G^* Properties - Without G^* Construction

M. YAMUNA
VIT University
Vellore, Tamilnadu
INDIA
myamuna@vit.ac.in

K. KARTHIKA
VIT University
Vellore, Tamilnadu
INDIA
karthika.k@vit.ac.in

Abstract: In any planar graph G there is a one to one correspondence between edges in G, its dual G^*. As the number of vertices and edges increase construction of G^* is complicated, when graph G^* is not required and only properties of G^* is a need, determining properties of G^* without actual construction of G^* is comfortable. In this paper we determine the domination number of G^*, $\overline{G^*}$, chromatic polynomial of G^*, spanning tree of G^*, number of spanning trees of G^* from G. Determining the domination number of G^*, $\overline{G^*}$ using matrix representation is presented. MATLAB code for determining the same is also provided.

Key–Words: Planar graph, γ-stable, Chromatic polynomial, Spanning tree.

1 Introduction

In graph theory, a planar graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other [6].

Let G^* denote the geometric dual graph of G obtained for a given embedding of G in the plane. Results of planar and dual graphs based on graph properties is interesting problem discussed by various authors. Maciej M. Syslo [7] has proved that, a graph G is outer-planar if and only if it has a dual G^*, having a vertex v such that $G^* - v$ contains no cycle. Arjana Zitnik in [1] provided the results on series parallel extensions of plane graphs to dual - Eulerian graphs. R. M. Tifenbach and S. J. Kirkland in [10] gave some class of graphs, whose adjacency matrices are nonsingular. Yuval Emek [15] furnished some algorithm on new insights regarding the connection between low average stretch spanning trees and planar duality. Gurami Tsitsiashvili et al [3] gave the asymptotic analysis of connectivity probability in random planar graphs and dual graphs.

Minimum dominating set can be used in proving various graph properties. Val Pinciu [11] has provided a recursive algorithm that finds a connected domination set for an outer-planar graph. M. Yamuna and K. Karthika in [12] discussed a constructive procedure for generating a spanning tree for any graph from its γ- set. Also they have provided a procedure for generating a minimum weighted spanning tree by using adjacency matrix. They have characterized planar graphs using γ-stable graphs in [14].

In this paper we focus on determining and discussing properties of the dual G^* of any planar graph G without actual construction of G^*. We also discussed the necessary and sufficient conditions for determining the domination number of G^*, $\bar{G}^*$ from G. In particular, we contribute to determine the domination number of G^* by using matrix representation. The corresponding MATLAB code for determining the same is also provided. In addition, the necessary and sufficient conditions for deciding if G^* is γ- stable using graph G are discussed. Similar to NG-type result bounds related to sum and product of a graph and its dual is provided. Modifying the routine procedure of determining the chromatic polynomial, a recursive method of determining the chromatic polynomial of G^* from G is given in the present paper. This recursive method leads to determining all possible spanning trees for G^* from G using Cayley's formula.

2 Terminology

We consider only simple connective undirected graphs $G = (V, E)$. The open neighborhood of vertex $v \in V(G)$ is defined by

$$N(v) = \{u \in V(G) | uv \in E(G)\},$$

while its closed neighborhood is the set

$$N[v] = N(v) \cup \{v\}.$$

We say that H is a subgraph of G, if $V(H) \subseteq V(G)$ and $uv \in E(H)$ implies $uv \in E(G)$. For any set S of vertices of G, the induced subgraph $\langle S \rangle$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $\langle S\rangle$ if and only if they are adjacent in G. An elementary contraction of a graph G is obtained by identifying two adjacent points u and v, that is by the removal of u and v and the addition of a new point w adjacent to those points to which u or v was adjacent. We say that $G_\bullet uv$ is obtained by contracting (u, v), where u adjacent to v. For properties related to graph theory we refer to F. Harary [4].

A set D of vertices in a graph $G = (V, E)$ is a dominating set if every vertex of $V - D$ is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set. The cardinality of any minimum dominating set for G is called the domination number of G and it is denoted by $\gamma(G)$. A γ- set denotes a dominating set for G with minimum cardinality. The private neighborhood of $v \in D$ is defined by

$$pn[v, D] = N(v) - N(D - \{v\}).$$

For properties related to domination we refer to T. W. Haynes, S. T. Hedetniemi, and P. J. Slater [5].

3 Results and Discussions

In this section we provide a necessary and sufficient condition to determine the domination number of G^* from G. Properties of G^* of a γ- stable graph G is discussed. Throughout this paper we consider simple planar graphs G for which its dual G^* is connected and simple.

Notation

Let $S = \{R_1, R_2, \cdots, R_q\}$ be the set of regions of G. Let $T = \{r_1, r_2, \cdots, r_q\}$ be the set of vertices in the regions $R_1, R_2, \cdots, R_q$ respectively, that is r_1 is the vertex in the region R_1, r_2 is the vertex in the region R_2, $\cdots$, r_q is the vertex in the region R_q respectively, that is $T = V(G^*)$.

There is a one to one mapping between the elements of S and T, that is, for every $R_i, i = 1, 2, \cdots, q$ in S there is a vertex r_i in $T, i = 1, 2, \cdots, q$. For every subset X of S, let us denote the corresponding subset in T by X^*, that is, if $X \subseteq S = \{R_i, R_p, R_j\}$, then $X^* \subseteq T^* = \{r_i, r_p, r_j\}$. For any $a \in E(G)$, let us denote the corresponding edge in G^* by a^*.

In Fig. 1, $R = \{R_3, R_7\}$ are a set of regions, so that the remaining regions are adjacent to at least one region in R. We observe that $\{r_3, r_7\}$ is a dominating set for G^*.

R_1 is adjacent to regions R_2, R_7 and R_8. In G^*, r_1 is adjacent to r_2, r_7 and r_8. So we observe that if R_i is adjacent to R_j in G, then r_i is adjacent to r_j in G^*.

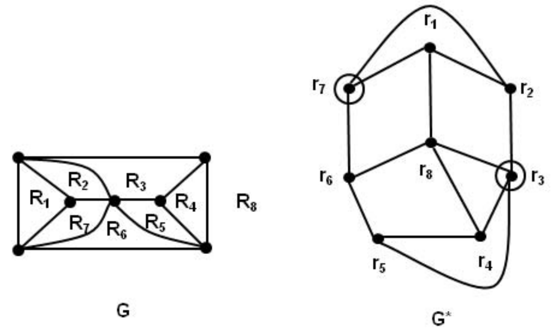


Figure 1:

Let $D \subseteq S$ be a set of regions such that every region in $S - D$ is adjacent to at least one region in D. Then there is a subset $D^* \subseteq T$, so that every vertex in $T - D^*$ is adjacent to at least one vertex in D^*. This means that D is a set of dominating regions in G and D^* is a dominating set in G^*. If D is the smallest set which satisfies this property, then D^* is a γ - set for G^*.

Throughout the paper, $D = \{R_1, R_2, \cdots, R_k\}$, denotes a minimum set of dominating regions for G and $D^* = \{r_1, r_2, \cdots, r_k\}$ the corresponding γ- set for G^*.

For any $R_i \in D$, let $Y = \{R_{e1}, R_{e2}, \cdots, R_{ek}\} \in S - D$ such that e_j is an edge common between R_i and R_j. Also every $R_{ej}, j = 1, 2, \cdots, k$ has no edge common to $D - \{R_i\}$, Y is a set of regions adjacent only to R_i with respect to D. Let us denote this set by $pr[R_i, D]$, that is $pr[R_i, D] = Y$.

A subset $Y \subseteq S$ of k- regions are adjacent if any two regions in Y are adjacent that is any two regions in Y have at least one edge in common. We shall denote the set Y as k- adjacent regions. Since G is a planar graph $|Y| \leq 4$.

From the discussion in Notation, the following Theorem 1 is obvious.

Theorem 1 *Let G be a graph and G^* be its dual. $\gamma(G^*) = k$ if and only if*

1. there is a set D of k- regions in G such that every region in $S - D$ is adjacent to at least one region in D.

2. there is no set $X \subseteq S$, that satisfies condition 1 such that $|X| < |D|$.

3.1 γ- Stable Graphs

For a given non-adjacent pair $\{x, y\}$ of vertices in a graph G, we denote by G_{xy} the graph obtained by deleting x and y and adding a new vertex xy adjacent to precisely those vertices of $G - x - y$ which were adjacent to at least one of x or y in G.

A graph G is said to be γ- stable if $\gamma(G_{xy}) = \gamma(G)$, for all $x, y \in V(G)$, x is not adjacent to y, where G_{xy} denotes the graph obtained by merging the vertices x, y [13].

R_1. A graph G is γ- stable if and only if every γ - set D of G is a clique.

R_2. If G is γ- stable, then $pn[u, D] \geq 2$, for all $u \in V(G)$ [13].

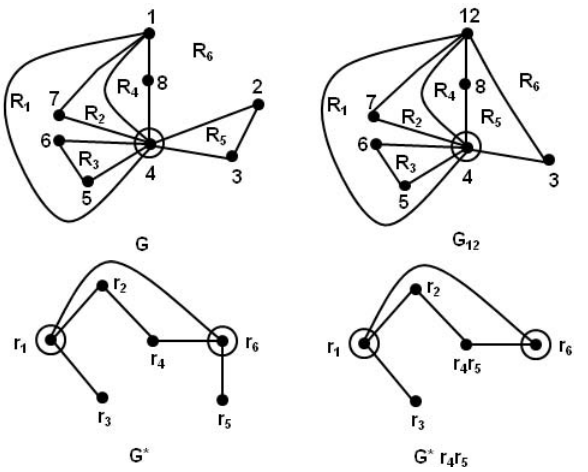


Figure 2:

In Fig. 2, G is a planar graph, G^* its dual. $G_{12}, G^*_{r_4r_5}$ denotes the graphs obtained by merging non-adjacent vertices. $\gamma(G) = \gamma(G_{12}) = 1$ and $\gamma(G^*) = \gamma(G^*_{r_4r_5}) = 2$, that is, G, G^* are γ- stable graphs.

For every γ- stable graph G its dual G^* need not to be γ- stable.

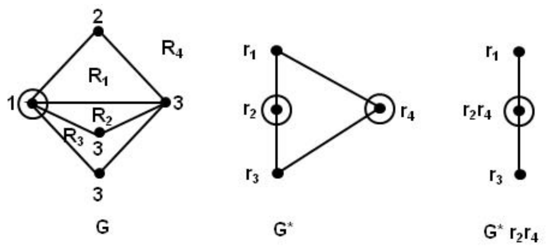


Figure 3:

In Fig. 3, G is γ- stable. $\gamma(G^*) = 2$, $\gamma(G^*_{r_2r_4}) = 1$ implies G^* is not γ- stable.

Theorem 2 below provides a necessary and sufficient condition for G and G^* to be γ- stable graphs.

Theorem 2 *Let G be a planar graph, $\gamma(G^*) = k$. G^* is γ- stable if and only if*

1. there is a set of k- adjacent regions $D \subseteq S$ such that every region in $S - D$ is adjacent to at least one region in D.

2. every set $D \subseteq S$ such that

[a.] $|D| = k$,

[b.] every region in $S - D$ is adjacent to at least one region in D,

is k- adjacent.

Proof. Let G be a planar graph, $\gamma(G^*) = k$. Assume that G^* is γ- stable. By R_1 , we know that every γ- set in G^* is complete. If condition 1 is not satisfied, that is, there is a set of k- regions, $D \subseteq S$ such that every region in $S - D$ is adjacent to at least one region in D such that the regions in D need not be k- adjacent, then there is a set D^* such that $|D^*| = k$, which is a dominating set for G^* but $\langle D^* \rangle$ is not a clique. This is not possible, since G^* is γ- stable. Every set D such that $|D| = k$ satisfies this property, that is, condition 2 is satisfied.

Conversely assume that Conditions 1 and 2 in Theorem are satisfied. By Condition 1, it follows from Notation that there is a dominating set D^* for G^* such that $|D^*| = k$, $\langle D^* \rangle$ is a clique. From condition 2 and Notation, it follows that every dominating set D^* such that $|D^*| = k$ is a clique. By our assumption we know that $\gamma(G^*) = k$, that is G^* is γ- stable. □

Similar to NG - type result, the bound for sum and product of a graph and its dual is provided in Theorem 3.

Theorem 3 *If G and G^* are γ- stable graphs, then*

1. $\gamma(G) + \gamma(G^*) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{5} \rfloor$;

2. $\gamma(G)\gamma(G^*) \leq \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{5} \rfloor$.

Proof. Claim 1 Let G^* be γ - stable graph. $\langle pr[R_i, D] \rangle$ is not a clique, for every region $R_i \in D$.

Proof: Since if G^* is γ - stable, by R_2 we know that $pn[u, D^*] \geq 2$. Let us assume that $pn[u, D^*] = 2$, for all $u \in D^*$. Let $D = \{R_1, R_2, \cdots, R_k\}$ be the dominating regions in G, so that $D^* = \{r_1, r_2, \cdots, r_k\}$ is a γ - set for G^*. Let private regions of $R_i = \{A_i, B_i\}$, for all $i = 1, 2, \cdots, k$. If for some R_i, A_i, B_i are adjacent, then $D - \{R_i\} \cup \{A_i\}$ or $D - \{R_i\} \cup \{B_i\}$ is a set of dominating region in G, implies $D^* - \{r_i\} \cup \{a_i\}$ or $D^* - \{r_i\} \cup \{b_i\}$ is a γ - set for G^* such that $\langle D^* - \{r_i\} \cup \{a_i\} \rangle$ or $\langle D^* - \{r_i\} \cup \{b_i\} \rangle$ is not a clique in G^*, a contradiction as G^* is γ - stable.

Claim 2: Let G be a γ- stable graph. Then $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.

Proof: If G is γ - stable, by R_2 we know that $pn[u, D] \geq 2$, for all $u \in D$.

$$n = D + pn[u, D] + \{2 - dominated\ vertices\}$$
$$\geq 3D + \{2 - dominated\ vertices\}$$
$$D \leq \frac{n - \{2 - dominated\ vertices\}}{3}$$

G may or may not have 2- dominated vertices. So $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$.

Since G and G^* are γ - stable graphs $1 \leq \gamma(G), \gamma(G^*) \leq 4$. The possible simple planar graphs with at least one region, when $n = 3, 4, 5$ is listed in Fig. 4 [4].

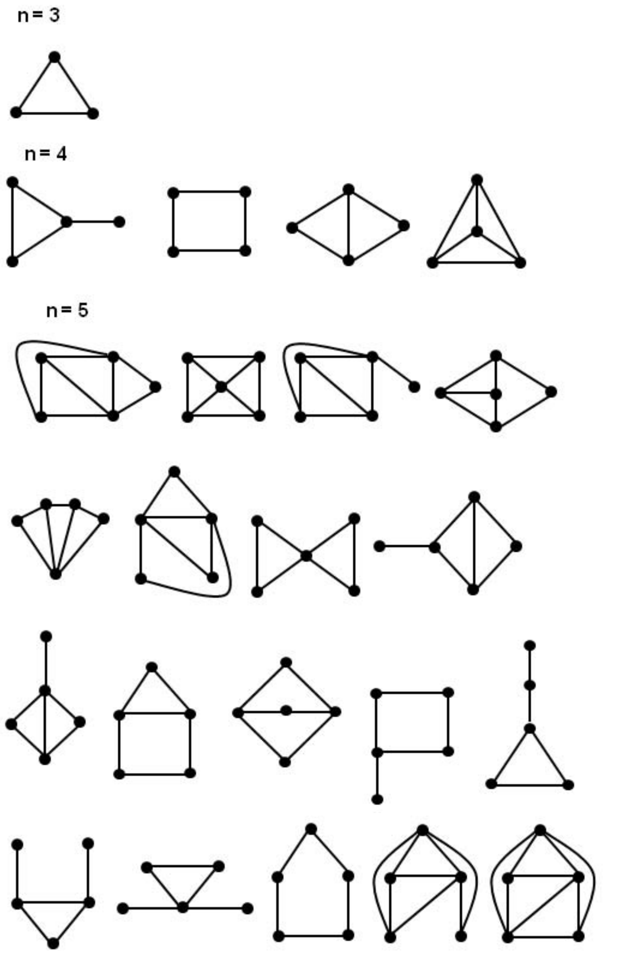


Figure 4:

As shown in Fig. 4, it is clear that the smallest possible planar graph with 4- adjacent regions is K_4. For the rest of the theorem, let us stick onto K_4.

Let $\gamma(G^*) = 4$. By Theorem 2, G has at least one set of 4- adjacent regions, say R_1, R_2, R_3 and R_4.

Since G^* is γ- stable, by R_2 we know that $pn[u, D^*] \geq 2$, for all $u \in D^*$. This means that each region $R_i, i = 1, 2, 3, 4$ has at least two private regions adjacent to it. Without loss of generality, let us assume that $pr[R_i, D] = \{A_i, B_i\}$, for all $i = 1, 2, 3, 4$. By claim 1, we know that A_i, B_i are not adjacent. Any smallest region is C_3, the private regions are as shown in Fig.5.

Fig. 5 represents one possible construction, where every private region is C_3. So, if $\gamma(G^*) = 4$,

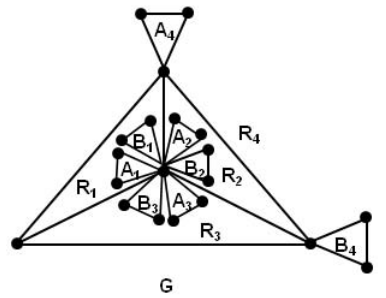


Figure 5:

then G has at least 20 vertices, implies $\gamma(G^*) \leq \lfloor \frac{n}{5} \rfloor$.

Similarly when $\gamma(G^*) = 3$, G has at least 16 vertices. When $\gamma(G^*) = 2$, G has at least 11 vertices. In all cases $\gamma(G^*) \leq \lfloor \frac{n}{5} \rfloor$. □

From Claim 1 and 2 we conclude that

1. $\gamma(G) + \gamma(G^*) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{5} \rfloor$.

2. $\gamma(G)\gamma(G^*) \leq \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{5} \rfloor$. □

Theorem 4 *Let G be a planar graph such that its dual G^* is γ- stable. For any γ- set D^* in G^**

[i.] $\gamma(G - a)^* = |D^*| - 1$.

[ii.] $(G - a)^*$ is γ- stable.

For all $a^* \in \langle D^* \rangle, a \in E(G)$.

Proof. [i.] For any planar graph G if a is an edge common to two regions, then $(G-a)^*$ can be obtained from G^* by deleting the corresponding edge a^* and merging the two end vertices of a^* in $G^* - a^*$ [8], that is $(G - a)^* = G^*_\bullet a^*$. Let $a^* = (u^* \ v^*)$, where $a^* \in D^*$. $D' = D^* - \{u^*, v^*\} \cup \{u^*v^*\}$ is a dominating set for $(G - a)^*$. If possible, let D'' be a γ - set for $(G - a)^*$ such that $|D''| < |D'|$. $D'' \cup \{u^*\}$ or $D'' \cup \{v^*\}$ is a γ - set for G^* such that $|D''| < |D^*|$, a contradiction. So, D' is a γ - set for $(G - a)^*$.

[ii.] If possible assume that $(G - a)^*$ is not γ-stable, implies there is at least one γ- set say D''' , which is not a clique. In G^*, D''' dominates $G^* - \{u^*\} - pn[u^*, D^*]$ or $G^* - \{v^*\} - pn[v^*, D^*]$.

Assume that D''' dominates $G^* - \{u^*\} - pn[u^*, D^*]$, implies $D''' \cup \{u^*\}$ is a γ- set for G^* such that

[a.] $|D'''| \cup \{u^*\} = |D^*|$,

[b.] $\langle D''' \cup \{u^*\} \rangle$ is not a clique,

a contraction as G^* is γ- stable, implies $(G - a)^*$ is γ - stable, for all $a^* \in \langle D^* \rangle$. □

4 Chromatic Polynomial of G^* from G

This section provides a recursive method of finding the chromatic polynomial of G^* from G and hence determining all possible spanning trees for G^*.

We aim to find the chromatic polynomial of G^* using graph G. A given graph G of n - vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows:

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n - vertices gives the number of ways of properly coloring the graph, using λ or fewer colors [8].

To determine the chromatic polynomial of a graph G we use two operations edge contraction and edge deletion.

If G is simple, then $P_n(\lambda)$ of $G = P_n(\lambda)$ of $(G - e) - P_n(\lambda)$ of $(G_\bullet\, e)$, for any edge e of G [8].

The following discussions and results is for modifying the original method of determining the chromatic polynomial for any graph, so that chromatic polynomial of G^* can be determined from G. $P_n(\lambda)$ of G is determined using edge removal and edge contraction. We modify these operations to find $P_n(\lambda)$ of G^* from G.

4.1 Edges in Series

Self loops and parallel edges do not contribute in determining $P_n(\lambda)$ of G^*. So, it is always sufficient to retain G^* as a simple graph. Any region in G may or may not have edges in series. Edges in series become parallel edges in G^*. Since it is sufficient that G^* is simple we can replace all edges in series by a single edge.

G_1 is a planar graph, G_2 is the graph obtained by removing the edges in series in G_1. Note that G_1^* is isomorphic to G_2^*, when the duals are simple graphs. The edges in series are between regions R_1 and R_5. In G_2, the edges in series are replaced by a single edge between R_1 and R_5. G_3 is another planar graph representation of G_2. In G_3^*, there is no edge between r_1 and r_2.

$$P_n(\lambda) \text{ of } G_2^* = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.$$

$$P_n(\lambda) \text{ of } G_3^* = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 14\lambda^2 + 5\lambda.$$

G_2^* and G_3^* do not have the same chromatic polynomial.

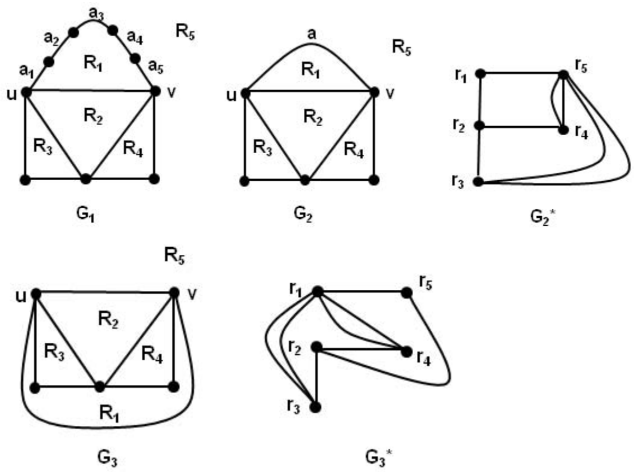


Figure 6:

From the above the discussions, we observe that when region adjacency is altered, the chromatic polynomial need not be the same. We need to modify the operations, so that there is no change in the chromatic polynomial.

To make this possible, edges in series between two regions will be replaced by a single edge between the same regions.

Type I Operation

Replace all the edges in series between any two regions R_i and R_j by a single edge drawn between the same regions R_i, R_j.

By recursively applying Type I operation on G, we observe that G is modified into a graph H without vertices of degree two. G^* is isomorphic to H^*, when G^* is a simple graph.

While finding the chromatic polynomial of G^*, we know that we apply $G^* - \{a^*\}$ and $G^*_\bullet u^*v^*$ on G^*. We modify and implement these two operations on H to generate two graphs H_1 and H_2, so that $H_1^* = H^* - \{a^*\}$ and $H_2^* = H^*_\bullet u^*v^*$, where $a^* = (u^*v^*)$.

4.2 Modified $G - \{a\}$

In $H^* - a^*$, we know that $E(H^* - a^*) = E(H^*) - 1$ and $V(H^* - a^*) = V(H^*)$, H^* is connected. We need to find a suitable operation on H to generate H_1, so that

1. $V(H_1^*) = V(H^*)$,
2. $E(H_1^*) = E(H^*) - \{a^*\}$,
3. H_1^* is connected.

Since we require $V(H_1^*) = V(H^*)$, we need to retain back all the regions. But $E(H^* - a^*) = E(H^*) - \{a^*\}$, that is the number of edges is reduced by one. To make these two possible we retain a as a fake edge in dual construction, that is while constructing the dual of H, edge a will not contribute an edge a^* to H^*, but it will be used as an edge to retain the vertex count.

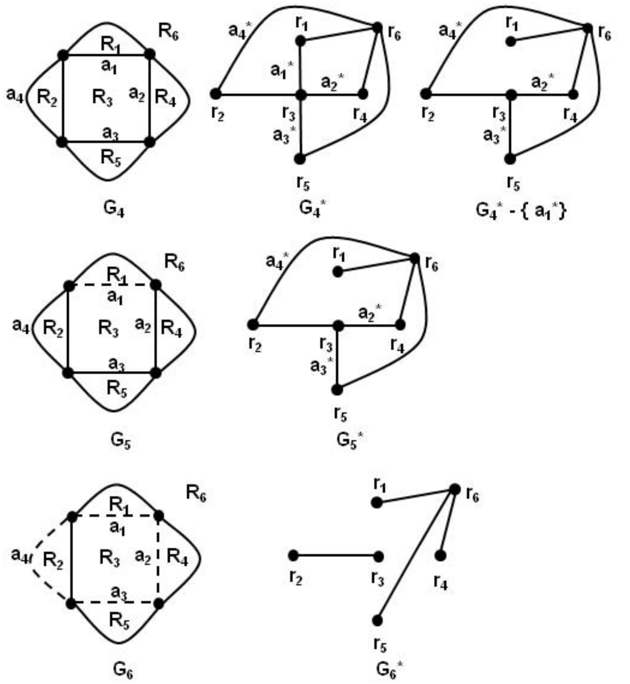


Figure 7:

In Fig. 7, G_4 represents the graph obtained by applying Type I operation, the required number of times. G_5 denotes the graph where a_1 is made a fake edge (represented in dotted lines). While constructing G_5^*, a_1 does not contribute a_1^* to G_5^*, but a_1 is used to retain back regions R_1 and R_3 as two separate regions. Note that G_5^* is equivalent to $G_4^* - a_1^*$.

G_6 denotes the graph by sifting edges $\{a_1, a_2, a_3, a_4\}$ to fake edges. $\{a_1, a_2, a_3, a_4\}$ is a circuit in G_6 forming a part of regions R_2 and R_3. When these edges become fake edges, regions R_2 and R_3 share no common edge with the remaining regions in dual construction, implies G_6^* is disconnected as seen in G_6^*. To overcome this while creating fake edges we take care that atleast one edge of every circuit is not considered as a fake edge.

Type II Operation $H - \{a\}$

Let H be the graph obtained by applying Type I operation on G. Let $a_{ij} = (u\, v) \in E(H)$, a_{ij} common to the regions R_i, R_j. Retain a_{ij} as a fake edge in H to generate graph H_1. When a_{ij} is retained as a fake edge we take care that at least one edge of every circuit in H is not declared as a fake edge. Construct the dual of H_1 as follows.

Let $S = \{R_1, R_2, \cdots, R_i, \cdots, R_j, \cdots, R_q\}$ be the set of regions in H. Let $T = \{r_1, r_2, \cdots, r_i, \cdots, r_j, \cdots, r_q\}$ be the set of vertices in the regions $\{R_1, R_2, \cdots, R_q\}$ respectively. For every edge a_{mn} between regions R_m, R_n, draw an edge between vertices r_m, r_n except for the edge a_{ij}.

1. $V(H_1^*) = V(H^*)$
2. $E(H_1^*) = E(H^* - \{a_{ij}^*\})$

On applying Type II operation we generate a new graph H_1, so that $H_1^* = H^* - a_{ij}^*$.

In the remaining part of the paper fake edges are denoted by the dash line.

4.3 Modified $G_\bullet uv$

In $H_\bullet^* u^* v^*$, we know that $V(H_\bullet^* u^* v^*) = V(H^*) - 1$ and $E(H_\bullet^* u^* v^*) = E(H^*) - k$, where $k \geq 1$.(Note that, after edge contraction we retain the graph simple.) We need a suitable operation on H, that make this is possible. We know that if a is an edge common to two regions, then $(G - a)^*$ is the graph obtained by deleting a^* and merging the two adjacent vertices in G^* [8]. Using this property we define Type III operation as follows.

Type III Operation

Let $H_2 = H - \{a_{ij}\}$, where $a_{ij} = (u\, v) \in E(H)$. We observe that $H_2^* = H_\bullet^* r_i r_j$.

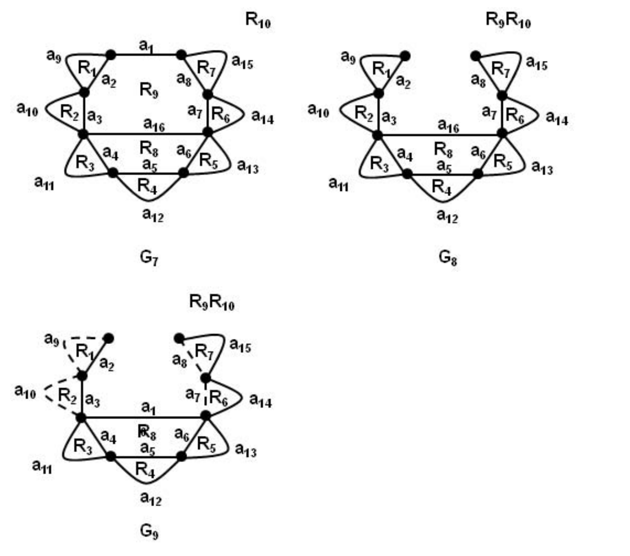


Figure 8:

G_7 is the graph obtained by recursively applying Type I operation. G_8 the graph obtained by applying Type III operation on the edge a_1. In G_8 we observe that regions R_1, R_2, R_6, R_7 share more than one edge common with the external region $R_9 R_{10}$. So G_8^* is not a simple graph, since we always retain the dual simple. We declare all except one edge between two regions as fake edges. In $G_9, a_9, a_{10}, a_8, a_7$ are declared as fake edges.

Type IV Operation

If by application of type III operation we generate a graph C such that $a_{1ij}, a_{2ij}, \cdots, a_{kij}$ are common edges between regions R_i and R_j declare one of $a_{mij}, m = 1, 2, \cdots k$ as a non fake edge and the remaining $(k-1)$ edges as fake edges as in Type II operation.

4.4 Tree Identification

By recursively applying Type I, II, III and IV operations on H we reach a graph E such that every non fake edge in E is the only non fake edge of some circuit in E. The fake edges form a spanning tree for E, the non fake edges are chords with respect to E. When we apply Type II operation on E as discussed in Section 4.2, E^* is a disconnected graph. The non fake edges is the smallest set of edges with this property, that is E^* is a minimally connected graph, implies E^* is a tree.

4.5 Complete Graph Identification

By recursively applying Type I, II, III and IV operations on H, we reach graph F such that number of regions in F is three or four. Verify if these regions are k - adjacent , k = 3 or 4. In such case F^* is a complete graph.

4.6 Chromatic Polynomial of G^*

Let G be any planar graph. Recursively applying Type I operation on G to generate H. We apply modified $G - \{e\}$ and $G_\bullet uv$ on H, to generate H_1, H_2, that is H_1 is the graph obtained by applying Type II operation on H, H_2 is the graph obtained by applying Type III operation on H (if necessary use Type I and Type IV operation). Continue this procedure recursively on H_1, H_2 and the remaining graph generated until we can not continue further. This means that either k - adjacent regions are identified $k = 3, 4$ or every edge that is not fake is the only non fake edge of some circuit. We now have a sequence of planar graphs $I_1, I_2, \cdots, I_k$ such that I_j^*, $j = 1, 2, \cdots, k$ is either a complete graph or a tree.

Type A

The graph generated by applying modified $G - \{e\}$ (Type II operation) on any graph.

Type B

The graph generated by applying modified $G_\bullet uv$ (applying Type I, Type IV if necessary) on any graph.

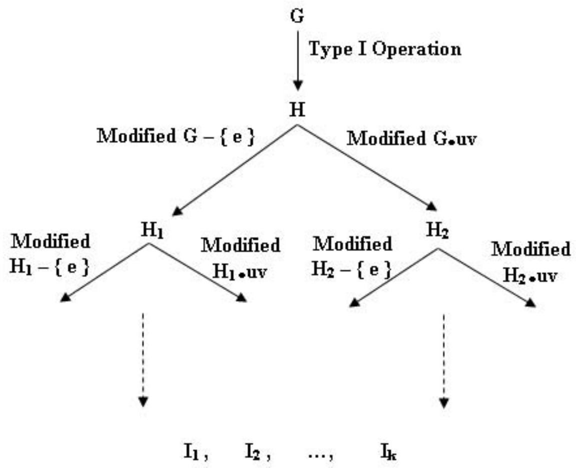
Note that $I_1, I_2, \cdots, I_k$ are either Type A or Type B graphs. Also H_1 is Type A graph, H_2 is Type B graph. We also know that Chromatic polynomial of H^* = Chromatic polynomial of H_1^* - Chromatic polynomial of H_2^* [2].

As discussed above we continue to apply modified $G - \{e\}$ and $G_\bullet uv$ recursively on H_1, H_2 to generate $I_1, I_2, \cdots, I_k$.

Number of vertices in I_j^* = Number of regions in I_j. Chromatic polynomial of

$$I_j^* = \begin{cases} \prod_{k=0}^{3} (\lambda - k) & \text{if } I_j \text{ is a 4-adjacent graph} \\ \prod_{k=0}^{2} (\lambda - k) & \text{if } I_j \text{ is a 3-adjacent graph} \\ \lambda(\lambda-1)^{(n-1)} & \text{otherwise} \end{cases}$$

$j = 1, 2, \cdots, k$. The procedure is summarized in Hierarchy - 1.



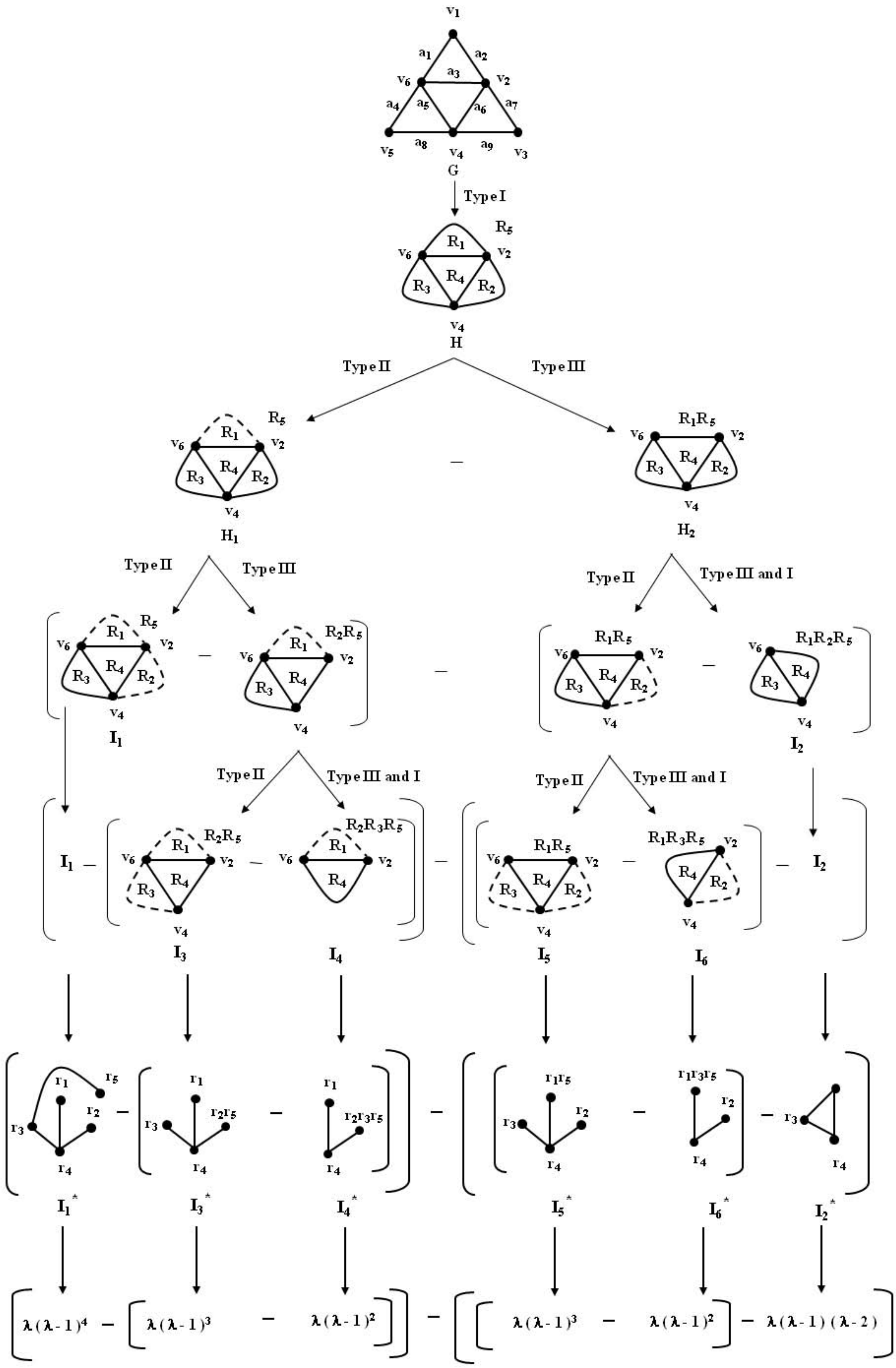
Hierarchy -1

Fig. 9 provides an example for finding the chromatic polynomial of G^*.

$$P_n(\lambda) \text{ of } G^* = \lambda(\lambda-1)^4 \\ -2\lambda(\lambda-1)^3 + 2\lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)$$

5 Spanning Tree for G^*

Let G be a planar graph and H be the graph obtained by applying Type I operation. Determine a spanning tree T for H. Declare these edges as fake edges, as discussed in Type II operation to generate H_1. Since T is a spanning tree for H all the non fake edges in H_1 are chords with respect to H. Since every chord



$$\textbf{Pn}(\lambda) \textbf{ of } G^* = \lambda(\lambda-1)^4 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)^2 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda(\lambda-1)^4 - 2\lambda(\lambda-1)^3 + 2\lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)$$

Figure 9:

generates a unique fundamental circuit every non fake edge is the only non fake edge of some circuit in H_1. H_1^* is a tree as discussed in Section 4.4, H_1^* is a spanning tree for G^*.

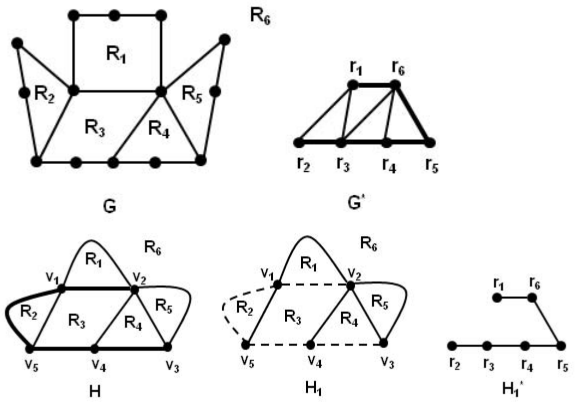


Figure 10:

In Fig. 10, G is a planar graph, G^* its dual (retained simple). Spanning tree of G^* is highlighted in bold lines. H obtained by applying Type I operation recursively (spanning tree for H is highlighted in bold lines). In H_1, the edges in spanning tree for H are declared fake edges. H_1^* is a spanning tree for G^* as seen in Fig. 10.

5.1 All Possible Spanning Trees for G^*

There is a simple and elegant recursive formula called Cayley's formula for the number of spanning trees in a graph.

Let $\tau(G)$ denote the number of spanning trees of G. If a is an edge of G, then $\tau(G) = \tau(G - a) + \tau(G_{\bullet}a)$ [2].

While generating the number of spanning trees of G, G need not to be a simple graph. So, while trying to find the number of spanning trees for G^*, G^* need not to be retained simple. Type I and Type IV operations are used to retain G^* a simple graph, which is not necessary in this case. As discussed in Section 4.2 and 4.3 , $G - \{a\}$ is equivalent to Type II and $G_{\bullet}\{a\}$ is equivalent to Type III operation respectively.

To obtain the number of spanning trees for G^* from G we can apply Type II and Type III operation recursively on G to generate a sequence of graphs $G_1, G_2, \cdots, G_k$, such that every non fake edge in $G_i, i = 1, 2, \cdots, k$ is a part of some circuit in G_i. G_i^* is a tree as discussed in Section 4.4. By Cayley's theorem $\tau(G^*) = k$.

Fig. 11 provides an example for generating the number of spanning trees of G^*. $\tau(G^*) = 8$

6 Matrix Representation

This section provides a method of determining the domination number of G^* using binary matrix defined on G, and a MATLAB code for the same.

Let G be graph with n - vertices.

$$N = [n_{ij}]_{n\times n} = \begin{cases} 1 & \text{if } i = j, \\ a_{ij} & \text{the } (i,j)^{th} \text{ entry in the adjacency matrix.} \end{cases}$$

Let $x = \langle x(v_1), x(v_2), \cdots, x(v_n)\rangle^T$ be a $\{0, 1\}$ vector. If x represents any dominating set, then $Nx \geq 1$, that is, in the resulting matrix Nx, all the entry values are non zero [9].

In this section we modify the above properties on a planar graph G with n- vertices, e-edges and k- regions $R_1, R_2, \cdots, R_k$ to identify all possible γ- sets of G^*. The number of vertices in G^* is equivalent to the number of regions in G. We define a region matrix R, with regions assumed to be ordered from $R_1, R_2, \cdots, R_k$ as

$$R = [r_{ij}]_{k\times k} = \begin{cases} 1 & \text{if there is an edge between } R_i \text{ and } R_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let N denote a $k \times k$ matrix, where

$$N = [n_{ij}]_{k\times k} = \begin{cases} 1 & \text{if } i = j, \\ r_{ij} & \text{the } (i,j)^{th} \text{ entry of the region matrix.} \end{cases}$$

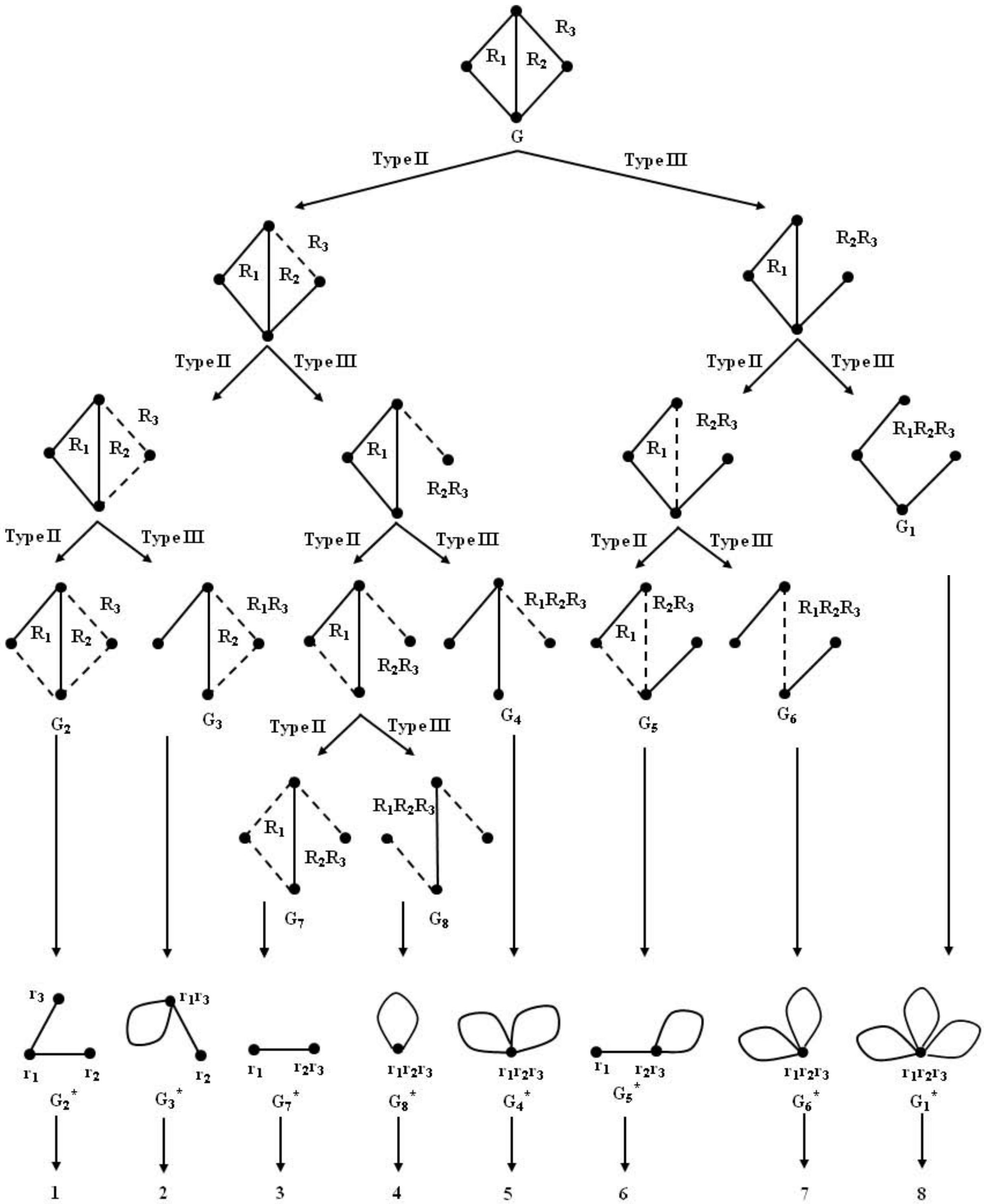
When two regions are adjacent in G, we know that the corresponding vertices are adjacent in G^*. Let $x = \langle x(R_1), x(R_2), \cdots, x(R_k)\rangle^T$ be a $\{0, 1\}$ vector. If x represents a set of dominating regions in G, then $Nx \geq 1$ [9].

Procedure

1. Consider all possible subsets with q regions, where $q = 1, 2, \cdots, k$. Label them as $S_1, S_2, \cdots, S_q$, where $q = 2^k - 1$. Let $X = \{x_1, x_2, \cdots, x_q\}$ be a set of $\{0, 1\}$ vectors defined by $x_i = \langle x(R_1), x(R_2), \cdots, x(R_k)\rangle^T$, where

$$x(R_i) = \begin{cases} 1 & \text{if } R_i \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

   Following the above notation if $k = 3$, then $S_1 = \{R_1\}$, $S_2 = \{R_2\}$, $S_3 = \{R_3\}$, $S_4 = \{R_1, R_2\}$, $S_5 = \{R_1, R_3\}$, $S_6 = \{R_2, R_3\}$,



Number of spanning trees of G* = 8

Figure 11:

$S_7 = \{R_1, R_2, R_3\}$. $x_1 = \langle 1,0,0\rangle^T, x_2 = \langle 0,1,0\rangle^T$, $x_3 = \langle 0,0,1\rangle^T$, $x_4 = \langle 1,1,0\rangle^T$, $x_5 = \langle 1,0,1\rangle^T$, $x_6 = \langle 0,1,1\rangle^T$, $x_7 = \langle 1,1,1\rangle^T$.

2. Nx_i is a column matrix. Let us denote this as vector $nx_i = \langle nx_i(R_1), nx_i(R_2), \cdots, nx_i(R_k)\rangle^T$.

3. Define a matrix of vectors V as $V = [R_{ij}]_{k\times q} = [x_1, x_2, \cdots, x_q]$, where each $x_i, i = 1, 2, \cdots, p$ denotes a vector defined in 1. Determine NV. This is a $k \times q$ matrix, where each column denotes vectors nx_i, that is the columns denote vector $nx_1, nx_2, \cdots, nx_q$.

4. Identify the first non - zero column in NV(say $nx_i, 1 \le i \le q$).

5. The numbers of 1's in x_i is the domination number of G^*.

In a planar graph G if there is no edge between R_i and R_j, then there is no edge between r_i and r_j in G^*, implies there is an edge between r_i and r_j in $\bar{G}^*$. To find the domination number of $\bar{G}^*$, the definition of region matrix R can be modified as

$$\bar{R} = [r_{ij}]_{k\times k} = \begin{cases} 1 & \text{if there is no edge between } R_i \text{ and } R_j, \\ 0 & \text{otherwise.} \end{cases}$$

Adopting the same procedure using matrix $\bar{R}$, the domination number of $\bar{G}^*$ can be determined.

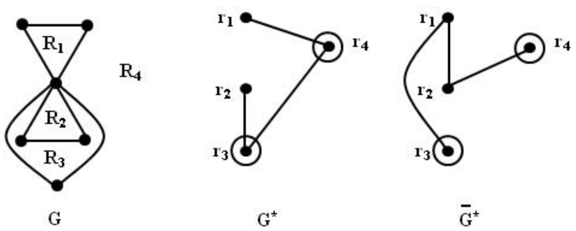


Figure 12:

In Fig. 12, G a planar graph, G^* its dual, $\bar{G}^*$ complement of G^*. $\gamma(G^*) = \gamma(\bar{G}^*) = 2$.

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$V = \begin{pmatrix} 0&0&0&1&0&0&0&1&1&1&0&1&1&1&1 \\ 0&0&1&0&0&1&1&0&0&1&1&0&1&1&1 \\ 0&1&0&0&1&0&1&0&1&0&1&1&0&1&1 \\ 1&0&0&0&1&1&0&1&0&0&1&1&1&0&1 \end{pmatrix}$$

$$NV = \begin{pmatrix} 1&0&0&1&1&1&0&2&1&1&1&2&2&1&2 \\ 0&1&1&0&1&1&2&0&1&1&2&1&1&2&2 \\ 1&1&1&0&2&2&2&1&1&1&3&2&2&2&3 \\ 1&1&0&1&2&1&1&2&2&1&2&3&2&2&3 \end{pmatrix}$$

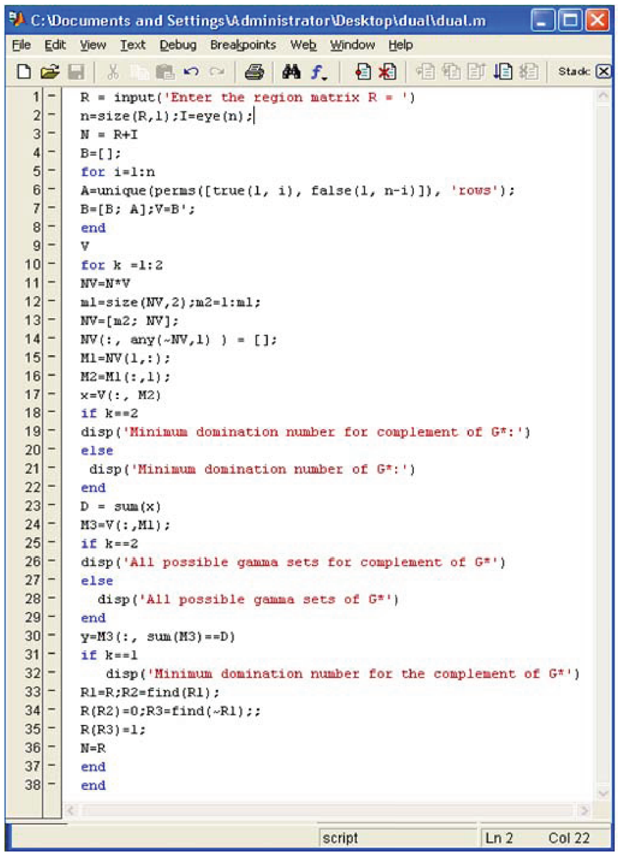
First nonzero column in NV is

$$Nx_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ where } x_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

The number of ones in x_5 is 2. So,$\gamma(G^*) = 2$. All possible γ- sets of G^*

$$y = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

A MATLAB Program for this procedure is provided in Snapshot 1.



Snapshot - 1

Domination number of $\bar{G}^*$

$$\bar{R} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$NV = \begin{pmatrix} 0&1&1&1&1&1&2&1&2&2&2&2&2&3&3 \\ 1&0&1&1&1&2&1&2&1&2&2&2&3&2&3 \\ 0&1&0&1&1&0&1&1&2&1&1&2&1&2&2 \\ 1&0&1&0&1&2&1&1&0&1&2&1&2&1&2 \end{pmatrix}$$

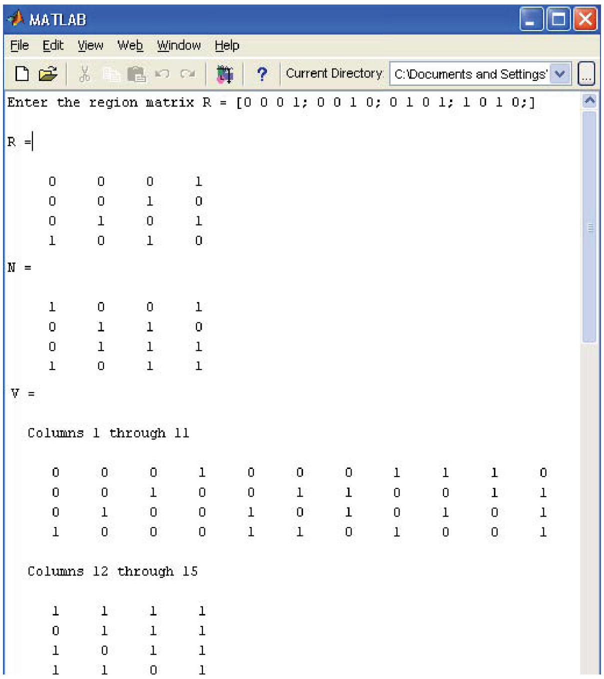
First nonzero column in NV is

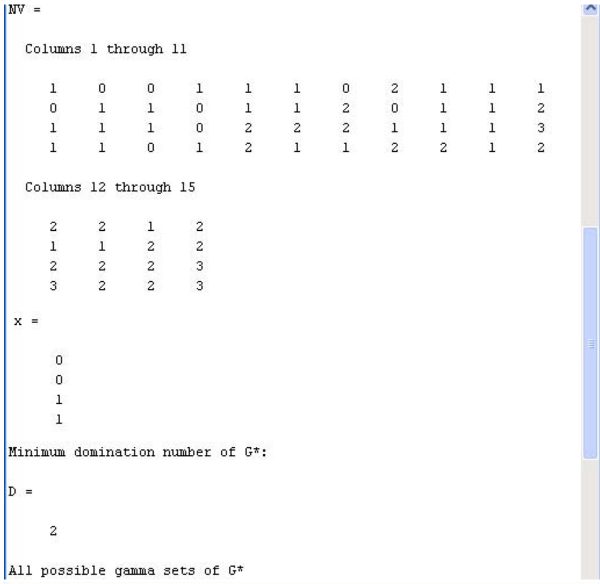
$$Nx_5 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \text{ where } x_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

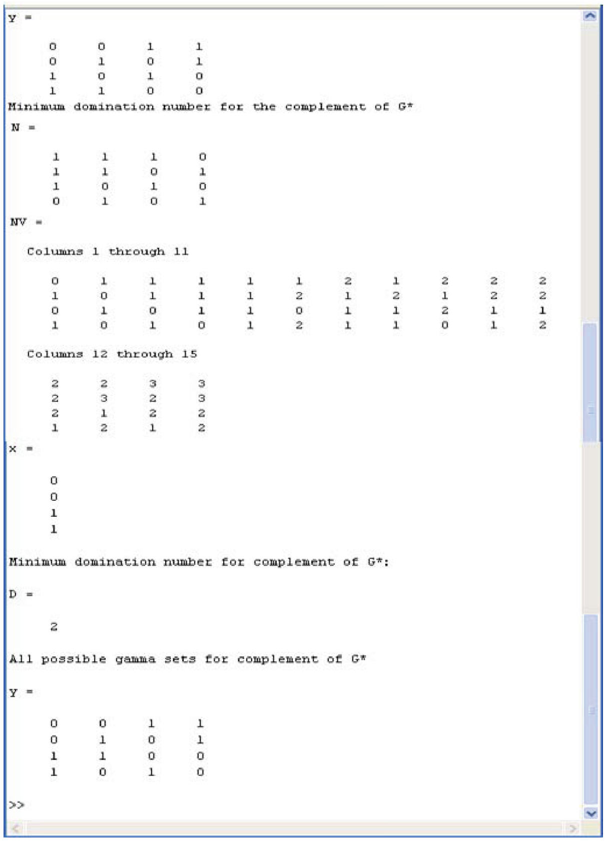
The number of ones in x_5 is 2. So,$\gamma(\bar{G^*}) = 2$.
All possible γ - sets of $\bar{G^*}$

$$y = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Snapshot 2 provides the output generated for the example in Fig. 12.







Snapshot - 2

From Snapshot 2, it can be verified that $\gamma(G^*) = \gamma(\bar{G^*}) = 2$. Also for G^*, $x = [0\ 0\ 1\ 1]^T$, that is, r_3, r_4 is a γ- set for G^* as seen in Fig. 12. Similarly for $\bar{G^*}, x = [0\ 0\ 1\ 1]^T$, that is, r_3, r_4 is a γ- set for $\bar{G^*}$ as shown in Fig. 12, which implies $\gamma(G^*) = \gamma(\bar{G^*}) = 2$.

From Snapshot 2, all possible γ - sets of G^* is $[0\ 0\ 1\ 1]^T$, $[0\ 1\ 1\ 0]^T$, $[1\ 0\ 0\ 1]^T$, $[1\ 1\ 0\ 0]^T$, that is the γ - sets are $\{r_3, r_4\}, \{r_2, r_3\}, \{r_1, r_4\}$, $\{r_1, r_2\}$. These sets are γ - sets for G^* as seen in Fig. 12. Similarly all possible γ - sets of $\bar{G^*}$ is $[0\ 0\ 1\ 1]^T$, $[0\ 1\ 1\ 0]^T$, $[1\ 0\ 0\ 1]^T$, $[1\ 1\ 0\ 0]^T$, that is the γ - sets are $\{r_3, r_4\}, \{r_2, r_3\}, \{r_1, r_4\}, \{r_1, r_2\}$. These sets are γ - sets for $\bar{G}^*$ as seen in Fig. 12.

7 Conclusion

Analyzing G^* without construction of G^* is comfortable in many cases. This paper contribute few properties of G^* from G. This can be extended further to furnish other properties of G^*

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