The Balance Sheet and the Assets-Claims on Assets Relationship in the Axiomatic Method

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Abstract: - The purpose of this study is to analyze the set structure of the balance sheet and assets-claims on assets relationship, considering the dual concept of monetary units, the axiomatic theory and accounting-specific axioms. The structure of the balance sheet and assets-claims on assets relationship are examined using a rationalistic, analytical and deductive method; this method uses the axiomatic set theory and predicate logic to define a set of axioms and the logical rationale to apply them to any deductive proof. The method includes accounting primitives and axioms to use in combination with those of the axiomatic theory. A direct proof is applied to test the balance sheet fit to a hereditary set structure according to the axiomatic theory, and proof by contraposition is employed to examine the assets-claims on assets equality by comparing their elements. Results show that balance sheet has a set structure that can be defined and analyzed with the axiomatic method and fits a hereditary set structure. Also, by comparing the elements of assets and claims on assets and considering their financial classification, it is shown that these sets do not contain the same elements and, consequently, they are not equal under the postulates of the axiomatic method.

Key-Words: Dual concept, accounting transactions, axiomatic method, assets, claims on assets, balance sheet.

1 Introduction

This paper addresses the issue of identifying a set structure to the balance sheet and testing the assets-claims on assets equality, using the axiomatic method.

The axiomatic method has been mainly used to create theories about the entire accounting system. The use of this method in accounting is significant [see 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, to name a few]; besides, the analysis of financial statements can include different types of logic, which is usually the language of the axiomatic method. Several types of logic, such as belief, circumscription, paraconsistent logic and dialogic, provide a different perspective on several accounting topics; among them the foundations of the accounting equation [14, 15, 16]. The axiomatic method is appropriate in any science to analyze structures [17], and so it is appropriate for analyzing the structure of financial statements, of which the balance sheet is a part.

In accounting, usually, its applications involve the axiomatization of the entire system. Many authors developed complex axiomatic systems to explain every aspect of accounting principles and practice. Nevertheless, the emphasis on creating entire accounting axiomatic systems led to difficulties in understanding the applications of this method. Moreover, these systems had to include every structural and functional aspect of the accounting system, resulting in the creation of many axioms, postulates, rules, theorems, deductive proofs, and oher logic tools.

However, another approach is to fit an existent axiomatic theory to the accounting system and use the axioms and rules of that theory to test the trustiness of the accounting assumptions. That is the approach used in this paper to analyze the balance sheet structure and assets-claims on assets equality. That has the advantage of avoiding creating new systems and axiomatic theories based on the author's preferences, which endure only for a short time.

Regarding the structure of financial statements, and more specifically the balance sheet, the dual aspects of accounting transactions provide a solid foundation for that structure. According to this principle, every accounting transaction is recorded in two accounts with different signs in a double classification system [1]. When it is extended to the assets-claims on assets relationship, it becomes an accounting assumption and, along with the double-entry bookkeeping system, is crucial to the organization of financial information.

Nonetheless, other approaches criticize the accounting principles, and propose an extension to a triple-entry bookkeeping system [see 18, 19, 20, 21, 22] or a different type of valuation of assets and claims on assets, such as the fair value approach [23, 24, 25, 26, see 27 for a critique of fair value accounting]. In doing so, they provide a different view of the dual aspects of accounting transactions, double-entry bookkeeping, and the assets-claims on assets relationship.

Therefore, two main issues arise, which are addressed in this paper; one of them is the use of a predominant axiomatic theory, along with some accounting-specific axioms, to analyze the structure of the balance sheet where the assets-claims on assets relationship dwells. The other issue is to test the assets-claims on assets equality with a deductive proof based on axiomatic theory and accounting-specific axioms.

2 Problem Formulation

The purpose of this research is to analyze the structure of the balance sheet and the assets-claims on assets equality, introducing the dual concept and the axioms and postulates of axiomatic theory along with accounting-specific axioms.

The dual concept, or duality principle, is the axiomatic form of the dual aspect of accounting transactions, which is a convention for registering the credits and debits. This convention is also the foundation of the double-entry bookkeeping system that fully supports the balance sheet.

One can consider the following distinctions: a) the duality principle or dual concept (it can be called the duality assumption as well) as an assumption or axiom that results in assets-claims on assets equality; b) the dual aspects of the accounting transactions as a convention (a rule) and a consequence of the duality concept; it is a definition in an axiomatic system; and c) the double-entry bookkeeping system as the rules governing the accountants' practice.

The accounting equation $A = L + E$ expresses that assets (A) are equal to liabilities (L) plus stockholders´ equity (E), i.e., assets are equal to claims on assets. Nevertheless, it is a mathematical equation, and despite the fact that the axiomatic method is widely involved in explaining mathematical operations, other mathematical methods should analyze the equation. On the contrary, the duality concept is an assumption, and it should be analyzed using the axiomatic method.

Justification exists for using the axiomatic method to analyze an accounting principle; this method is one of the most important components of classical science [17] and provides a logical structure to a subject by using an axiom system [17, 28] as well as a scientific explanation of the bases of any field of knowledge. The axiomatic method is the ideal in precision and provides a structure for concepts and propositions using deductive logic [2].

The axiomatic method has been used in accounting on many occasions, usually to create a new and entire axiomatic system for accounting theory and practice [see 2, 6, 7, 8, 9, 11, 12, 13].

Nevertheless, this paper introduces a major difference to its use; instead of creating a new accounting-specific axiomatic system, as most of the authors do, it takes an existing, well-known, and not accounting-specific axiomatic theory to analyze the structure of the balance sheet and the dual concept. The reason for doing so is to test the use of an established axiomatic theory in analyzing the accounting system structure and take advantage of its deductive mechanisms.

Despite the fact that accounting-specific axiomatic systems are well-defined, they meet their goals of explaining the assumptions of accounting only partly, and they introduce a significant variation in their postulates, definitions, and theorems; no matter how good they are, no consensus exists about which one is appropriate to axiomatize the accounting principles [1, 11]. Moreover, they are created to explain the assumptions and practices of accounting and not to analyze them critically, which results in a variety of theories, depending on the author's preferences, and most of them pervade only for a short time. This situation is why this paper favors fitting a well-known axiomatic theory over creating a new one and reducing the accounting axioms to just a few.

The Zermelo-Fraenkel (ZF) axiomatic theory, used in this research, gives a robust and logical structure to the analysis.

Another main difference, regarding the utilization of the axiomatic method in this paper, is that it is neither applied to mathematical expressions nor double-entry bookkeeping, but only to one of the accounting assumptions. The reason for that is that double-entry bookkeeping is the practice associated with the dual aspects of accounting transactions, and the accounting equation is the ultimate mathematical expression of the assets-claims on assets equality, so every one of them is a different levels of analysis [see 29 for an analysis without this distinction]. It is necessary to separate these topics of analysis to make it possible to identify their analytical demands properly and select the analysis that leads to reaching more solid conclusions.

Also, it must be noted that in this research, the conceptual definition of accounting terms was deliberately avoided. Many other studies involving axiomatic theory include a large number of definitions of accounting terms and operations [see 2, 8], but the viewpoint adopted in this paper is that definitions are a matter of other instances; it is a different discussion level that does not play any role in this analysis. The existent definitions created by international associations are accepted, and no additional ones are needed.

Finally, other issues, such as measurement theory applied to accounting, the validity of the addition property, or the representation and uniqueness theorem [see 11] are not analyzed in this paper. These aspects of accounting theory must be explained as different topics and with another type of analysis to avoid confusion.

In short, this paper analyzes the assets-claims on assets equality as an assumption of the accounting system from the viewpoint of axiomatic theory. To this end, the problem is twofold: first, to fit the balance sheet to an existent axiomatic theory, and second to test the assets-claims on assets equality, taking into account the dual concept and the axioms of accounting and axiomatic theory.

2.1 Methodology

The methodological approach in this paper is analytical, rationalistic, and deductive. The axiomatic theory used in this paper has those characteristics; the method uses axiomatic set theory along with predicate logic to develop rationales and conclusions. It involves a set of axioms and the logical rationale to apply them to any proof.

The Zermelo-Fraenkel (ZF) axiomatic theory, utilized in this analysis, comprises well-defined axioms to build logical operations with a predicate logic language. Initially, Zermelo created this system because advances in set theory did not involve a proper definition of sets [30]; Fraenkel made some adjustments to the theory and added the replacement axiom [31]. This axiomatic theory remains as the most prevalent, and it deals with infinite and finite sets.

3 Problem Solution

3.1 Primitives and Axioms of the Zermelo–Fraenkel theory

In ZF theory [see 32], the primitives are membership $\in$ and set $\{x_i\}$. Membership $\in$ primitive expresses the inclusion of a set x into another set y, and in this sense, x is a member of y; set $\{x_i\}$ primitive expresses that a set exists. ZF theory deals only with sets; thus, the elements of a set are, in turn, sets; it does not accept elements not linked to any set (urelements).

There are several versions of the original ZF theory; in what follows, I will give one extended form of this theory. The ZF theory comprises the following axioms: 1) Axiom of extensionality, which introduces set equality; 2) Axiom of empty set that defines a null set, 3) Axiom of separation (or axiom of specification) that allows for creating subsets by defining some properties of its members by a formula; 4) Axiom of power set that introduces a set comprising all the subsets of another set; 5) Axiom of union to create a set that contains as elements the elements of the elements of another set; 6) Axiom of choice that determines the existence of a set that contains one and only one of the elements of another set; 7) Axiom of infinity that introduces the existence of the infinite set; 8) Axiom of pairing, which defines that for every set pair, another set contains both of them; 9) Axiom of replacement, which defines the image of another set as a set; and 10) Axiom of regularity (or axiom of foundation) that expresses that every non-empty set contains one element that is disjoint with that set.

Nevertheless, in this paper, only three of these axioms are used. They are: a) Axiom of specification, to create sets based on a formula; b) Axiom of union, to properly group some sets into another set; and c) Axiom of extensionality to identify the equality of sets. These axioms will be explained all along during the analysis.

The ZF theory also accepts the definition of a subset as a set that is a member of another set. This definition could also be another axiom. The ZF theory, its extensions, and its axioms have been widely analyzed [see 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46].

3.2 Accounting primitives, definitions, and axioms

Primitives. According to ZF theory, some sets exist, so in the accounting system, and accepting the framework of the ZF theory some sets exist too. It would be, therefore, unnecessary to define the existence of an accounting set as a primitive. Nevertheless, one of the characteristics of these sets is that some or all of them can be empty.

The axiomatic method in accounting requires additional accounting primitives and axioms. The primitive in this system is the monetary unit u_i, used in the financial statements as set or set member.

As stated elsewhere [see 28], accounting science has many potential primitives or undefined terms, but this paper only introduces the monetary unit, which is the one utilized in the analysis.

Definitions. In what follows, it must be noted that no new definitions are created to fit the requirements of the analysis; the applied accounting definitions are those of the discipline. Hence, no definitions are introduced in this paper, and operations with monetary units in credit and debit accounts, and assets and claims on assets accounts are those accepted and defined by accounting science, so no new definitions are needed.

In this regard, a monetary unit is characterized as an asset or claim on asset as follows:

u_A: monetary unit considered an asset under an accepted definition.
u_C: monetary unit considered a claim on assets under an accepted definition.

It might be that some accounting topics needs defining, but, in that case, another different analysis would be needed.

The accounting axioms are the following:

Accounting axiom 1. The elements of any nonempty set of assets and claims on assets are sets that contain sets of monetary units. This axiom means that the lowest level nonempty sets are always sets of monetary units. Therefore:

$$\forall A \forall C \forall u_i[(\forall A_i \forall C_i\,(u_i \in A \mid u_i \in C) \rightarrow (u_i \in A_i \mid u_i \in C_i)] \quad (1)$$

with A = assets, C = claims on assets, A_i = element (subset) of assets, C_i = element (subset) of claims on assets, and u_i = monetary units. A special type of set is the single monetary unit $\{u_i\}$.

These sets can be empty sets and, even though they do not have a single monetary unit, their existence is a requirement of the accounting recording system.

The monetary unit can be in the legal tender or any other unit; it does not make any difference to the analysis and does not need additional definition. Once the monetary unit is chosen it is the same for all sets.

For the purpose of this paper, the accounts in financial statements comprise a finite number of monetary units; however, it must be noted that the issue of whether the financial statements sets are finite or infinite deserves much more attention. Nevertheless, this paper will not address that issue and will take these sets as finite, which does not make any difference to the analysis.

Accounting axiom 1 is in agreement with ZF theory, which only deals with sets and not with elements that are not sets. This axiom also expresses an application of the monetary unit assumption in financial reporting. This assumption states that financial statements must only include events in monetary units (Both the Financial Accounting Standard Board [FASB] and the International Standard Accounting Board [IASB] accept this standard accounting assumption).

Accounting axiom 2. Every monetary unit $\{u_i\}$ is different from another monetary unit $\{u_j\}$.

$$\forall u_i \forall u_j [u_i \neq u_j] \quad (2)$$

This axiom is necessary, because if the monetary units were equal, a set containing ten monetary units would be equal to a set containing just one. Therefore, by this axiom, to any pair of monetary units $\{u_i\}$ and $\{u_j\}$:

$$\forall u_i \forall u_j \forall x_i[(u_i \in x_i \wedge u_j \in x_i) \rightarrow u_i \neq u_j] \quad (3)$$
$$\forall u_i \forall u_j \forall x_i \forall y_i[(u_i \in x_i \wedge u_j \in y_i) \rightarrow u_i \neq u_j]. \quad (4)$$

In the ZF theory, a set exists, which is one of the primitives of the system. The existence of a set x is usually represented by $\forall x\,(x = x)$, so a set equal to itself is the same set, a single set. Hence, Accounting axiom 2 avoids any confusion in the formulae.

Accounting axiom 3. Every monetary unit has the property of being an asset and a claim on asset set, simultaneously. That is:

$$\forall u_i\, \exists! C_i\, \exists! A_i\, \exists A\, \exists C\, [u_i \in A \wedge u_i \in C \rightarrow (u_i \in A_i \wedge u_i \in C_i)]. \quad (5)$$

Therefore, a monetary unit $\{u_i\}$ can belong to two different sets A_i and C_i simultaneously. This axiom represents the dual concept or duality principle, as an assumption and axiom, the duality assumption.

Some issues need clarifying about this axiom. It is not equal to double-entry bookkeeping, the practice of the dual aspects of the accounting transactions. Double-entry bookkeeping refers to the practice of recording transactions in the accounting system, and it is based on the credit-debit account distinction. Moreover, credit and debit accounts are located on both sides of the assets-claims on assets equality, i.e., both sides have credit and debit accounts.

On the contrary, assets and claims on assets items have separated locations in the balance sheet, and they are the final result of the double-entry bookkeeping operations.

The debit-credit relationship is a tautology [1] and supports double-entry bookkeeping. In this vein, the assets-claims on assets relationship might be a tautology too. However, this could be so because the analysis lacks taking into account the item structure in the balance sheet and the dual concept.

The tautological credit-debit relationship led to introduce the credit-debit equality as a theorem, on some occasions [see 8 as an example]. However, in this research, due to the reasons named earlier, this is not so.

On the other hand, sometimes the accounting equation takes the role of an axiom [see 28], and other times it is a theorem [2, 12]. However, the viewpoint in this research is that the accounting equation is the mathematical expression resulting from the Accounting axiom 3, and not the axiom itself; the mathematical analysis of the accounting equation must be conducted with other methods.

Finally, to test the hypothesis, that assets are equal to claims on assets, the ZF theory and accounting axioms are the basis for the analysis.

3.3 The set structure of assets and claims on assets under the axiomatic method

In financial statements, and, more specifically on the balance sheet, assets (A) are equal to claims on assets (C). That is so because all of the financial resources of an organization come from institutions, companies, or individuals, and they have the right to make a claim on these resources. That is the rationale for this relationship; however, both groups refer to the only capital that exists.

To test this relationship under the ZF axiomatic theory and accounting axioms, the analysis will focus first on the set structure of the balance sheet.

From now on, the letters u, x, y, z, C, A, L, and E, are used to name sets, with no reference to elements not included in a set. The terms A and C of the balance sheet refer to sets.

Theorem 1. If the balance sheet accounts are considered sets, then the structure of the balance sheet fits a hereditary set structure according to the axioms of the ZF axiomatic theory and accounting axioms.

The proof of Theorem 1 will consist of a direct proof, i.e., by applying the axioms of the ZF theory and accounting axioms, along with the primitive terms, it will be shown that it is possible to consider the balance sheet a hereditary set structure. Hereditary sets are those whose elements are always sets. As previously commented, in ZF theory all sets comprise other sets, so all of them are hereditary sets. Hence, matching the content of the balance sheet account (or item) structure with a hereditary set structure, using the axioms of the ZF theory and accounting axioms, will prove that what Theorem 1 states is true.

According to Accounting axiom 1, every monetary unit is located in some accounts (sets), and according to Accounting axiom 3, they are in both assets and claims on assets. Accordingly, every monetary unit is in asset and a claim on assets accounts.

Now, the sets A (assets) and C (claims on assets), in any financial statements, need defining by formulae. The specification axiom allows the identification of subsets under certain conditions. This axiom states that:

$$\forall z\, \forall w_1\, \forall w_{2\, \ldots}\, \forall w_n\, \exists y\, \exists x[x \in y \leftrightarrow (x \in z \wedge \phi)]. \quad (6)$$

It means that a formula ϕ allows identification of subset y such that it contains every element x of the set z that has the property defined in the formula ϕ.

The sets A and C are subsets of the sets A_s and C_s, respectively. These sets A_s and C_s are also assets and claims on assets, respectively, but they are more comprehensive sets and comprise groups of companies, the industry, the country, or any other combination. In this sense, the sets A and C are subsets of other sets.

Therefore, applying the specification axiom to A and C:

$$\forall A_s\, \exists A\, \exists u_A\, [u_A \in A \leftrightarrow (u_A \in A_s \wedge \phi_A)] \quad (7)$$

where ϕ_A: u_A is a monetary unit of the company´s assets. In the same form,

$$\forall C_s\, \exists C\, \exists u_C\, [u_C \in C \leftrightarrow (u_C \in C_s \wedge \phi_C)] \quad (8)$$

where ϕ_C: u_C is a monetary unit of the company´s claims on assets.

Claims on assets comprise liabilities, and stockholder´s equity accounts (subsets). Then, applying the specification axiom again to create the subsets L (liabilities) and E (stockholder´s equity) of C,

$$\forall C\, \exists L\, \exists u_L\, [u_L \in L \leftrightarrow (u_L \in C \wedge \phi_L)] \tag{9}$$

where ϕ_L: u_L is a monetary unit of the company´s liability, and

$$\forall C\, \exists E\, \exists u_E\, [u_E \in E \leftrightarrow (u_E \in C \wedge \phi_E)] \tag{10}$$

where ϕ_E: u_L is a monetary unit of the company´s stockholder´s equity.

For the sake of clarity, the analysis will address only a few items of the balance sheet, but the conclusions are easily extended to the whole system. Therefore, by the specification axiom, one can create subsets in such a way that set A contains the subsets current assets A_c and non-current assets A_{nc}. Current assets A_c, in turn, comprises cash A_{cc} and accounts receivable A_{car}, whereas non-current assets A_{nc} contains long-term investments A_{nclti}, property, plant, and equipment A_{ncppe}, and intangible assets A_{ncia}.

As already mentioned, the formula ϕ of the specification axiom allows the inclusion of monetary units in sets or subsets. This formula applies to any set or subset of the balance sheet.

Financial statements allocate items to other items, and ZF set theory assumes the definition of a subset as a set that is a member of another set. This definition is useful here; in predicate logic and set language, the definition of a subset is in the following form:

$$(x \subseteq y) \leftrightarrow (\forall z\, (z \in x \rightarrow z \in y)). \tag{11}$$

That means that if a set x contains a set z and y contains x, then y contains z, and x is a subset of y. Regarding monetary units, and keeping in mind that the ZF theory includes only sets,

$$(Xu_i \subseteq Yu_j) \leftrightarrow (\forall Zu_n\, (Zu_n \in Xu_i \rightarrow Zu_n \in Yu_j)). \tag{12}$$

where u_i, u_j, and u_n represents monetary units. In the formula, Xu_i, Yu_j, and Zu_n are sets that include monetary units, and it means that Xu_i is a subset of Yu_j because every element Zu_n of Xu_i is contained in Yu_j.

Thus, total assets is a set A that consists of sets containing other sets:

$$A = \{\{A_c\}, \{A_{nc}\}\} \tag{13}$$
$$A_c = \{\{A_{cc}\}, \{A_{car}\}\} \tag{14}$$
$$A_{nc} = \{\{A_{nclti}\}, \{A_{ncppe}\}, \{A_{ncia}\}\}. \tag{15}$$

The definition of the subset allows the following structure to be built:

$$(A_{cc} \subseteq A_c) \leftrightarrow (\forall u_i\, (u_i \in A_{cc} \rightarrow u_i \in A_c)) \tag{16}$$
$$(A_{car} \subseteq A_c) \leftrightarrow (\forall u_i\, (u_i \in A_{car} \rightarrow u_i \in A_c)) \tag{17}$$
$$(A_{ncia} \subseteq A_{nc}) \leftrightarrow (\forall u_i\, (u_i \in A_{ncia} \rightarrow u_i \in A_{nc})) \tag{18}$$
$$(A_{ncppe} \subseteq A_{nc}) \leftrightarrow (\forall u_i\, (u_i \in A_{ncppe} \rightarrow u_i \in A_{nc})) \tag{19}$$
$$(A_{nclti} \subseteq A_{nc}) \leftrightarrow (\forall u_i\, (u_i \in A_{nclti} \rightarrow u_i \in A_{nc})) \tag{20}$$
$$(A_c \subseteq A) \leftrightarrow (\forall A_i\, (A_i \in A_c \rightarrow A_i \in A)) \tag{21}$$
$$(A_{nc} \subseteq A) \leftrightarrow (\forall A_i\, (A_i \in A_{nc} \rightarrow A_i \in A)). \tag{22}$$

In this structure, A_i is any subset of A_c or A_{nc}. Likewise, the set L contains subsets, such as current liabilities L_c and non-current liabilities L_{nc}. Current liabilities L_c include, in turn, subsets such as accounts payable L_{cap} and unearned revenues L_{cur}, whereas non-current liabilities L_{nc} contains the set mortgage payable L_{ncmp} and notes payable L_{ncnp}. The set owners' equity E includes issued capital E_{ic}, common stocks E_{cs}, and retained earnings E_{re}. These sets, as in the total asset set, are in the form:

$$L = \{\{L_c\}, \{L_{nc}\}\} \tag{23}$$
$$L_c = \{\{L_{cap}\}, \{L_{cur}\}\} \tag{24}$$
$$L_{nc} = \{\{L_{ncmp}\}, \{L_{ncnp}\}\} \tag{25}$$
$$E = \{\{E_{ic}\}, \{E_{cs}\}, \{E_{re}\}\}. \tag{26}$$

According to the definition of subset, these sets are:

$$(L_{cap} \subseteq L_c) \leftrightarrow (\forall u_i\, (u_i \in L_{cap} \rightarrow u_i \in L_c)) \tag{27}$$
$$(L_{cur} \subseteq L_c) \leftrightarrow (\forall u_i\, (u_i \in L_{cur} \rightarrow u_i \in L_c)) \tag{28}$$
$$(L_{ncmp} \subseteq L_{nc}) \leftrightarrow (\forall u_i\, (u_i \in L_{ncmp} \rightarrow u_i \in L_{nc})) \tag{29}$$
$$(L_{ncnp} \subseteq L_{nc}) \leftrightarrow (\forall u_i\, (u_i \in L_{ncnp} \rightarrow u_i \in L_{nc})) \tag{30}$$
$$(L_c \subseteq L) \leftrightarrow (\forall L_{ci}\, (L_{ci} \in L_c \rightarrow L_{ci} \in L)) \tag{31}$$
$$(L_{nc} \subseteq L) \leftrightarrow (\forall L_{nci}\, (L_{nci} \in L_{nc} \rightarrow L_{nci} \in L)). \tag{32}$$

Furthermore,

$$(E_i \subseteq E) \leftrightarrow (\forall u_i\, (u_i \in E_i \rightarrow u_i \in E)). \tag{33}$$

The sets and subsets $\{A\}$, $\{A_c\}$, $\{A_{nc}\}$, $\{A_{cc}\}$, $\{A_{cr}\}$, $\{A_{lti}\}$, $\{A_{ppe}\}$, $\{A_{ia}\}$, $\{L\}$, $\{L_c\}$, $\{L_{nc}\}$, $\{L_{ap}\}$, $\{L_{ur}\}$, $\{L_{mp}\}$, $\{L_{np}\}$, $\{E\}$, $\{E_{ic}\}$, $\{E_{cs}\}$, and $\{E_{re}\}$ are created by formulae; this grouping has three levels for assets and liabilities and two for equity.

Another application of the subset definition leads to define the set C as comprising the subsets L and E, in the form:

$$(E \subseteq C) \leftrightarrow (\forall E_i\,(E_i \in E \rightarrow E_i \in C)) \quad (34)$$

$$(L \subseteq C) \leftrightarrow (\forall L_i\,(L_i \in L \rightarrow L_i \in C)). \quad (35)$$

Therefore,

$$C = \{\{L\}, \{E\}\}. \quad (36)$$

This balance sheet structure that includes just a few items agrees with the concept of hereditary sets that characterizes the ZF axiomatic theory, i.e., sets comprising other sets.

It is important to note that the resulting set structure is not a partition. The concept of partition was used to create axiomatic models of accounting [see 1, 9]. However, even including several refinements of partitions, which would be more complex to understand, it does not represent the real structure of hereditary sets in the balance sheet.

A partition results in different sets, and their union is the original set. However, it comprises no empty sets, and the sets share no element: they are disjoint. On the contrary, the balance sheet structure can comprise multiple empty sets, and they are not disjoint, as assets and claims on assets share all the monetary units.

Moreover, an equivalence relation could create equivalence classes as a partition, and the axiom of choice would provide representatives of those classes, but even so, it would not lead to an adequate multi-level structure, such as that of the balance sheet.

The use of the standard axioms of the ZF theory provides a more intuitively understandable structure of the hereditary sets, avoiding the inconveniences of a partition.

Finally, it was shown that the structure of financial statements fits a hereditary set structure by using the axioms of the ZF axiomatic theory and accounting axioms.

3.4 The relationship between assets and claims on assets

The following theorem expresses the assets-claims on assets relationship:

Theorem 2. Let consider the balance sheet accounts a hereditary set structure according to the axioms of the ZF axiomatic theory and accounting axioms. Now, if the set of assets is equal to the set of claims on assets, then they should have the same elements.

It is important to note that Theorem 2 refers to the equality of sets and not to the equality of a number of monetary units to the same monetary units. The fact that monetary units are located on both assets and claims on assets can be misleading, and the analysis could fail to take into account the item structure in the balance sheet and also the Accounting axiom 3 that supports that practice. It is the theorem that goes under scrutiny using the axiomatic method, to test the hypothesis that assets are equal to claims on assets.

The type of proof to be conducted is by contraposition, i.e., if the set of assets is equal to the set of claims on assets, then they should have the same elements. In case they have not, then they are not equal.

In all of the previous analyses, the lowest level sets contain the single monetary unit sets $\{u_i\}$. However, these sets $\{u_i\}$ have no financial meaning because they lack proper identification in the balance sheet. They acquire financial meaning by their inclusion in the next higher category, such as A_{cc} …. E_{re}. The Accounting axiom 1 states that sets in the accounting system are sets that contain sets of monetary units. That allows for aggregating sets of monetary units into higher order sets, which can be done by the union axiom of the ZF theory.

The axiom of union says that the union of sets is a set that contains the elements of the elements of another set. The formal expression of this axiom is

$$\forall X\, \exists Y\, \forall z\, \forall w\, [(w \in z \wedge z \in X) \rightarrow w \in Y] \quad (37)$$

It means that if a set X contains subsets z and these elements contain subsets w, the union of the elements w of the subsets z of the set X is another set Y. In the case of L (liabilities), current liabilities L_c and non-current liabilities L_{nc}, it is:

$$\forall L\, \exists L_u\, \forall L_j\, \forall L_i\, [(L_i \in L_j \wedge L_j \in L) \rightarrow L_i \in L_u]. \quad (38)$$

The set L contains the subsets L_j (L_c and L_{nc}); L_i is every element of the sets L_c and L_{nc}, and L_u is the union of the elements of the elements of all of L_j. That is, the set L_u includes all the subsets L_i of L_c and L_{nc}. With the definition of subset, L_u is included in set C:

$$(L_u \subseteq C) \leftrightarrow (\forall L_i\, (L_i \in L_u \rightarrow L_i \in C)) \quad (39)$$

where L_i is any subset of L_u.

Likewise, there are two sets on the claims on assets side: one is L_u and the other is E; E contains all its subsets defined above. Set C contains both sets. The union C_u of these sets is:

$$\forall C\ \exists C_u\ \forall C_j\ \forall C_i\ [(C_i \in C_j \wedge C_j \in C) \rightarrow C_i \in C_u] \quad (40)$$

where C is the set that contains the sets C_j (L_u and E) and C_i any subset of L_u and E. Then, the set C_u comprises all C_i elements of L_u and E.

The union of the subsets of A is:

$$\forall A\ \exists A_u\ \forall A_j\ \forall A_i\ [(A_i \in A_j \wedge A_j \in A) \rightarrow A_i \in A_u] \quad (41)$$

where A contains the subsets A_j (A_c and A_{nc}); A_i is any element of the sets A_c and A_{nc}; A_u is the union of the elements of the A_j subsets. That is, the set A_u includes all the subsets of A_c and A_{nc}.

As a result, there are two sets, A_u and C_u, which contain all the subsets of assets and all the subsets of claims on assets, respectively. These subsets are the lowest level sets with financial meaning because they have relevant item labels, such as cash, accounts receivable, accounts payable, mortgage payable, and so on. They contain all the subsets of monetary units $\{u_i\}$.

The Accounting axiom 3 states that every monetary unit is simultaneously located in both assets and claims on assets. Therefore, one can look for the type of relationship between assets and claims on assets, taking into account the set structure they have. The test to be conducted is:

$$A_u = C_u. \quad (42)$$

According to the axiom of extensionality, the equality of sets is:

$$\forall x\ \forall y\ [\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)]. \quad (43)$$

This formula means that set x is equal to set y if for every z, whenever z is a subset of x, z is a subset of y, and, conversely, whenever z is a subset of y, z is a subset of x. Then, for A_u and C_u:

$$\forall A_u\ \forall C_u[\forall x_i(x_i \in A_u \leftrightarrow x_i \in C_u) \rightarrow A_u = C_u)]. \quad (44)$$

Accordingly, for A_u and C_u to be equal, they need to have the same subsets x_i. It means that the subsets A_i must be equal to the subsets C_i.

Also, for the subsets C_i and A_i to be equal, all of the monetary units $\{u_i\}$ in a set C_i should only be in a set A_i. Therefore, there must be a subset A_i for each C_i, such that both of them have the same elements $\{u_i\}$. Consequently, using the sets C_i and A_i of C_u and A_u respectively, for every C_i to be equal to an A_i:

$$\forall A_i\ \forall C_i\ [\forall u_i(u_i \in A_i \leftrightarrow u_i \in C_i) \rightarrow A_i = C_i)]. \quad (45)$$

Assuming that the C_i subsets are equal to the A_i subsets means that the $\{u_i\}$ elements should be the same in each set. That is, the monetary unit sets $\{u_i\}$ in a set C_i are also in a single set A_i, and both sets have to have the same elements. Hence, and by the Accounting axiom 3 and the existence primitive of the ZF theory, an identity function f_{I} exists, f_{I}: $C_i \rightarrow A_i$, so that for every $\{u_{ci}\}$ of C_i there is an $\{u_{ai}\}$ of A_i, such as $f_{\mathrm{I}}(\{u_{ci}\}) = \{u_{ai}\}$, with C_i the domain, and A_i the range of the function. If f_{I} is bijective, then $A_i = C_i$; if f_{I} is not bijective, then $A_i \neq C_i$.

In order to accomplish the requirement of the bijection function, every set C_i must have an image A_i, and the images A_i and A_j of any two C_i and C_j must be different in A_u, i.e. $f(C_i) \neq f(C_j)$ (injective function). Furthermore, all the sets A_i must have a pre-image or reverse image C_i in C_u (surjective function), and $f^{-1}(C_i) \neq f^{-1}(C_j)$.

However, it is not a requirement of the Accounting axiom 3 to have the subsets of monetary units $\{u_i\}$ of every C_i located in a unique set A_i. If that is the case, then the domain is C_{ix}, a subset of C_i, $C_{ix} \subset C_i$, and the function is f_{Ix}: $C_{ix} \rightarrow A_i$ and not f_{I}; it can be called a restriction of f_{I}; then $C_i \neq A_i$. Moreover, in case f_{I}: $C_i \rightarrow A_{ix}$, with A_{ix} a subset of A_i, $A_{ix} \subset A_i$,, f_{I} would not be a bijection, and $C_i \neq A_i$.

In general:

$$\forall C_i\ \forall A_i\ \forall A_j\ [\forall u_i \forall u_j\ (u_i \in C_i) \rightarrow \exists A_i\ (\ u_i \in A_i\ ) \rightarrow \exists u_j\ (u_j \in C_i \wedge u_j \in A_j)]. \quad (46)$$

Therefore, the requirement that the C_i subsets are equal to the A_i subsets is not a requirement of the Accounting axiom 3, and it leads to the conclusion that at least one set C_i could be not equal to any set A_i, and A_u and C_u would not have the same members. Moreover, if a set C_i is not equal to any A_i, it means that at least another set C_j is not equal to any A_i. Therefore, as a consequence, they are not equal, i.e.

$$A_u \neq C_u. \quad (47)$$

Another proof is that if C_i and A_i are equal, then the set created by selecting some members of any A_i that are also members of a single C_i, should be equal to that A_i. In case they are not equal, then the assets set is not equal to the claims on assets set. This proof is obtained by creating a new set by the axiom of specification.

This set is:

$$\forall A_i\, \forall w_1\, \forall w_2 \ldots \forall w_n\, \exists C_{ie}\, \exists u_i\, [u_i \in A_{ie} \leftrightarrow (u_i \in A_i \wedge \phi)] \quad (48)$$

where ϕ = elements $\{u_i\}$ of A_i that are also members of a particular C_i. That means that the application of the property ϕ to the elements of a set A_i will restrict the elements of a new set called A_{ie} to those that are also members of a given C_i.

Then, by the axiom of extension, in the case of $A_u = C_u$, it should be that:

$$\forall A_i\, \forall A_{ie}\, [\forall u_i\, (u_i \in A_i \leftrightarrow u_i \in A_{ie}) \rightarrow A_i = A_{ie})] \quad (49)$$

where A_{ie} contains the elements of a particular A_i that are elements of a single C_i. However, by the Accounting axiom 3, it might happen that $A_i \neq A_{ie}$, because some monetary units $\{u_i\}$ of the set A_i are in a different C_i. This case is similar to the previous one; as every u_{ie} in A_{ie} is the same than an u_i of $A_{i,,}$ to be both sets equal an identity function f_I must exist, f_I: $A_i \rightarrow A_{ie}$, so that for every $\{u_i\}$ of A_i there is an $\{u_{ie}\}$ of A_{ie}, such as $f_I(\{u_i\}) = \{u_{ie}\}$, with A_i the domain, and A_{ie} the range of the function. If f_I is bijective, then $A_i = A_{ie}$; if f_I is not bijective, then $A_i \neq A_{ie}$.

Though, as it is not a requirement to have the subsets of monetary units $\{u_i\}$ of C_i located in a unique set A_i,, the domain can be A_{ix}, a subset of A_i, $A_{ix} \subset A_i$, and the function can be f_{Ix}: $A_{ix} \rightarrow A_{ie}$ and not f_I, it can be called a restriction of f_I; then $A_i \neq A_{ie}$. Moreover, in case f_I: $A_i \rightarrow A_{iex}$, with A_{iex} a subset of A_{ie}, $A_{iex} \subset A_{ie}$, f_I would not be a bijection, and $A_i \neq A_{ie}$.

Therefore:

$$\forall A_u\, \forall C_u\, [\forall C_i\, \forall A_i\, ((C_i \in C_u \wedge A_i \in A_u) \rightarrow \exists u_i((u_i \in C_i \wedge u_i \notin A_i) \vee (u_i \notin C_i \wedge u_i \in A_i)))] \quad (50)$$

Then, as at least one A_i is different to one A_{ie}, the assets set is not equal to the claims on assets set. In fact, if a set A_{ie} is not equal to any A_i, it means that at least another set A_{je} is not equal to any A_i as well. Therefore, as the consequence of the condition is not satisfied, then the assets set is not equal to the claims on assets set. In a general formal expression, the conclusion is:

$$\forall C_i\, [\forall A_i\, (C_i \in C_u \wedge A_i \in A_u) \rightarrow C_i \neq A_i]. \quad (51)$$

Consequently,

$$A_u \neq C_u. \quad (52)$$

Assets and claims on assets are not equal when taking into account the different structures they have.

However, a new application of the axiom of union would produce a set with the union of all sets containing monetary unit sets $\{u_i\}$. The application of this axiom to the set of assets is:

$$\forall A_u\, \exists A_{uu}\, \forall A_i\, \forall u_i\, [(u_i \in A_i \wedge A_i \in A_u) \rightarrow u_i \in A_{uu}]. \quad (53)$$

The result is a set A_{uu} consisting of all the subsets of the type $\{u_i\}$. It is also possible to perform this operation on all financial obligations C_u in the following manner:

$$\forall C_u\, \exists C_{uu}\, \forall C_i\, \forall u_i\, [(u_i \in C_i \wedge C_i \in C_u) \rightarrow u_i \in C_{uu}]. \quad (54)$$

Again, the result is a set C_{uu} consisting of all the subsets of the type $\{u_i\}$. The sets A_{uu} and C_{uu} have all the monetary units $\{u_i\}$ because they are not included in any other item and, according to the axiom of extension,

$$\forall A_{uu}\, \forall C_{uu}\, [\forall u_i\, (u_i \in A_{uu} \leftrightarrow u_i \in C_{uu}) \rightarrow A_{uu} = C_{uu})]. \quad (55)$$

All subsets $\{u_i\}$ are members of the sets A_{uu} and C_{uu}, so an identity function f_I exists, f_I: $C_{uu} \rightarrow A_{uu}$, so that for every $\{u_{ci}\}$ of C_{uu} there is an $\{u_{ai}\}$ of A_{uu}, and $f_I(\{u_{ci}\}) = \{u_{ai}\}$, with C_{uu} the domain, and A_{uu} the range of the function. In this case, f_I is bijective, and thus:

$$A_{uu} = C_{uu}. \quad (56)$$

However, these sets do not fit the requirement of the Accounting axiom 1, as they are just amounts of monetary unit without a proper financial statement classification. Although these sets are equal, this result is meaningless in financial accounting. A number of monetary units are always equal to the same number of monetary units when their different financial classifications are removed. Yet, this classification is the essence of financial accounting.

4 Conclusion

The application of axiomatic theory to the balance sheet leads to the findings that the assets and claims on assets can be characterized and analyzed as a set with an existing axiomatic theory combined with a small number of accounting axioms, avoiding the creation of new theories.

Also, it yielded the conclusion that assets and claims on assets are not equal when considering their different set structures and the dual concept of monetary units. Moreover, it demonstrated that considering assets set equal to claims on assets is based on an analysis that does not take into account what the Accounting axiom 1 states. When this axiom is introduced, which leads to the Theorem 1 and the set structure of the financial statements, it is clear that the equality would fail to agree with the axiom.

Finally, the analysis took only a few items on the balance sheet to get to these results; however, the same results would have been reached with any number of items or levels. The obtained results must be understood within the framework of the axiomatic theory, but they have strong implications for the financial accounting.

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