac{0.8I(t)}{1+I(t)}, \\ \frac{dR(t)}{dt} = \frac{0.8I(t)}{1+I(t)} - 0.02R(t) - 0.5R(t-\tau_2), \end{cases} \tag{22}$$

which satisfies $R_0 = 4.0761 > 1$.

Then, we obtain the unique positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ of system (22) by the software package Matlab. By computing, we obtain $A_{10} = 0.0227$, $A_{11} = 0.3545$, $A_{12} = 1.0645$. Thus, the condition (H_{11}) holds when $\tau_1 = \tau_2 = 0$. So, all the roots of Eq.(6) have negative real parts in the absence of delay.

When $\tau_1 > 0, \tau_2 = 0$, by some computations, we obtain that Eq.(8) has a unique positive root $\omega_{10} = 0.4301$. Then, we have $\tau_{10} = 3.6108$ and $f_1'(v_{1*}) = 0.7534 > 0$. That is, the conditions $(H_{21})-(H_{22})$ hold and the characteristic equation (7) has a pair of purely imaginary roots $\pm i\omega_{10}$. The numerical simulation in Figure 1 shows that the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ is asymptotically stable when $\tau_1 = 3.525$ which is smaller than τ_{10}. Figure 2 shows that periodic oscillations occur when τ_1 is larger than τ_{10}, such as $\tau_1 = 3.715$. The results show that if we shorten the latent period, we will control the disease. Similarly, we have $\omega_{20} = 1.0336$, $\tau_{20} = 3.1676$ for $\tau_1 = 0$, $\tau_2 > 0$. The corresponding waveform and the phase plots are shown in Figures 3-4. As can be seen from Figure 3, the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ is asymptotically stable when $\tau_2 = 3.05 \in [0, \tau_{20})$. However, when τ_2 passes through the critical value τ_{20}, the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$. This property can be seen from Figure 4. The results show that if we shorten the temporary immunity period we will control the disease.

When $\tau_1 = \tau_2 = \tau > 0$, we obtain $\omega_0 = 0.6026$, $\tau_0 = 3.0107$ by some complex computations. For $\tau = 2.74 < \tau_0 = 3.0107$, the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ is asymptotically stable and this property can be illustrated by Figure 5. In this case, we can control the disease. However, Once τ passes through the critical value τ_0, the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ loses its stability and a Hopf bifurcation occurs, and the corresponding waveform and phase plots are shown in Figure 6. As can be seen from Figure 6, when $\tau = 3.15 > \tau_0 = 3.0107$ the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ is unstable and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ and the disease will be out of control in this case.

Lastly, we obtain $\omega_2^* = 0.9220$, $\tau_2^* = 3.0724$ for $\tau_2 > 0$ and $\tau_1 = 2.05 \in (0, \tau_{10})$. Let $\tau_2 = 2.97 \in (0, \tau_2^*)$, we can know that the positive equilibrium $E_*(3.6645, 11.6531, 1.4169)$ is asymptotically stable as depicted in Figure 7. Namely, we can control the disease when the value of τ_2 is smaller than the critical value of τ_2^*. When $\tau_2 = 3.146$ which is larger than τ_2^*, periodic oscillations occur and this property can be illustrated by Figure 8. Also, this phenomenon shows that the disease will be out of control when the value of τ_2 is larger than $\tau_2^* = 3.0724$. In addition, we have $\lambda'(\tau_2^*) = 0.0088 - 0.0195i$ and $C_1(0) = -0.0634 - 0.0031i$ by some complex computations. Further, we obtain $\mu_2 = 7.2045 > 0$, $\beta_2 = -0.1268 < 0$, $T_2 = 0.0507 > 0$. Thus, according to Theorem 6, we can conclude that the Hopf bifurcation of system (22) is supercritical, the bifurcating periodic solutions are stable and the period of the bifurcating periodic solutions increases.

5 Conclusion

In this paper, an SIRS epidemic model with saturation recovery and two delays is proposed based on the model in [7]. Compared with the model considered in [7], we incorporate not only the latent period of the epidemic but also the temporary immunity period of the recovered individuals into the model considered in [7]. Namely, the model proposed in this paper is more general. We mainly consider the effects of the two delays on the proposed model in the present paper. By analyzing the distribution of the eigenvalues of the corresponding transcendental characteristic equation of its linearized equation, we find the critical values for the occurrence of Hopf bifurcation. When the Hopf bifurcation occurs, the propagation of the disease is out of control. Therefore, In order to control and even eliminate the propagation of the disease, the two delays in the model should remain less than the corresponding critical value. Furthermore, explicit formulae are derived to determine the direction and the stability of the Hopf bifurcation by using the normal form theory and center manifold theorem. In order to verify the theoretical analysis, a numerical example is also included.

Acknowledgements: We would like to thank the anonymous referees for their careful reading of the original manuscript and their many valuable comments and suggestions that greatly improve the presentation of this work. The research was supported by National Natural Science Foundation of China (No. 11261010).

References:

[1] L. J. S. Allen and P. Driessche, The basic reproduction number in some discrete-time epidemic models, *J. Diff. Equ. Appl,* 14, 2008, pp. 1127–1147.

[2] D. Xiao and S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, *Math. Biosci,* 208, 2007, pp. 419–429.

[3] Z. Hu, Z. Teng and H. Jiang, Stability analysis in a class of discrete SIRS epidemic models, *Nonlinear Anal. Real World Appl,* 13, 2012, pp. 2017–2033.

[4] Z. J. Liu, Dynamics of positive solutions to SIR and SEIR epidemic models with saturated incidence rates, *Nonlinear Anal: Real. World. Appl,* 14, 2013, pp. 1286–1299.

[5] Y. Z. Wang, D. W. Huang, S. D. Zhang and H. J. Liu, Dynamic behavior in a HIV infection model for the delayed immune response, *WSEAS Transaction on Mathematics,* 10, 2011, pp. 398–407.

[6] X. B. Liu and L. J. Yang, Stability analysis of an SEIQV epidemic model with saturated incidence rate, *Nonlinear Anal: Real. World. Appl,* 13, 2012, pp. 2671–2679.

[7] H. Wan and J. A. Cui, Rich Dynamics of an Epidemic Model with Saturation Recovery, *J. Appl. Math,* vol. 2013, Article ID 314958, 9 pages.

[8] Y. Muroya, H. X. Li and T. Kuniya, Complete global analysis of an SIRS epidemic model with graded cure and incomplete recovery rates, *J. Math. Anal. Appl,* 410, 2014, pp. 710–732.

[9] K. J. Zhuang and H. L. Zhu, Stability and bifurcation analysis for an improved HIV model with time delay and cure rate, *WSEAS Transaction on Mathematics,* 8, 2013, pp. 960-869.

[10] R. Xu and Z. Ma, Stability of a delayed SIRS epidemic model with a nonlinear incidence rate, *Chaos, Solitons £Fractals,* 41, 2009, pp. 2319–2325.

[11] C. J. Xu and Q. M. Zhang, Qualitative analysis for Lotka-Volterra model with time delays, *WSEAS Transaction on Mathematics,* 13, 2014, pp. 603-614.

[12] T. K. Kar and A. Ghorai, Dynamic behaviour of a delayed predator-prey model with harvesting, *Appl. Math. Comput,* 217, 2011, pp.9085-9104.

[13] C. Bianca, M. Ferrara and L. Guerrini, Hopf bifurcations in a delayed-energy-based model of capital accumulation, *Applied Mathematics £Information Sciences,* 7, 2013, pp. 139-143.

[14] C. Bianca, M. Ferrara and L. Guerrini, Qualitative analysis of a retarded mathematical framework with applications to living systems, *Abstr. Appl. Anal,* vol. 2013, Article ID 736058, 7 pages.

[15] C. Bianca, M. Ferrara and L. Guerrini, The time delays' effects on the qualitative behavior of an economic growth model, *Abstr. Appl. Anal,* vol.2013, Article ID 901014, 10 pages.

[16] Y. L. Song and J. J. Wei, Bifurcation analysis for Chens system with delayed feedback and its application to control of chaos, *Chaos, Solitons £Fractals,* 22, 2004, pp. 75–91.

[17] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory and Applications of Hopf Bifurcation,* Cambridge University Press, Cambridge, 1981.

[18] L. Gori, L. Guerrini and M. Sodini,Hopf bifurcation in a Cobweb model with discrete time delays, *Discrete Dyn. Nat. Soc,* vol.2014,Article ID 137090, 8 pages.