The k-path vertex cover of some product graphs

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Abstract: For a graph G and a positive integer k, a subset S of vertices of G is called a k-path vertex cover if every path of order k in G contains at least one vertex from S. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$. In this paper, we give some bounds and the exact values in special cases for ψ_k of the Cartesian, and lexicographic products of some graphs.

Key–Words: Vertex cover; k-path vertex cover; Cartesian product; lexicographic product

1 Introduction

Let x be a real number, denoted by $\lfloor x \rfloor$ the maximum integer no more than x, and denoted by $\lceil x \rceil$ the minimum integer no less than x. For any integers $a < b$, let $[a, b]$ denote the set of integers $\{a, a+1, \cdots, b\}$ for simplicity. We use $V(G), E(G)$ to denote the vertex set and the edge set of graph G, respectively. The order of a path P is the number of vertices on P while the length of a path is the number of edges of P.

In recent years, many parameters and graph classes were studied. For example, in [30], Zuo et al. gave the exact values of the linear $(n-1)$-arboricity of some Cartesian product graphs, in [31], Zuo showed that a Conjecture holds for all unicyclic graphs and all bicyclic graphs, in [28], Xue, Zuo et al. studied the hamiltonicity and path t-coloring of Sierpińński-like graphs, in [13], Jin and Zuo gave the further ordering bicyclic graphs with respect to the Laplacian spectra radius, in [16], Lai et al. gave a survey for the more recent developments of the research on supereulerian graphs and the related problems, and in [32], Zuo et al. studied the equitable colorings of Cartesian product graphs of wheels with complete bipartite graphs.

For a graph G and a positive integer k, a subset S of the vertex set of G is called a k-path vertex cover if every path of order k in G contains at least one vertex from S. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$. The motivation for the k-path vertex cover arises from secure communications in wireless sensor networks in [19]. The topology of wireless sensor networks can be represented by a graph, in which vertices represent sensor devices and edges represent communication channels between pairs of sensor devices. Traditional security techniques cannot be applied directly to wireless sensor networks since sensor devices are limited in their computation, energy, and communication capabilities. Furthermore, they are often deployed in accessible areas, where they can be captured by an attacker. Generally speaking, a standard sensor device is not taken into account as tamper-resistant and it is unnecessary to make all devices of a sensor network tamper-proof due to increasing cost. Hence, the design of wireless sensor networks safety contracts has become a challenge in security research. We focus on the Canvas scheme [8, 19, 20, 23] which should provide data integrity in a sensor network. The scheme combines the properties of cryptographic primitives and the network topology. The model distinguishes between two kinds of sensor element protected and unprotected. The attacker is incapable to copy secrets from a protector. This property can be realized by making the protector tamper-resistant or placing the protector at a safe location, where trapping is problematic. On the other hand, an unprotected device can be catched by the assailant, who can also copy secrets from the device and gain control over it. During the deployment and initialization of a sensor network, it should be ensured, that at least one defended node exists on each path of the length $k-1$ in the communication graph [19]. The matter to minimize the cost of the network by minimizing the number of protectors is expressed in [19].

The model of communications in wireless sensor networks is just equivalent the traffic control which was formulated in [25]. The increasing cars and buses result in more and more traffic accidents, hence posing the installment of cameras to be in an urgent state.

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If every crossing is installed with several cameras, the cost would be enormous and unnecessary since the installing fees can vary greatly due to different factors. Hence we need to install cameras at certain crossings that can ensure that a driver will encounter at least one camera within n crossings, and, at the same time, guarantee the lowest cost. This practical issue can, then, be turned into the vertex cover P_n problem.

The concept of k-path vertex cover is a generalization of the vertex cover. Clearly $\psi_2(G)$ coincides with the size of a minimum vertex cover, moreover

$$\psi_2(G) = |V(G)| - \alpha(G),$$

where $\alpha(G)$ is the independence number of graph G. This gives an interesting connection to the well studied independence number [11, 12, 22, 27].

A subset of vertices in graph G is called a dissociation set if it induce a subgraph with maximum degree at most 1. The number of vertices in a maximum cardinality set in G is called the dissociation number of G and is denoted by $diss(G)$. It is obvious that

$$\psi_3(G) = |V(G)| - diss(G).$$

It was shown that determining the dissociation number of G is NP-hard in the class of bipartite graphs [29]. The dissociation number problem was studied in [1, 2, 5, 9]. We can see a survey for this results in [21]. Some approximation algorithms for $\psi_3(G)$ were studied in [24, 25, 26] and an exact algorithm for computing $\psi_3(G)$ in running time $O(1.5171^n)$ for a graph of order n was presented in [14]. Also, a polynomial time randomized approximation algorithm with an expected approximation ratio of $\frac{23}{11}$ for the minimum 3-path vertex cover was presented.

It was shown that for any fixed integer $k \geq 2$ the computing $\psi_k(G)$ problem is in general NP-hard but for tree the problem can be solved in linear time, as shown in [3]. The authors also gave some upper bounds on the value of $\psi_k(G)$ and provide several estimations and the exact values of $\psi_k(G)$.

The concept of the k-path vertex cover was also studied in different graph products. The Cartesian product $G \boxtimes H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1 = u_2$ and $v_1v_2 \in E(G)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$.

The lexicographic product $G \circ H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u_1, v_1), (u_2, v_2)$ are adjacent whenever $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

For the Cartesian product of two paths, an asymptotically tight bound for ψ_3 and the exact value for ψ_3 was given in [4]. Also, an upper bound for ψ_3 and a lower bound of ψ_k of regular graphs were presented. Some bounds for the Cartesian product of two paths were improved in [17] and extended to the strong product of paths. In [17] some results for the lexicographic product of arbitrary graphs were also presented. For the lexicographic product of two arbitrary graphs, the bounds were presented for ψ_k, furthermore, ψ_2 and ψ_3 were exactly determined in [3]. Recently, a lower and an upper bounds for ψ_k of the rooted product graphs were presented in [18], moreover, ψ_2 and ψ_3 were exactly determined.

It is obvious that the following two results hold.

Lemma 1. *If positive integers $k \geq 2$ and $k \leq n$, then*

$$\psi_k(P_n) = \lfloor \tfrac{n}{k} \rfloor,$$
$$\psi_k(C_n) = \lceil \tfrac{n}{k} \rceil,$$
$$\psi_k(K_n) = n - k + 1.$$

Lemma 2. *If H is a subgraph of G and k is a positive integer, then*

$$\psi_k(G) \geq \psi_k(H).$$

This is trivial since we can obtain one k-path vertex cover $S \cap V(H)$ of H from every k-path vertex cover S of G for every subgraph H of G.

Lemma 3. *[4] For $k \geq 4$, $n \geq 2\lceil\sqrt{k}\rceil$, and $m \geq 3\lceil\sqrt{k}\rceil$, the following holds*

$$\psi_k(P_n \boxtimes P_m) \geq \frac{mn}{24\lceil\sqrt{k}\rceil}.$$

In this paper, we will present several results on ψ_k for Cartesian product and lexicographic product of some graphs.

2 Main results

Let G and H be arbitrary graphs, for a fixed vertex $v \in V(H)$, we refer to the set $V(G) \times \{v\}$ as a G-layer. Similarly $\{u\} \times V(H)$, for a fixed vertex $u \in V(G)$, is an H-layer. Whenever referring to a specific G- or H- layer, we denote them by G^v or ${}^uH$, respectively. It is clear that in the Cartesian and lexicographic products, a G-layer or H-layer is isomorphic to G or H, respectively.

Clearly, $\psi_1(G) = |V(G)|$ and $\psi_k(G) = 0$ for any graph G and each integer $k > |V(G)|$, so we always suppose that $2 \leq k \leq |V(G)|$ for $\psi_k(G)$ in the following.

Lemma 4. *If $n \geq 2$ and $\lceil\frac{n}{2}\rceil + 1 \leq k \leq n + 1$, then*

$$\psi_k(P_2 \boxtimes K_n) = n.$$

Proof. Firstly we will construct a k-path vertex cover with n vertices to prove that

$$\psi_k(P_2 \boxtimes K_n) \leq n.$$

Let

$$S_1 = \{(u_1, v_j) \in V(P_2 \boxtimes K_n) \mid k \leq j \leq n\}$$

with $|S_1| = n - k + 1$, and

$$S_2 = \{(u_2, v_j) \in V(P_2 \boxtimes K_n) \mid 1 \leq j \leq k - 1\}$$

with $|S_2| = k - 1$. It is easy to see that $S = S_1 \cup S_2$ is a k-path vertex cover, since the largest connected subgraph of $P_2 \boxtimes K_n$ with all vertices uncovered is isomorphic to K_{k-1}. So we have

$$\psi_k(P_2 \boxtimes K_n) \leq |S| = n.$$

Secondly we will prove that

$$\psi_k(P_2 \boxtimes K_n) \geq n.$$

Assume to the contrary that T is a k-path vertex cover of the graph $P_2 \boxtimes K_n$, with $|T| \leq n - 1$. Clearly there exist two vertices $(u_1, v_j), (u_2, v_j) \notin T$, where $1 \leq j \leq n$. Therefore, lying in the layer ${}^{u_1}K_n$, all the vertices which are not covered by T can form a path P_1 with the terminate vertex (u_1, v_j). And lying in the layer ${}^{u_2}K_n$, all the vertices which are not covered by T can form a path P_2 with the original vertex (u_2, v_j). Set

$$P = P_1 + (u_1, v_j)(u_2, v_j) + P_2.$$

Since

$$|V(P)| = 2n - |V(T)| \geq 2n - (n - 1) = n + 1 \geq k,$$

we have a path of order at least k with no vertices belong to T, a contradiction. □

Lemma 5. *If $n \geq 2$ and $k \geq n + 2$, then*

$$\psi_k(P_a \boxtimes K_n) \geq n,$$

where $a = \lceil \frac{k-1}{n} \rceil + 1$.

Proof. We will prove our result by contradiction. Assume to the contrary that S is a k-path vertex cover of the graph $P_a \boxtimes K_n$ with $|S| \leq n - 1$. Let (u_i, v_{j_i}) and (u_i, v_{l_i}) be the first vertex and the last vertex, which lie in P and belong to the layer ${}^{u_i}K_n$, respectively, where $1 \leq i \leq a$, $1 \leq j_i \leq n$ and $1 \leq l_i \leq n$. Since

$$\begin{aligned} &|V(P_a \boxtimes K_n)| - |V(S)| \\ &\geq n(\lceil \tfrac{k-1}{n} \rceil + 1) - (n - 1) \\ &\geq n(\tfrac{k-1}{n} + 1) - (n - 1) = k, \end{aligned}$$

we only need to show that all the vertices of $P_a \boxtimes K_n$, not covered by S, can form a path P. Moreover, if $(u_i, v_{j_i}) \neq (u_i, v_{l_i})$, then all the vertices that lie in P and between (u_i, v_{j_i}) and (u_i, v_{l_i}) should belong to $V({}^{u_i}K_n)$. We will show our result by induction for a.

Claim 1. For $a = 3$, we can get a contradiction.

We will deal with the result in three cases.

Case 1. Suppose that all the vertices which belong to S lie in the same layer ${}^{u_i}K_n$ completely for some $i \in [1, 3]$. It is easy to prove that all the vertices of $P_a \boxtimes K_n$, not covered by S, can induce a path since there is a vertex $(u_i, v_j) \notin S$ for some $1 \leq j \leq n$, a contradiction.

Case 2. Suppose that all the vertices which belong to S lie in two layers ${}^{u_b}K_n$ and ${}^{u_c}K_n$ completely, where $1 \leq b \neq c \leq 3$.

Clearly, in this case, $n \geq 3$ and there are at least 2 vertices on each layer which are not covered by S. If all the vertices which belong to S lie in the two layers ${}^{u_1}K_n$ and ${}^{u_2}K_n$, then there exist two vertices (u_1, v_j), $(u_2, v_j) \notin S$ for some j with $1 \leq j \leq n$. Therefore, lying in the layer ${}^{u_1}K_n$, all the vertices which are not covered by S can form a path P_1 with the terminate vertex (u_1, v_j); Lying in the layer ${}^{u_2}K_n$, all the vertices which are not covered by S can form a path P_2 with the original vertex (u_2, v_j) and the terminate vertex (u_2, v_l); All the vertices which lie in the layer ${}^{u_3}K_n$ can form a path P_3 with the original vertex (u_3, v_l), where $1 \leq l \leq n$ and $l \neq j$. Set

$$P = P_1 + (u_1, v_j)(u_2, v_j) + P_2 + (u_2, v_l)(u_3, v_l) + P_3.$$

Then all the vertices of $P_a \boxtimes K_n$ which are not covered by S induce a path P, a contradiction.

If all the vertices which belong to S lie in the two layers ${}^{u_2}K_n$ and ${}^{u_3}K_n$, then we can get a contradiction similarly.

If all the vertices which belong to S lie in the two layers ${}^{u_1}K_n$ and ${}^{u_3}K_n$, then $n \geq 3$ and there exist vertices $(u_1, v_j), (u_3, v_l) \notin S$, where $1 \leq j \neq l \leq n$. Therefore, lying in the layer ${}^{u_1}K_n$, all the vertices which are not covered by S can form a path P_1 with the terminate vertex (u_1, v_j); All the vertices which lie in the layer ${}^{u_2}K_n$ can form a path P_2 with the original vertex (u_2, v_j) and the terminate vertex (u_2, v_l); Lying in the layer ${}^{u_3}K_n$, all the vertices which are not covered by S can form a path P_3 with the original vertex (u_3, v_l). Set

$$P = P_1 + (u_1, v_j)(u_2, v_j) + P_2 + (u_2, v_l)(u_3, v_l) + P_3.$$

Then, all the vertices of $P_a \boxtimes K_n$ which are not covered by S form a path P with order at least k, a contradiction, too.

Case 3. Suppose that $S \cap^{u_i} K_n \neq \emptyset$ for each $i \in [1,3]$. Then $n \geq 4$ and there exist four vertices $(u_1, v_j), (u_2, v_j), (u_2, v_l)$ and (u_3, v_l) which are not in S for some $1 \leq j \neq l \leq n$, since $|S| \leq n-1$. Therefore, lying in the layer ${}^{u_1}K_n$, all the vertices which are not covered by S can form a path P_1 with the terminate vertex (u_1, v_j); Lying in the layer ${}^{u_2}K_n$, all the vertices which are not covered by S can form a path P_2 with the original vertex (u_2, v_j) and the terminate vertex (u_2, v_l); Lying in the layer ${}^{u_3}K_n$, all the vertices which are not covered by S can form a path P_3 with the original vertex (u_3, v_l). Set

$$P = P_1 + (u_1, v_j)(u_2, v_j) + P_2 + (u_2, v_l)(u_3, v_l) + P_3.$$

Then, not covered by S, all the vertices of $P_a \boxtimes K_n$ form a path P with order at least k, and thus we also get a contradiction.

Claim 2. Assume that the lemma is true for $a = q \geq 3$. Then we can get a contradiction for $a = q+1$.

In the following we will deal with the result in three cases for $a = q+1$.

Case 1. If $S \subseteq (V(P_{q+1} \boxtimes K_n) - V({}^{u_{q+1}}K_n))$, then, by the induction hypothesis, we assume that, all the vertices of $(V(P_{q+1} \boxtimes K_n) - V({}^{u_{q+1}}K_n))$ that are not covered by S can form a path P_1 with the terminate vertex (u_q, v_l), where $1 \leq l \leq n$. All the vertices which lie in the layer ${}^{u_{q+1}}K_n$ can form a path P_2 with the original vertex (u_{q+1}, v_l). Set

$$P = P_1 + (u_q, v_l)(u_{q+1}, v_l) + P_2.$$

Then all the vertices of $P_{q+1} \boxtimes K_n$ which are not covered by S form a path P with order at least k, a contradiction.

Case 2. If $S \subseteq (V(P_{q+1} \boxtimes K_n) - V({}^{u_1}K_n))$, then we can get a contradiction similarly.

Case 3. Assume that $S \cap^{u_1} K_n \neq \emptyset$ and $S \cap^{u_{q+1}} K_n \neq \emptyset$. By the induction hypothesis, all the vertices which are not covered by S and lie in layers ${}^{u_i}K_n$ can form a path P_1 with the terminate vertex lying in the layer ${}^{u_q}K_n$, where $1 \leq i \leq q$. Let (u_{q-1}, v_j) be the last vertex lying in P_1 and belonging to the layer ${}^{u_{q-1}}K_n$. Denote by V_1 the vertex set of (u_{q-1}, v_j) together with all the vertices which lie in P_1 and precede (u_{q-1}, v_j). Set $P_2 = P_1[V_1]$, and let (u_{q-1}, v_j) be the terminate vertex of P_2, where $1 \leq j \leq n$. Since $|V(P_2)| \geq 1$, there are at most $n-2$ vertices being covered by S and lying in the two layers ${}^{u_q}K_n$ and ${}^{u_{q+1}}K_n$. So, there exist vertices (u_q, v_l), $(u_{q+1}, v_l) \notin S$, where $1 \leq l \leq n$ and $l \neq j$. Lying in the layer ${}^{u_q}K_n$, all the vertices which are not covered by S can form a path P_3 with the original vertex (u_q, v_j) and the terminate vertex (u_q, v_l); Lying in the layer ${}^{u_{q+1}}K_n$, all the vertices which are not covered by S can form a path P_4 with the original vertex (u_{q+1}, v_l). Set

$$\begin{aligned} P = {} & P_2 + (u_{q-1}, v_j)(u_q, v_j) + P_3 \\ & + (u_q, v_l)(u_{q+1}, v_l) + P_4. \end{aligned}$$

Then, all the vertices of $P_a \boxtimes K_n$ which are not covered by S form a path P with order at least k, a contradiction, too. □

Theorem 6. *For $m \geq 2$ and $n \geq 2$, the following holds*

(1) If $2 \leq k \leq \lceil \frac{n}{2} \rceil$, then

$$\psi_k(P_m \boxtimes K_n) = m(n-k+1).$$

(2) If $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n+1$, then

$$\psi_k(P_m \boxtimes K_n) = \begin{cases} \frac{mn}{2}, & \text{if } m \text{ is even}, \\ \frac{(m+1)n}{2} - k + 1, & \text{if } m \text{ is odd}. \end{cases}$$

(3) Let $n+1 < k < n\lceil \frac{m-1}{2} \rceil + 1$. If

$$mn \equiv l(mod\ (n+k-1))$$

for $l \in [1, k-1] \cup \{0\}$, then

$$n\lfloor \frac{m}{\lceil \frac{k-1}{n} \rceil + 1} \rfloor \leq \psi_k(P_m \boxtimes K_n) \leq n\lfloor \frac{mn}{n+k-1} \rfloor.$$

Moreover, if $k \equiv 1(mod\ n)$, then

$$\psi_k(P_m \boxtimes K_n) = n\lfloor \frac{mn}{n+k-1} \rfloor.$$

If $mn \equiv l(mod\ (n+k-1))$ for $l \in [k, n+k-2]$, then

$$\begin{aligned} & n\lfloor \tfrac{m}{\lceil \frac{k-1}{n} \rceil + 1} \rfloor \leq \psi_k(P_m \boxtimes K_n) \\ & \leq mn - (k-1)\lfloor \tfrac{mn}{n+k-1} \rfloor. \end{aligned}$$

(4) If $n\lceil \frac{m-1}{2} \rceil + 1 \leq k \leq (m-1)n+1$, then

$$\psi_k(P_m \boxtimes K_n) = n.$$

(5) If $(m-1)n+2 \leq k \leq mn$, then

$$\psi_k(P_m \boxtimes K_n) = mn - k + 1.$$

Proof. (1) Firstly we will construct a k-path vertex cover with $m(n-k+1)$ vertices to prove that

$$\psi_k(P_m \boxtimes K_n) \leq m(n-k+1).$$

Let

$$S_i = \{(u_i, v_j) \in V(P_m \boxtimes K_n) | k \leq j \leq n\}$$

for odd i and

$$S_i = \{(u_i, v_j) \in V(P_m \boxtimes K_n) | 1 \leq j \leq n-k+1\}$$

for even i, where $1 \leq i \leq m$. Clearly $|S_i| = n-k+1$. It is obvious that $S = \cup_{i=1}^{m} S_i$ is a k-path vertex cover, since the largest connected subgraph of $P_m \boxtimes K_n$ with all vertices uncovered is isomorphic to K_{k-1}. Therefore,

$$\psi_k(P_m \boxtimes K_n) \leq |S| = m(n-k+1).$$

Secondly, since each layer ${}^{u_i}K_n$ is isomorphic to K_n for $1 \leq i \leq m$, we need at least $\psi_k(K_n)$ vertices to cover each K_n-layer and we have m such layers. Therefore,

$$\psi_k(P_m \boxtimes K_n) \geq m\psi_k(K_n) = m(n-k+1)$$

by Lemma 1.

(2) Firstly, we will construct a k-path vertex cover S to prove that

$$\psi_k(P_m \boxtimes K_n) \leq \begin{cases} \frac{mn}{2}, & \text{if } m \textit{ is even}, \\ \frac{(m+1)n}{2} - k + 1, & \text{if } m \textit{ is odd}. \end{cases}$$

Let

$$S_i = \{(u_i, v_j) \in V(P_m \boxtimes K_n) | k \leq j \leq n\}$$

with $|S_i| = n-k+1$ for odd i and

$$S_i = \{(u_i, v_j) \in V(P_m \boxtimes K_n) | 1 \leq j \leq k-1\}$$

with $|S_i| = k-1$ for even i, where $1 \leq i \leq m$. It is clear that $S = \cup_{i=1}^{m} S_i$ is a k-path vertex cover since the largest connected subgraph of $P_m \boxtimes K_n$ with all vertices uncovered is isomorphic to K_{k-1}. Therefore, we have

$$\psi_k(P_m \boxtimes K_n) \leq |S| = \begin{cases} \frac{mn}{2}, \ if\ m\ is\ even, \\ \frac{(m+1)n}{2} - k + 1, \\ \qquad if\ m\ is\ odd. \end{cases}$$

Now we show that $\psi_k(P_m \boxtimes K_n) \geq |S|$. If $m = 2$, then the conclusion is true by Lemma 4. Suppose that $m \geq 3$ in the following. We delete the edges between the two layers ${}^{u_{2i}}K_n$ and ${}^{u_{2i+1}}K_n$, where $2 \leq 2i < 2i+1 \leq m$.

If m is even, then the graph $P_m \boxtimes K_n$ can be partitioned into $\frac{m}{2}$ disjoint subgraphs which are isomorphic to $P_2 \boxtimes K_n$. We need at least $\psi_k(P_2 \boxtimes K_n)$ vertices to cover each subgraph that is isomorphic to $P_2 \boxtimes K_n$. Therefore,

$$\psi_k(P_m \boxtimes K_n) \geq \frac{m}{2}\psi_k(P_2 \boxtimes K_n) = \frac{mn}{2}.$$

If m is odd, then the graph $P_m \boxtimes K_n$ can be partitioned into $\frac{m-1}{2}$ disjoint subgraphs which are isomorphic to $P_2 \boxtimes K_n$ and a subgraph that is isomorphic to K_n. We need at least $\psi_k(P_2 \boxtimes K_n)$ vertices to cover each subgraph that is isomorphic to $P_2 \boxtimes K_n$ and at least $\psi_k(K_n)$ vertices to cover the subgraph that is isomorphic to K_n. Therefore,

$$\begin{aligned} \psi_k(P_m \boxtimes K_n) &\geq \tfrac{m-1}{2}\psi_k(P_2 \boxtimes K_n) + \psi_k(K_n) \\ &= \tfrac{(m+1)n}{2} - k + 1. \end{aligned}$$

(3) Let vertex (u_i, v_j) label $n(i-1)+j$, and

$$\begin{aligned} S = \{&(u_i, v_j) | 1 \leq i \leq m, 1 \leq j \leq n, \\ &n(i-1)+j \equiv l(mod\ (n+k-1)), \\ &l \in [k, n+k-2] \cup \{0\}\}. \end{aligned}$$

Clearly, the labels of vertices in $V(P_m \boxtimes K_n)$ are $[1, mn]$, and $|S| \geq n$ since

$$n + k - 1 \leq n + n\lceil \frac{m-1}{2} \rceil \leq n + \frac{mn}{2} \leq mn.$$

We shall show that S is a k-path vertex cover of $P_m \boxtimes K_n$. Let T be the vertex set of any n consecutive label vertices and $T \neq V({}^{u_1}K_n)$. For any such vertex set T, we have $|T \cap V(P_m^{v_j})| = 1$ for every $1 \leq j \leq n$. Then, for a fixed such vertex set T, there exist i with $1 \leq i \leq m-1$, such that $V(T) \subset V({}^{u_i}K_n \cup {}^{u_{i+1}}K_n)$. There are at most $\lfloor \frac{mn}{n+k-1} \rfloor$ groups of such vertex set T of n consecutive label vertices in S. Remove all vertices $(u_i, v_j) \in S$ and all edges incident with them from $P_m \boxtimes K_n$. We can get at most $\lceil \frac{mn}{n+k-1} \rceil$ components with order at most $k-1$, so S is a k-path vertex cover of $P_m \boxtimes K_n$. If $mn \equiv l(mod\ (n+k-1))$ for $l \in [1, k-1] \cup \{0\}$, then

$$\psi_k(P_m \boxtimes K_n) \leq |S| = n\lfloor \frac{mn}{n+k-1} \rfloor.$$

If $mn \equiv l(mod\ (n+k-1))$ for $l \in [k, n+k-2]$, then

$$\psi_k(P_m \boxtimes K_n) \leq |S| = mn - (k-1)\lfloor \frac{mn}{n+k-1} \rfloor.$$

Let

$$\begin{aligned} a &= \lceil \tfrac{k-1}{n} \rceil + 1 \leq \lceil \tfrac{n\lceil \frac{m-1}{2} \rceil + 1 - 1}{n} \rceil + 1 \\ &= \lceil \tfrac{m-1}{2} \rceil + 1 \leq \tfrac{m}{2} + 1 \leq m. \end{aligned}$$

If $m = a$, then the lower bound is true. Suppose that $m \geq a+1$. We delete the edges between the two layers ${}^{u_{ai}}K_n$ and ${}^{u_{ai+1}}K_n$, where $a \leq ai < ai+1 \leq m$. We can partition the graph $P_m \boxtimes K_n$ into r disjoint subgraphs which are isomorphic to $P_a \boxtimes K_n$, where $r = \lfloor \frac{m}{\lceil \frac{k-1}{n} \rceil + 1} \rfloor$. We need at least $\psi_k(P_a \boxtimes K_n)$

vertices to cover each subgraph that is isomorphic to $P_a \boxtimes K_n$. According to Lemma 5, we have

$$\psi_k(P_m \boxtimes K_n) \geq r\psi_k(P_a \boxtimes K_n) \geq n\lfloor \frac{m}{\lceil \frac{k-1}{n} \rceil + 1} \rfloor.$$

Clearly, if $mn \equiv l(mod\ (n+k-1))$ for $l \in [1, k-1] \cup \{0\}$, then the two bounds of $\psi_k(P_m \boxtimes K_n)$ are equal when $k \equiv 1(mod\ n)$, hence we can get

$$\psi_k(P_m \boxtimes K_n) = n\lfloor \frac{mn}{m+k-1} \rfloor$$

in this case.

(4) Let

$$x = \lceil \frac{m-1}{2} \rceil + 1$$

and

$$S = \{(u_x, v_j) \in V(P_m \boxtimes K_n) | 1 \leq j \leq n\}.$$

It is easy to see that S is a k-path vertex cover of $P_m \boxtimes K_n$, since the largest connected subgraph of $P_m \boxtimes K_n$ with all vertices uncovered is isomorphic to $P_{x-1} \boxtimes K_n$ and

$$|V(P_{x-1} \boxtimes K_n)| = n\lceil \frac{m-1}{2} \rceil \leq k-1.$$

Therefore,

$$\psi_k(P_m \boxtimes K_n) \leq |S| = n.$$

On the other hand, set $a = \lceil \frac{k-1}{n} \rceil + 1$. Since

$$a \leq \lceil \frac{(m-1)n+1-1}{n} \rceil + 1 = m,$$

$$P_a \boxtimes K_n \subseteq P_m \boxtimes K_n.$$

According to Lemmas 2 and 5, we have

$$\psi_k(P_m \boxtimes K_n) \geq \psi_k(P_a \boxtimes K_n) \geq n.$$

(5) Let $S = \{(u_1, v_j) \in V(P_m \boxtimes K_n) | 1 \leq j \leq mn-k+1\}$. It is easy to see that S is a k-path vertex cover of $P_m \boxtimes K_n$, since the order of the largest connected subgraph of $P_m \boxtimes K_n$ with all vertices uncovered is at most $k-1$. Therefore, $\psi_k(P_m \boxtimes K_n) \leq |S| = mn-k+1$.

On the other hand, the process of proving $\psi_k(P_m \boxtimes K_n) \geq mn-k+1$ is similar to the proof of Lemma 5. □

As seen in the previous theorems, it is very hard to determine exact results for product of fixed graphs G and H. Next we give some lower bounds of $\psi_k(P_m \boxtimes P_n)$ for general m and n.

Theorem 7. *If $m \geq 2, n \geq 2$ and $k \geq 2$ are positive integers, then*

$$\psi_k(P_m \boxtimes P_n) \geq \max\{2\lfloor \frac{m}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{n}{2} \rfloor + \lfloor \frac{mn - (2\lfloor \frac{k}{2} \rfloor + 2)\lfloor \frac{m}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{n}{2} \rfloor}{k} \rfloor, 2\lfloor \frac{n}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{m}{2} \rfloor + \lfloor \frac{mn - (2\lfloor \frac{k}{2} \rfloor + 2)\lfloor \frac{n}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{m}{2} \rfloor}{k} \rfloor\}.$$

Proof. As seen in Fig.1, we partition the graph $P_m \boxtimes P_n$ into x disjoint subgraphs which are isomorphic to C_y, where $x = \lfloor \frac{m}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{n}{2} \rfloor$ and $y = 2\lfloor \frac{k}{2} \rfloor + 2$. The remain vertices can construct a path of order z, where $z = mn - xy$. We need at least $\psi_k(C_y)$ vertices to cover each subgraph that is isomorphic to C_y and at least $\psi_k(P_z)$ vertices to cover the subgraph that is isomorphic to P_z. Therefore,

$$\psi_k(P_m \boxtimes P_n) \geq x\psi_k(C_y) + \psi_k(P_z) = 2x + \lfloor \frac{z}{k} \rfloor = 2\lfloor \frac{m}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{n}{2} \rfloor + \lfloor \frac{mn - (2\lfloor \frac{k}{2} \rfloor + 2)\lfloor \frac{m}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{n}{2} \rfloor}{k} \rfloor.$$

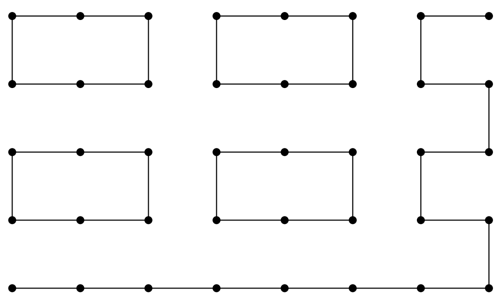

Figure 1: A partition of $P_m \boxtimes P_n$.

Similarly, we can obtain

$$\psi_k(P_m \boxtimes P_n) \geq 2\lfloor \frac{n}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{m}{2} \rfloor + \lfloor \frac{mn - (2\lfloor \frac{k}{2} \rfloor + 2)\lfloor \frac{n}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{m}{2} \rfloor}{k} \rfloor.$$

□

Lemma 3 and Theorem 7 give two lower bounds of $\psi_k(P_n \boxtimes P_m)$. When k is relatively small, the lower bound of Theorem 7 is better than Lemma 3.

Corollary 8. *If m is a positive integer, then*

$$\psi_5(P_2 \boxtimes P_m) = 2\lfloor \frac{m}{3} \rfloor.$$

Proof. If $m < 3$, the conclusion is true. Suppose that $m \geq 3$. We will construct a 5-path vertex cover of order $2\lfloor \frac{m}{3} \rfloor$. Let

$$S = \{(u_i, v_j) | 1 \leq i \leq 2, j \equiv 0(mod\ 3)\}.$$

It is clear that S is a 5-path vertex cover, since the largest connected subgraph of $P_2 \boxtimes P_m$ induced with all vertices uncovered is isomorphic to $P_2 \boxtimes P_2$. So, we have obtained

$$\psi_5(P_2 \boxtimes P_m) \leq |S| = 2\lfloor \frac{m}{3} \rfloor.$$

On the other hand, according to Theorem 7, we have $\psi_5(P_2 \boxtimes P_m) \geq 2\lfloor \frac{m}{3} \rfloor$. □

Corollary 9. *Let $m \geq 2, n \geq 2$ and $k \geq 2$ be positive integers. For any positive number ε, we have*

$$\frac{\psi_k(P_m \boxtimes P_n)}{|V(P_m \boxtimes P_n)|} \geq \frac{1}{\lfloor \frac{k}{2} \rfloor + 1} - \varepsilon,$$

when m and n are large sufficiently.

Proof. Let

$$m = a(\lfloor \frac{k}{2} \rfloor + 1) + c$$

and

$$n = 2b + d,$$

where $c \in [0, \lfloor \frac{k}{2} \rfloor]$ and $d = 0$ or 1. According to Theorem 7, we have

$$\begin{aligned} \frac{\psi_k(P_m \boxtimes P_n)}{|V(P_m \boxtimes P_n)|} &\geq 2\lfloor \frac{m}{\lfloor \frac{k}{2} \rfloor + 1} \rfloor \lfloor \frac{n}{2} \rfloor \frac{1}{mn} \\ &= 2\frac{m-c}{\lfloor \frac{k}{2} \rfloor + 1} \frac{n-d}{2} \frac{1}{mn} \\ &= (1 - \frac{c}{m} - \frac{d}{n} + \frac{cd}{mn}) \frac{1}{\lfloor \frac{k}{2} \rfloor + 1} \\ &\to \frac{1}{\lfloor \frac{k}{2} \rfloor + 1} (m, n \to +\infty). \end{aligned}$$

Therefore, we have

$$\frac{\psi_k(P_m \boxtimes P_n)}{|V(P_m \boxtimes P_n)|} \geq \frac{1}{\lfloor \frac{k}{2} \rfloor + 1} - \varepsilon,$$

when m and n are large sufficiently. □

Next we give some lower bounds for $\psi_k(P_m \boxtimes P_n^2)$.

Theorem 10. *Let $k \geq 2$, $m \geq min\{\frac{k}{2}, 2\}$ and $n \geq k$ be positive integers. Then*

$$\psi_k(P_m \boxtimes P_n^2) \geq \begin{cases} 4\lfloor \frac{n}{\lceil \frac{k}{2} \rceil + 2} \rfloor \lfloor \frac{m}{2} \rfloor + \lfloor \frac{mn - (2\lceil \frac{k}{2} \rceil + 4)\lfloor \frac{n}{\lceil \frac{k}{2} \rceil + 2} \rfloor \lfloor \frac{m}{2} \rfloor}{k} \rfloor, & for\ n \equiv 1 (mod\ (\lceil \frac{k}{2} \rceil + 2)), \\ 4\lfloor \frac{n}{\lceil \frac{k}{2} \rceil + 2} \rfloor \lfloor \frac{m}{2} \rfloor + \lceil \frac{mn - (2\lceil \frac{k}{2} \rceil + 4)\lfloor \frac{n}{\lceil \frac{k}{2} \rceil + 2} \rfloor \lfloor \frac{m}{2} \rfloor}{k} \rceil, & otherwise. \end{cases}$$

Proof. Let $H = P_2 \boxtimes P_a^2$, where $a = \lceil \frac{k}{2} \rceil + 2$.

Claim 1. If positive integers $k \geq 2$ and $a = \lceil \frac{k}{2} \rceil + 2$, then $\psi_k(H) = 4$, where $H = P_2 \boxtimes P_a^2$.

If $2 \leq k \leq 4$, then $\psi_k(H) = 4$, so we suppose that $k \geq 5$.

Firstly we construct a k-path vertex cover S with $|S| = 4$ to prove that $\psi_k(H) \leq 4$. Let $S = \{(u_1, v_2), (u_1, v_3), (u_2, v_2), (u_2, v_3)\}$. Remove from G the vertex $(u_i, v_j) \in S$ and all edges incident with (u_i, v_j). We can get two disjoint subgraphs P_2 and $P_2 \boxtimes P_{a-3}^2$. Then S is a k-path vertex cover of H, since

$$|V(P_2 \boxtimes P_{a-3}^2)| = 2\lceil \frac{k}{2} \rceil - 2 \leq 2\frac{k+1}{2} - 2 = k - 1.$$

Therefore, $\psi_k(H) \leq |S| = 4$.

Since $C_{2a} \subset H$, $\psi_k(H) \geq \psi_k(C_{2a}) = 2$. Suppose T is a minimum k-path vertex cover of H with $|T| \geq 2$. Assume two different vertices (u_i, v_j) and (u_p, v_q) belong to T and $T_1 = \{(u_i, v_j), (u_p, v_q)\}$, where $1 \leq i, p \leq 2$ and $1 \leq j, q \leq a$. We can show that the vertices which belong to $V(H) \setminus T_1$ always form a circle of order $2a - 2$ or $2a - 3$. Since $\psi_k(C_{2a-2}) = \psi_k(C_{2a-3}) = 2$, we need at least two more vertices that lie in each constructed circle to belong to T. Therefore, $\psi_k(H) = |T| \geq |T_1| + 2 = 4$ and then the claim is proved.

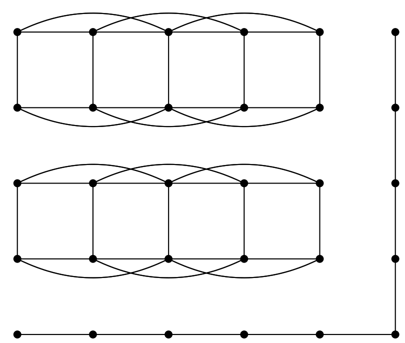

Figure 2: A partition of $P_m \boxtimes P_n^2$ for $n \equiv 1(mod\ (\lceil \frac{k}{2} \rceil + 2))$.

As seen in Fig.2, if

$$n \equiv 1(mod\ (\lceil \frac{k}{2} \rceil + 2)),$$

then we can partition the graph $P_m \boxtimes P_n^2$ into x disjoint subgraphs which are isomorphic to H and a path of order y, where

$$x = \lfloor \frac{n}{\lceil \frac{k}{2} \rceil + 2} \rfloor \lfloor \frac{m}{2} \rfloor$$

and

$$y = mn - x(2\lceil\frac{k}{2}\rceil + 4).$$

We need at least $\psi_k(H)$ vertices to cover each subgraph that is isomorphic to H and at least $\psi_k(P_y)$ vertices to cover the subgraph that is isomorphic to P_y. Therefore,

$$\psi_k(P_m \boxtimes P_n^2) \geq x\psi_k(H) + \psi_k(P_y) = 4x + \lfloor\frac{y}{k}\rfloor$$
$$= 4\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor + \lfloor\frac{mn-(2\lceil\frac{k}{2}\rceil+4)\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor}{k}\rfloor.$$

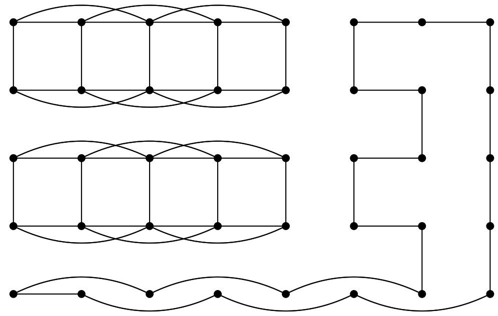

Figure 3: A partition of $P_m \boxtimes P_n^2$ for $n \not\equiv 1(mod\ (\lceil\frac{k}{2}\rceil + 2))$.

As seen in Fig.3, if

$$n \not\equiv 1(mod\ (\lceil\frac{k}{2}\rceil + 2)),$$

then we can partition the graph $P_m \boxtimes P_n^2$ into x disjoint subgraphs which are isomorphic to H and a cycle of order y, where

$$x = \lfloor\frac{n}{\lceil\frac{k}{2}\rceil + 2}\rfloor\lfloor\frac{m}{2}\rfloor$$

and

$$y = mn - x(2\lceil\frac{k}{2}\rceil + 4).$$

If $m \equiv 0(mod\ 2)$ and $n \equiv 0(mod\ (\lceil\frac{k}{2}\rceil + 2))$, then $y = 0$; otherwise, $y \geq \min\{n, 2m\} \geq k$. We need at least $\psi_k(H)$ vertices to cover each subgraph that is isomorphic to H and at least $\psi_k(C_y)$ vertices to cover the subgraph that is isomorphic to C_y. Therefore,

$$\psi_k(P_m \boxtimes P_n^2) \geq x\psi_k(H) + \psi_k(C_y) = 4x + \lceil\frac{y}{k}\rceil$$
$$= 4\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor + \lceil\frac{mn-(2\lceil\frac{k}{2}\rceil+4)\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor}{k}\rceil.$$

□

Corollary 11. *Let $k \geq 2$, $m \geq 2$ and $n \geq 2$ be positive integers. Then*

$$\psi_k(P_m \boxtimes P_n^2) \geq 4\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor + \lfloor\frac{mn-(2\lceil\frac{k}{2}\rceil+4)\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor}{k}\rfloor.$$

Proof. We can get the result as the proof of Theorem 10 for $n \equiv 1(mod\ (\lceil\frac{k}{2}\rceil + 2))$, similarly. □

Corollary 12. *Let $k \geq 2$, $m \geq 2$ and $n \geq 2$ be positive integers. For any positive number ε, we have*

$$\frac{\psi_k(P_m \boxtimes P_n^2)}{|V(P_m \boxtimes P_n^2)|} \geq \frac{2}{\lceil\frac{k}{2}\rceil + 2} - \varepsilon,$$

when m and n are large sufficiently.

Proof. Let $n = a(\lceil\frac{k}{2}\rceil + 2) + c$ and $m = 2b + d$, where $c \in [0, \lceil\frac{k}{2}\rceil + 1]$ and $d = 0$ or 1. According to Corollary 11, we have

$$\begin{aligned}\frac{\psi_k(P_m \boxtimes P_n^2)}{|V(P_m \boxtimes P_n^2)|} &\geq 4\lfloor\frac{n}{\lceil\frac{k}{2}\rceil+2}\rfloor\lfloor\frac{m}{2}\rfloor\frac{1}{mn} \\ &= 4\frac{n-c}{\lceil\frac{k}{2}\rceil+2}\frac{m-d}{2}\frac{1}{mn} \\ &= 2(1 - \frac{c}{n} - \frac{d}{m} + \frac{cd}{mn})\frac{1}{\lceil\frac{k}{2}\rceil+2} \\ &\to \frac{2}{\lceil\frac{k}{2}\rceil+2}(when\ m, n \to +\infty).\end{aligned}$$

Therefore, for any positive number ε, we have obtained

$$\frac{\psi_k(P_m \boxtimes P_n^2)}{|V(P_m \boxtimes P_n^2)|} \geq \frac{2}{\lceil\frac{k}{2}\rceil + 2} - \varepsilon,$$

when m and n are large sufficiently. □

Finally, we give some results for the lexicographic product of a path and a complete graph. We can obtain the following result by Lemma 5 since $P_a \boxtimes K_n$ is a subgraph of $P_a \circ K_n$. We will present a much easier proof than Lemma 5.

Lemma 13. *If $n \geq 2$ and $k \geq n + 2$, then*

$$\psi_k(P_a \circ K_n) \geq n,$$

where $a = \lceil\frac{k-1}{n}\rceil + 1$.

Proof. Assume to the contrary that S is a k-path vertex cover of the graph $P_a \circ K_n$ with $|S| \leq n - 1$. Let $S_i = S \cap V^{u_i}K_n$ with $|S_i| = n_i$, where $1 \leq i \leq a$. Therefore, lying in the layer ${}^{u_i}K_n$, all the vertices that are not covered by S_i can construct a path P_i of order $n - n_i$. Let the original vertex of P_i be (u_i, v_i) and the terminate vertex of P_i be (u_i, w_i), where $1 \leq i \leq a$. Set

$$P = P_1 + (u_1, w_1)(u_2, v_2) + P_2 + (u_2, w_2)(u_3, v_3) + P_3 + \cdots + (u_{a-1}, w_{a-1})(u_a, v_a) + P_a.$$

Therefore

$$\begin{aligned}|V(P)| &= \sum_{i=1}^{a} P_i \\ &= an - \sum_{i=1}^{a} n_i \\ &= (\lceil \tfrac{k-1}{n} \rceil + 1)n - (n-1) \\ &\geq (\tfrac{k-1}{n} + 1)n - (n-1) \\ &= k.\end{aligned}$$

We have a path of order at least k with no vertices in S, a contradiction. □

Theorem 14. *For $m \geq 2$ and $n \geq 2$, the following results hold.*

(1) If $2 \leq k \leq n+1$, then

$$\psi_k(P_m \circ K_n) = \begin{cases} \frac{m(2n-k+1)}{2}, if\ m\ is\ even, \\ \frac{(m+1)(2n-k+1)}{2} - n, if\ m\ is\ odd. \end{cases}$$

(2) If $n+1 < k < n\lceil \frac{m-1}{2} \rceil + 1$, then

$$n\lfloor \frac{m}{\lceil \frac{k-1}{n} \rceil + 1} \rfloor \leq \psi_k(P_m \circ K_n) \leq n\lfloor \frac{m}{\lfloor \frac{k-1}{n} \rfloor + 1} \rfloor.$$

Moreover, if $k \equiv 1(mod\ n)$, then

$$\psi_k(P_m \circ K_n) = n\lfloor \frac{mn}{n+k-1} \rfloor.$$

(3) If $n\lceil \frac{m-1}{2} \rceil + 1 \leq k \leq (m-1)n + 1$, then

$$\psi_k(P_m \circ K_n) = n.$$

(4) If $(m-1)n + 2 \leq k \leq mn$, then

$$\psi_k(P_m \circ K_n) = mn - k + 1.$$

Proof. (1) Firstly we will construct a k-path vertex cover S to obtain the upper bound. Let

$$S_i = \{(u_i, v_j) \in V(P_m \circ K_n) | k \leq j \leq n\}$$

with $|S_i| = n - k + 1$ for odd i and

$$S_i = \{(u_i, v_j) \in V(P_m \circ K_n) | 1 \leq j \leq n\}$$

with $|S_i| = n$ for even i, where $1 \leq i \leq m$. It is clear that $S = \cup_{i=1}^{m} S_i$ is a k-path vertex cover since the largest connected subgraph of $P_m \circ K_n$ with all vertices uncovered is isomorphic to K_{k-1}. Therefore,

$$\psi_k(P_m \circ K_n) \leq |S| = \begin{cases} \frac{m(2n-k+1)}{2}, if\ m\ is\ even, \\ \frac{(m+1)(2n-k+1)}{2} - n, \\ \qquad if\ m\ is\ odd. \end{cases}$$

Now we will show the lower bound. It is easy to see that $P_2 \circ K_n \cong K_{2n}$ and thus $\psi_k(P_2 \circ K_n) = 2n - k + 1$. If $m = 2$, then the conclusion is true. Suppose $m \geq 3$, we delete the edges between the two layers ${}^{u_{2i}}K_n$ and ${}^{u_{2i+1}}K_n$, where $2 \leq 2i < 2i+1 \leq m$.

If m is even, then the graph $P_m \circ K_n$ can be partitioned into $\frac{m}{2}$ disjoint subgraphs which are isomorphic to $P_2 \circ K_n$. Hence, we have

$$\psi_k(P_m \circ K_n) \geq \frac{m}{2}\psi_k(P_2 \circ K_n) = \frac{m(2n-k+1)}{2}.$$

If m is odd, then the whole graph $P_m \circ K_n$ can be partitioned into $\frac{m-1}{2}$ disjoint subgraphs which are isomorphic to $P_2 \circ K_n$ and a subgraph that is isomorphic to K_n. Therefore,

$$\begin{aligned}\psi_k(P_m \circ K_n) &\geq \tfrac{m-1}{2}\psi_k(P_2 \circ K_n) + \psi_k(K_n) \\ &= \tfrac{(m+1)(2n-k+1)}{2} - n.\end{aligned}$$

(2) Let

$$\begin{aligned}a &= \lceil \tfrac{k-1}{n} \rceil + 1 \leq \lceil \tfrac{n\lceil \frac{m-1}{2} \rceil + 1 - 1}{n} \rceil + 1 \\ &= \lceil \tfrac{m-1}{2} \rceil + 1 \leq \tfrac{m}{2} + 1 \leq m.\end{aligned}$$

If $m = a$, then the lower bound is true. Suppose $m \geq a+1$, we delete the edges between the two layers ${}^{u_{ia}}K_n$ and ${}^{u_{ia+1}}K_n$, where $1 \leq ia < ia+1 \leq n$. The graph $P_m \circ K_n$ can be partitioned into $\lfloor \frac{m}{a} \rfloor$ disjoint subgraphs which are isomorphic to $P_a \circ K_n$. We need at least $\psi_k(P_a \circ K_n)$ vertices to cover each subgraph that is isomorphic to $P_a \circ K_n$. According to Lemma 13, we have

$$\psi_k(P_m \circ K_n) \geq \lfloor \frac{m}{a} \rfloor \psi_k(P_a \circ K_n) \geq n\lfloor \frac{m}{\lceil \frac{k-1}{n} \rceil + 1} \rfloor.$$

On the other hand, let $b = \lfloor \frac{k-1}{n} \rfloor + 1 \leq a \leq m$ and $r = \lfloor \frac{m}{b} \rfloor \geq 1$. Set $S_i = \{(u_{ib}, v_j) \in V(P_m \circ K_n) \mid 1 \leq j \leq n\}$ with $|S_i| = n$ for $1 \leq i \leq r$. It is clear that $S = \cup_{i=1}^{r} S_i$ is a k-path vertex cover of $P_m \circ K_n$, since the order of the largest connected subgraph of $P_m \circ K_n$ with all vertices uncovered is at most

$$n(b-1) = n\lfloor \frac{k-1}{n} \rfloor \leq k-1.$$

So,

$$\psi_k(P_m \circ K_n) \leq |S| = nr = n\lfloor \frac{m}{\lfloor \frac{k-1}{n} \rfloor + 1} \rfloor.$$

Clearly, when $k \equiv 1(mod\ n)$, the two bounds of $\psi_k(P_m \circ K_n)$ are equal, hence we can obtain that

$$\psi_k(P_m \circ K_n) = n\lfloor \frac{mn}{n+k-1} \rfloor$$

in this case.

(3) Let $x = \lceil \frac{m-1}{2} \rceil + 1$ and $S = \{(u_x, v_j) \in V(P_m \circ K_n) | 1 \leq j \leq n\}$. It is easy to see that S is a k-path vertex cover of $P_m \circ K_n$ since the largest connected subgraph of $P_m \circ K_n$ with all vertices uncovered is isomorphic to $P_{x-1} \circ K_n$ and

$$|V(P_{x-1} \circ K_n)| = n(x-1) = n\lceil \frac{m-1}{2} \rceil \leq k-1.$$

Therefore,

$$\psi_k(P_m \circ K_n) \leq |S| = n.$$

On the other hand, set

$$a = \lceil \frac{k-1}{n} \rceil + 1.$$

Since

$$a = \lceil \frac{k-1}{n} \rceil + 1 \leq \lceil \frac{(m-1)n+1-1}{n} \rceil + 1 = m,$$

$$P_a \circ K_n \subseteq P_m \circ K_n.$$

According to Lemmas 2 and 13, we have

$$\psi_k(P_m \circ K_n) \geq \psi_k(P_a \circ K_n) \geq n.$$

(4) Let

$$S = \{(u_1, v_j) \in V(P_m \circ K_n) | 1 \leq j \leq mn-k+1\}.$$

It is easy to see that S is a k-path vertex cover of $P_m \circ K_n$, since the order of the largest connected subgraph of $P_m \circ K_n$ with all vertices uncovered is at most $k-1$. Therefore,

$$\psi_k(P_m \circ K_n) \leq |S| = mn-k+1.$$

On the other hand, it is easy to prove that

$$\psi_k(P_m \circ K_n) \geq mn-k+1$$

as Lemma 13 similarly. □

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