The Assets-Claims on Assets Equivalence in the Axiomatic Method

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Abstract: The purpose of the study is to analyze the assets-claims on assets equivalence based on the dual concept of monetary units and the axiomatic method. The methodology is analytical, rationalistic and deductive; it uses axiomatic theory with set theory and predicate logic to test set equivalence. The axiomatic theory involves a set of axioms, which are used in combination with accounting axioms to develop a proof of the assets-claims on the assets considered as finite sets. The analysis uses a bijective function based on the dual concept of monetary units, and proof by contraposition to test the fulfillment of the requirement of a bijective function. Results show that assets cardinality is not equal to claims on assets cardinality when taking into account the dual concept of monetary units, and as a consequence assets and claims on assets are not equivalent.

Keywords: Dual concept, axiomatic method, cardinality, equivalence, balance sheet.

1 Introduction

The paper addresses the issue of assets-claims on assets equivalence, using the concept of cardinality, the dual concept of monetary units and the axiomatic method.

The axiomatic method is appropriate for any science to analyze structures [1] and assumptions, it is also appropriate to analyze accounting assumptions. The main use of this method in accounting has been to create entire systems comprising concepts, theorems and rules; by doing so, much of the accounting practice and theory was axiomatized. Hence, this method was relevant and extensively used in accounting [see 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, to name a few, see 13, 14 for some discussion].

In addition, different types of logic, which is usually the language of the axiomatic method, are of interest in the analysis of financial statements. Introducing belief, circumscription and paraconsistent logics, and dialogic logic in the analysis of management reports, notes to financial statements and relationships among account items, gives another perspective to financial information [15, 16, 17].

With regard to the dual concept of monetary units, it is associated with the dual aspects of accounting transactions and double-entry bookkeeping. The dual aspects of accounting transactions indicate that every accounting transaction must be recorded in two accounts with different signs, in a double classification system [2]. Double-entry bookkeeping is the practice associated with this rule, and the dual concept is the principle governing the rule and practice; its ultimate expression is the accounting equation.

The accounting equation expresses the assets-claims on assets equality and is a consequence of the dual concept of monetary units. In this sense, the axiomatic method using logic and set theories [8] showed the importance of the duality approach [8, pp. 101–105]. Moreover, new, complete and consistent axiomatic systems were built [10, 11, 12] and the dual aspect of accounting transactions was preserved [see 3].

Although the accounting equation is the crucial basis for the balance sheet, its logical foundations have been revisited [see 15, 16, 17]. Additionally, other approaches exist that put into question the principles of accounting by proposing a triple-entry bookkeeping [see 18, 19, 20, 21, 22] or fair value accounting [23, 24, 25, 26; see 27 for a critique]. Those approaches are, somehow, a critique of the dual concept that supports the accounting equation.

Nevertheless, the accounting equation is a mathematical expression, and it needs to be analyzed by mathematical methods other than the axiomatic method; i.e. one mathematical method involves a combination of several functions and a coordinate transformation [see 28]; in contrast, the dual concept of monetary units is a type of assumption, an accounting principle, and requires analysis by the axiomatic method. Separating this analysis would contribute to simplifying the understanding of the different role of every concept and results in more robust findings.

As previously mentioned, the axiomatic method has been extensively used in accounting with a focus on creating new, entire and complex accounting systems. That was confusing and difficulties arose in understanding the benefits of using the axiomatic method and the quality of the obtained conclusions.

Another approach is to fit an existing general axiomatic theory to the accounting system; this is the approach used in this paper. The axioms, rules and other logical tools of the existing and nonaccounting theory are applied to accounting topics. The advantage is to avoid creating theories that endure only for a short time and introduce a large variety on the accounting basis. In addition, there is no consensus among authors about how to axiomatize the accounting discipline [2, 10].

Using an existing and well-known axiomatic theory, a test about the assets and claims on assets set equality showed that these sets are not equal [29, 30]. Those researches focused on defining the balance sheet with a set structure and testing the equality of subsets of both sets, i.e. the subsets of both assets and claims on assets should have the same elements to be equal. As the result of these researches was that assets set is not equal to claims on assets set, it was a first argument questioning the assets-claims on assets equality that supports the balance sheet.

Following the findings of that research, and as a second argument, it might happen that, even though they are not equal, those sets have the same number of elements, which is called “equivalence” in set theory, so a test regarding their equivalence is needed. A first test of the assets-claims on assets equivalence showed that these sets were not equivalent [31]; however that test lacked the analysis of the different relations between assets and claims on assets and their corresponding proofs, which are included in this paper.

Therefore, the goal of this paper is to use a predominant axiomatic theory, along with some accounting-specific axioms, to analyze the assets-claims on assets equivalence with a deductive proof based on the axiomatic and accounting axioms. This proof will take into account the different relationships that might exists between assets and claims on assets.

The structure of the paper is as follows. The problem description is presented in Section 2; also, in that section, the methodological approach to the problem and its justification are described, as well as the advantages and differences of this approach compared with others. In Section 3, a description of the axiomatic method used in this paper, along with the accounting primitives and axioms is provided. Besides, this section contains the analysis of the assets-claims on assets equivalence, based on three different conditions that might exists in financial statements. Finally, Section 4 contains the main conclusions of the paper.

2 Problem Formulation

As previously mentioned, the axiomatic method is useful for analyzing the assumptions of accounting and the assets-claims on assets equivalence. In this sense, the dual concept of monetary units along with the dual aspects of accounting transactions still deserve more attention and analysis with the axiomatic method.

Therefore, the purpose of this paper is to examine the assets-claims on assets equivalence, using the axiomatic method and the dual concept of monetary units.

2.1 Methodology

This research uses an analytical, rationalistic and deductive methodology; it uses the axiomatic method with set theory and predicate logic to develop rationales and conclusions. The axiomatic method involves a set of axioms, and the logical rationale to apply them to any proof. Predicate logic was the language used to formulate the axiomatic theory of Zermelo and Fraenkel (ZF) [see 32, 33] that supports this analysis.

The ZF axiomatic theory, utilized in this analysis, has a set of well-defined axioms that allows the formation of logical operations with a predicate logic language. Zermelo created this system due to the lack of advances in set theory, which resulted in an improper definition of sets [32]; Fraenkel made some adjustments and added the replacement axiom [34]. This axiomatic theory remains the most prevalent, and deals with infinite and finite sets.

On the other hand, the dual concept, or duality principle, is the axiomatic form of the dual aspect of accounting transactions; the latter is a convention to register the transactions in credit and debit accounts, and this convention is the foundation of the double-entry bookkeeping system that supports the balance sheet.

One can consider the following distinctions: a) the duality principle, duality concept or dual concept (it can be called the duality assumption as well) as an assumption or axiom that leads to assets-claims on assets equality; b) the dual aspects of accounting transactions as a convention (a rule) and a result of the duality concept; it is a definition in the axiomatic system; and c) the double-entry bookkeeping system as the set of rules governing the practice of accountants.

The mathematical expression of the assets-claims on assets equality is the accounting equation $A = L + E$, with assets (A) equal to liabilities (L) plus stockholders' equity (E). In other words, assets are equal to claims on assets, as both liabilities and stockholders' equity are claims on assets. Nevertheless, as mentioned earlier, this mathematical equation is better analyzed by other methods than the axiomatic method. In contrast, the duality concept is an assumption, and it should be analyzed using the axiomatic method.

The justification to use the axiomatic method to analyze an accounting principle refers to its qualities [see 1, 3, 35] and its extensive use [see 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Nevertheless, this paper introduces a major difference to previous uses of the axiomatic method: it takes an existing, well-known and not accounting-specific axiomatic theory to do the analyses. The purpose of doing so is to use an established axiomatic theory and benefit from its deductive mechanisms.

Moreover, other researches tend to explain the assumptions and practice of accounting, and not to analyze them critically. In this sense, the use of ZF axiomatic theory provides a robust analysis, without any commitment to theoretical content.

Another main difference is that the axiomatic method is neither applied to mathematical expressions nor double-entry bookkeeping, but only to one of the accounting assumptions. As previously stated, they are different concepts and analysis levels [see 29 for an analysis without this distinction], but it is necessary to identify their various analytical demands and select the analysis that delivers the most robust conclusions.

Also, this research deliberately avoids giving any definition of accounting terms. Many other studies using the axiomatic theory give definitions, as well as accounting terms and operations [see 3, 7, 8, 9, 11, 12], but the viewpoint of this paper is to adopt the existing definitions created by international associations, and no additional ones are needed. Definitions are other instances matter.

Moreover, for the same reasons, this research does not deal with measurement theory concepts; for example, the validity of the addition and the representation and uniqueness theorem [see 10] are not analyzed in this paper. These topics require a different level of analysis to avoid confusion.

Finally, the method used in this paper has the following advantages: a) the use of a well-stablished axiomatic theory, avoiding creating axiomatic systems ad-hoc; b) the definition of only an accounting primitive and three axioms, on a logical base; and c) a conceptual distinction among the dual concept of monetary unit, the dual aspect of accounting transactions and double-entry bookkeeping, that simplifies the analysis.

In short, the objective of this paper is to analyze the assets-claims on assets equivalence from the viewpoint of the axiomatic theory taking into account the axioms of accounting and axiomatic theory.

3 Problem Solution

3.1 Primitives and axioms of the Zermelo–Fraenkel theory

The ZF theory has two primitives, membership $\in$ and set $\{x_i\}$. The membership $\in$ primitive represents the inclusion of a set x in another set y, i.e. x is a member of y; the set $\{x_i\}$ primitive expresses that a set exists. This theory comprises some well-defined axioms to deal only with sets. The ZF theory does not deal with elements not linked to any set (urelements). In this sense, the members of a set are always, in turn, sets. This theory can be applied to infinite and finite sets.

There are different versions of the original ZF theory; in general, the following axioms are included: 1) Axiom of extensionality, which defines set equality; 2) Axiom of empty set, which introduces the null set; 3) Axiom of separation (or axiom of specification), which defines subsets by identifying some properties of its members; 4) Axiom of power set, which defines a set that includes all the subsets of another set; 5) Axiom of union to create a set that contains the elements of the elements of another set; 6) Axiom of choice, which introduces the existence of a set that contains one and only one of the elements of every set; 7) Axiom of infinity, which defines the existence of an infinite set; 8) Axiom of pairing, which states that for every set pair, they are subsets of another set; 9) Axiom of replacement, which describes the image of another set as a set; and 10) Axiom of regularity (or axiom of foundation), which states that for every nonempty set, one element of that set exists so that it is disjoint with that set.

This research only uses the axioms of union, specification and replacement. They will be explained in formal language throughout the analysis.

The ZF theory accepts the definition of a subset as a set that is a member of another set; this could be considered another axiom. The ZF theory, its extensions, and its axioms have been widely analyzed [see 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50].

3.2 Accounting axioms

In ZF theory, the existence of sets is a primitive, so in the accounting system with the framework of ZF theory sets exist too. There is no need to define the existence of an accounting set as a primitive. However, one of the characteristics of accounting sets is that some or all of them can be empty. In some accounts of the ZF theory, the existence of the null set is an axiom, as it was considered in this paper.

The use of the axiomatic method in accounting requires a list of accounting axioms and primitives or undefined terms.

Although accounting science could have many primitives [see 35], in this paper the only primitive is the monetary unit u_i.

Definitions. In this research, no new definitions are introduced, and those operating in the analysis are the ones provided by the discipline. Hence, the usual operations with monetary units in credit and debit accounts, and assets and claims on assets accounts are those defined by the accounting science.

In this sense, a monetary unit is considered an asset or claim on asset in the following manner:

u_A: monetary unit considered an asset under an accepted definition.
u_C: monetary unit considered a claim on assets under an accepted definition.

New definitions would require a different type of analysis.

The accounting axioms are the following:

Accounting axiom 1. The elements of any nonempty set of assets and claims on assets are sets that contain sets of monetary units. Therefore

$$\forall A\, \forall C \forall u_i[(\forall A_i \forall C_i\, (u_i \in A \mid u_i \in C) \rightarrow (u_i \in A_i \mid u_i \in C_i)] \quad (1)$$

with A = assets, C = claims on assets, A_i = elements (subsets) of assets, C_i = elements (subsets) of claims on assets, and u_i = monetary units. A special type of set is the single monetary unit $\{u_i\}$. This axiom expresses that a set X (A_i or C_i) has sets of monetary units, i.e. $X = (\{u_i\}, \{u_i\}, \{u_i\})$.

The monetary unit can be in the legal tender or any other unit; it does not make any difference to the analysis, so it does not need additional definition. The only requirement is that the type of monetary unit must be the same for all sets.

Accounting axiom 2. Every monetary unit $\{u_i\}$ is different to another monetary $\{u_j\}$ unit.

$$\forall u_i\, \forall u_j [u_i \neq u_j] \quad (2)$$

This axiom is necessary, because if the monetary units were equal, a set containing ten monetary units would be equal to a set containing just one. Therefore, to any pair of monetary units $\{u_i\}$ and $\{u_j\}$

$$\forall u_i\, \forall u_j\, \forall x_i [(u_i \in x_i \wedge u_j \in x_i) \rightarrow u_i \neq u_j] \quad (3)$$
$$\forall u_i\, \forall u_j\, \forall x_i\, \forall y_i [(u_i \in x_i \wedge u_j \in y_i) \rightarrow u_i \neq u_j] \quad (4)$$

Accounting axiom 3. This axiom represents the dual concept of monetary units, or duality principle. Every monetary unit is allocated to a single assets set and claim on assets set, simultaneously. That is

$$\forall u_i\, \exists! C_i\, \exists! A_i\, \exists A\, \exists C\, [u_i \in A \wedge u_i \in C \rightarrow (u_i \in A_i \wedge u_i \in C_i)] \quad (5)$$

Therefore, a monetary unit $\{u_i\}$ can belong to two different sets A_i and C_i simultaneously.

This axiom is based on double-entry bookkeeping and the dual aspects of accounting transactions. Double-entry bookkeeping is based on the distinction between credit and debit accounts and these accounts are located on both sides of the assets-claims on assets equality, i.e. both sides have credit and debit accounts.

In contrast, assets and claims on assets items have separate locations in the balance sheet and they are the final result of the double-entry bookkeeping operations and the dual aspects of the accounting transactions.

The debit-credit relationship is a tautology [2], and the assets-claims on assets relationship might be a tautology too. However, that happens when the analysis does not take into account either the different structures of assets and claims on assets or the dual concept.

The tautological credit-debit relationship has led, on some occasions, to the introduction of the credit-debit equality as a theorem [see 8 as an example]. Nevertheless, in this research, and despite the fact that it could be considered in that form, the credit-debit equality is a consequence of the need to keep the assets-claims on assets equality.

At other times, the accounting equation takes the role of an axiom [see 35] or a theorem [3, 11]. However, the viewpoint in this research is that the accounting equation is the mathematical expression resulting from the Accounting axiom 3, and not the axiom itself; the mathematical analysis of the accounting equation must be conducted with other methods.

Finally, to test the assets-claims on the assets equivalence, the ZF theory and accounting axioms are the basis for the analysis.

From now on, the letters *u, x, y, z, C, A, L* and *E* are used to name sets, with no reference to elements not included in a set. The terms *A* and *C* on the balance sheet refer to sets.

3.3 The structure of assets and claims on assets

The balance sheet structure needs to be defined as a set structure to test the assets-claims on assets sets equivalence.

In a previous paper this structure was identified [for a complete description and a direct proof of how the structure of the balance set fits a set structure see 30]. In what follows a brief description of the procedure is provided.

Assets and claims on assets comprise sets that contain monetary units (sets) (Accounting axiom 1) and not sets that contain other sets containing monetary units (sets). This means that no aggregation accounts exist in this structure, although those accounts could be created [see 30]. This structure only takes on the lowest-level accounts on the balance sheet. However, these lowest-level sets A_i and C_i are included in *A*, assets set, or in *C*, claims on assets set.

The axiom of specification combined with the axiom of union creates this structure consisting only of the lowest-level accounts, included in *A* or *C*. The details of the creation of this structure are not relevant in this research [see 30 for a full description], but some explanations will be provided.

The specification axiom enables allocation of monetary units to any set A_i of assets or C_i of claims on assets; according to the Accounting axiom 3, every monetary unit is in both of them.

In the case of assets the specification axiom is

$$\forall A_i \exists A \exists u_a\, [u_a \in A \leftrightarrow (u_a \in A_i \wedge \phi_a)] \tag{6}$$

with ϕ_A: u_a monetary unit considered an A_i asset under an accepted definition. It is a property that all elements of a set A_i must have.

In the same manner, to claims on assets

$$\forall C_i\, \exists C\, \exists u_c\, [u_c \in C \leftrightarrow (u_c \in C_i \wedge \phi_C)] \tag{7}$$

with ϕ_C: u_c monetary unit considered a C_i claim on assets under an accepted definition. Again, this is a property that all elements of a set C_i must have.

There are definitions of what an asset or claim on assets is; however, these definitions are not relevant to the analysis, as long as they are consistent throughout the balance sheet.

The axiom of union allows the grouping of sets into another set. The formal expression of this axiom is

$$\forall X\, \exists Y\, \forall z\, \forall w\, [(w \in z \wedge z \in X) \rightarrow w \in Y] \tag{8}$$

This means that if a set *X* contains subsets *z* and these elements contain subsets *w*, the union of the elements *w* of the subsets *z* of the set *X* is another set *Y*.

The final union C_u of these sets is

$$\forall C\, \exists C_u\, \forall C_j\, \forall C_i\, [(C_i \in C_j \wedge C_j \in C) \rightarrow C_i \in C_u] \tag{9}$$

where *C* is the set that contains the sets C_j, which are liabilities and stockholders'equity (L_u and *E*) and C_i any subset of L_u and *E*. Therefore, the set C_u comprises all C_i elements of L_u and *E*.

The union of the subsets of *A* is

$$\forall A\, \exists A_u\, \forall A_j\, \forall A_i\, [(A_i \in A_j \wedge A_j \in A) \rightarrow A_i \in A_u] \tag{10}$$

where *A* contains the subsets A_j, which are current and noncurrent assets (A_c and A_{nc}); A_i is any element of the sets A_c and A_{nc}; and A_u is the union of the elements of the A_j subsets. Therefore, the set A_u

includes all the subsets of A_c and A_{nc}. From now on, C_u and A_u are C and A, respectively.

By the Accounting axioms 2 and 3, the sets C_i are disjoint sets, and no monetary unit $\{u_{ci}\}$ is a member of C_i and C_j simultaneously. The same rationale is valid for the sets A_i, so they are disjoint too. However, C_i and A_i contain the same monetary units, and they are not disjoint.

The accounting axioms do not require that all the monetary units of a single C_i be in a specific A_i (accounting Axiom 3); thus, the monetary units of a single C_i could be in several A_i or those of a single A_i could be in several C_i.

Finally, the structures of assets and claims on assets sets fit the concept of hereditary sets, or sets comprising sets, which is one of the characteristics of the ZF theory. In addition, all of them are finite and enumerable sets; the latter is inherent to ZF theory [43].

3.4 Analysis of the assets and claims on assets equivalence

To test the assets-claims on assets equivalence the following theorem is introduced:

Theorem. If the assets and claims on the assets sets are equivalent then they have the same cardinality.

The proof by contraposition will test the assets-claims on assets equivalence. The equality of cardinalities is the measure of the equivalence in this research. The equality of cardinalities requires the existence of a bijective function. Accordingly, if the requirements of a bijective function are not met, then the cardinalities are not equal, and assets and claims on assets sets are not equivalent.

As previously stated the structure of the balance sheet fits a set structure according to the axioms of the ZF axiomatic theory and accounting axioms. If assets (A) and claims on assets (C) are equivalent, then the value of A will be equal to the value of C, although they have different elements or subsets. The value of a set is called its cardinality, and it refers to the number of elements that a set has, no matter what these elements are. The notion of cardinality is logically privileged [51].

When comparing two sets, their cardinality can be equal, or one of them can be greater than the other; this is so because of the trichotomy property of the non-negative integers [52]. In addition, Zermelo proved that well-ordered sets combined with the trichotomy of ordinals result in the trichotomy of the cardinals [52]. However, by suppressing the axiom of choice that relation is no longer so [53]; this axiom is related to some properties of cardinality [54] and is necessary to compare cardinalities [55].

The cardinality of each A_i and C_i is the number of monetary units they have. The cardinality of a set with n monetary units is depicted as $|n|$ because every monetary unit is unique (Accounting axiom 2). Thus, the cardinality of a set or an item in the balance sheet is the amount of monetary units allocated to it.

A cardinality property is that the cardinality of the union of disjoint sets is the sum of the cardinality of each set. As shown, the sets C_i are disjoint, and the cardinality of the set C is the sum of the cardinalities of its m subsets C_i

$$|C| = \sum_{i=1}^{m} |C_i| \tag{11}$$

It is the same rationale for the cardinality of A, which contains all the subsets A_i. It is the sum of the cardinalities of its n disjoint subsets A_i

$$|A| = \sum_{i=1}^{n} |A_i| \tag{12}$$

It should be noted that if C set comprises the two sets Liabilities and Equity, and A comprises the two sets current and noncurrent assets, the cardinality of both of them, A and C, will be 2 (two elements each), which is not the value sought in this research.

To use the cardinality of the set union in the balance sheet correctly, it has to operate with all the lowest-level accounts A_i and C_i, without any aggregation. The cardinality of interest is the total monetary value of the sets A and C, which contain subsets A_i and C_i, which, in turn, contain subsets of monetary units $\{u_i\}$; thus, the cardinality relationship to test is

$$|A| = |C| \tag{13}$$

Thus, for the cardinality of the total assets to be equal to that of the total claims on assets, it must be

$$\sum_{i=1}^{n} |A_i| = \sum_{i=1}^{m} |C_i| \tag{14}$$

with $m \neq n$ in the usual arrangement of the balance sheet.

The cardinality equality of two sets requires a bijection so that they have the same number of elements; this is called "equinumerosity." Thus, there must be a bijective function f that relates C to A to determine that they have the same cardinality. Consequently, there must be a function f from C to A that links a member A_i to a member C_i, in such a way that A_i is an image of C_i by the function f.

According to the replacement axiom, the image of a set is contained within another set; this axiom is as follows:

$$\forall A \forall w_{1 \ldots\ldots} \forall w_n \, [\forall x(x \in A \rightarrow \exists! \, y \, f) \rightarrow \exists B \forall x \, (x \in A \rightarrow \exists y(y \in B \wedge f))] \tag{15}$$

with f: a function between sets. That means that every element x of the set A is related to an element y of the set B by f. The application to assets-claims on assets equality is

$$\forall C \forall w_{i \ldots\ldots} \forall w_n \, [\forall C_i(C_i \in C \rightarrow \exists! \, A_i \, f) \rightarrow \exists A \forall A_i \, (C_i \in C \rightarrow \exists A_i(A_i \in A \wedge f))] \tag{16}$$

This axiom guarantees that every C_i in C must have an image A_i in A. As previously mentioned, the function f must be a bijection. The images A_i and A_j of any two C_i and C_j must be different in A, i.e. $f(C_i) \neq f(C_j)$ (injective function). Furthermore, all the sets A_i must have a pre-image or reverse image C_i in C (surjective function), and $f^{-1}(C_i) \neq f^{-1}(C_j)$. If the function f does not meet these conditions, it is not bijective, and the cardinalities of C and A are not the same.

In order to have both sets C and A with the same cardinality, the following three conditions are possible: 1) C_i and A_i must have the same monetary units; 2) C_i and A_i do not have the same monetary units, but they must have the same number of monetary units; 3) C_i and A_i need neither to have the same monetary units nor the same number of monetary units. The analysis of the three conditions follows.

Analysis of condition 1. C_i and A_i must have the same monetary units.

The assets-claims on assets equivalence assumes that all the monetary units of claims on assets are in assets too (Accounting axiom 3). Thus by the dual concept of monetary units (which relies on the dual aspect of accounting transactions) (Accounting axiom 3), and by the existence primitive of the ZF theory, a crucial function is the identity function f_{I} that links every C_i to the A_i that has the same monetary units.

In that case, for each pair C_i, A_i there must be a function f_{I}: $C_i \rightarrow A_i$, with C_i the domain, and A_i the range of the function and for every $\{u_{ci}\} \in C_i$ there is an $\{u_{ai}\} \in A_i$, and $f_{\mathrm{I}}(\{u_{ci}\}) = \{u_{ai}\}$. If f_{I} is bijective, then $A_i = C_i$ for every pair A_i, C_i; if f_{I} is not bijective, then $A_i \neq C_i$. Therefore, the function f_{I}: $C \rightarrow A$ must be bijective.

However, it is not a requirement of the Accounting axiom 3 to have the subsets of monetary units $\{u_i\}$ of every C_i located in a unique set A_i. In case they are not, then the domain is C_{ix}, a subset of C_i, $C_{ix} \subset C_i$, and the function is $f_{\mathrm{I}x}$: $C_{ix} \rightarrow A_i$ and not f_{I}; therefore, $C_i \neq A_i$ and $|C_i| \neq |A_i|$. Moreover, in case f_{I}: $C_i \rightarrow A_{ix}$, with A_{ix} a subset of A_i, $A_{ix} \subset A_i$, f_{I} would not be a bijection, and $C_i \neq A_i$ and $|C_i| \neq |A_i|$.

In any of these cases, if some C_i or A_i without image or pre-image exists, then the function f_{I}: $C \rightarrow A$ is not a bijection. By the same rationale, the domain of this function could be restricted to C_x, a subset of C, $C_x \subset C$, and the function would be $f_{\mathrm{I}x}$: $C_x \rightarrow A$ and not f_{I}; therefore $C \neq A$ and $|C| \neq |A|$. Furthermore, in case f_{I}: $C \rightarrow A_x$, with A_x a subset of A, $A_x \subset A$, f_{I} would not be a bijection, and $C \neq A$ and $|C| \neq |A|$.

In general

$$\forall C_i \, \forall A_i \, \forall A_j \, [\forall u_i \forall u_j \, (u_i \in C_i) \rightarrow \exists A_i \, (\, u_i \in A_i \,) \rightarrow \exists u_j \, (u_j \in C_i \wedge u_j \in A_j)] \tag{17}$$

Therefore, as the requirement that the C_i subsets have the same elements as an A_i is not a requirement of Accounting axiom 3, it leads to the conclusion that at least one set C_i could not have the same cardinality as any set A_i. Therefore

$$\forall A_u \, \forall C_u \, [\forall C_i \, \forall A_i \, ((C_i \in C_u \wedge A_i \in A_u) \rightarrow \exists u_i((u_i \in C_i \wedge u_i \notin A_i) \vee (u_i \notin C_i \wedge u_i \in A_i)))] \tag{18}$$

Moreover, if a set C_i does not have the same elements as any A_i, it means that at least another set C_j will not have the same elements as A_i.

Analysis of condition 2. C_i and A_i do not have the same monetary units, but they must have the same number of monetary units.

In that case $u_{ci} \in C_i$, $u_{ai} \in A_i$ and it might happens that $\{u_{ci}\} \neq \{u_{ai}\}$; a function f exists, f: $C \rightarrow A$, so that for every pair C_i A_i, $f(C_i) = A_i$, with C the domain, and A the range of the function. If f is bijective, then $|A| = |C|$; if f is not bijective, then $|A| \neq |C|$. Therefore, the function f: $C \rightarrow A$ must be bijective.

However, the Accounting axiom 3 does not require of C_i and A_i to have the same number of monetary units. If they do not, the domain could be

C_x, a subset of C, $C_x \subset C$, and the function would be f_x: $C_x \rightarrow A$ and not f; thus, $|C| \neq |A|$. Moreover, in case f: $C \rightarrow A_x$, with A_x a subset of A, $A_x \subset A$, f would not be a bijection, and $|C| \neq |A|$.

Analysis of condition 3. C_i and A_i have neither the same monetary units nor the same number of monetary units.

In that case a function exists such as, f: $C \rightarrow A$, so that for every pair C_i A_i, $f(C_i) = A_i$, with C the domain, and A the range of the function. If f is bijective, then $|A| = |C|$; if f is not bijective, then $|A| \neq |C|$. Therefore, the function f: $C \rightarrow A$ must be bijective.

Though, the Accounting axiom 3 do not requires having all the monetary units of C distributed into all C_i accounts and there might be some empty C_i account; similarly, it is not a requirement to have all the monetary units of A distributed into all A_i accounts and there might be some empty A_i accounts as well. If the case is that the monetary units of C are distributed into all C_i accounts but they are not into all A_i accounts, then f: $C \rightarrow A_x$, with A_x a subset of A, $A_x \subset A$, ad f would not be a bijection, and $|C| \neq |A|$. If the case is that the monetary units of A are distributed into all A_i accounts but they are not into all C_i accounts, then the domain is C_x, a subset of C, $C_x \subset C$, and the function is f_x: $C_x \rightarrow A$ and not f, then $|C| \neq |A|$.

Accounting axiom 3 determines the allocation of every monetary unit to C_i and A_i; however, this axiom does not require that the monetary units of a single C_i be all allocated to a single A_i; also, it does not require of C_i and A_i to have the same number of monetary units and be nonempty accounts either.

Consequently, the function f is not a bijection, and that is in contradiction with the requirement of a bijective function for C and A, which is equal to having the same cardinality. The lack of a bijective function leads to the conclusion that

$$|A| \neq |C| \tag{19}$$

The cardinalities of assets and claims on assets are not equal. The dual concept of monetary units, along with the dual aspects of accounting transactions (accounting Axiom 3), the structures of assets and claims on assets, and the function that allocates the same elements to sets C_i and A_i lead to a cardinality change.

4 Conclusion

The objective of this research was to test the equivalence of the assets and claims on assets sets, based on the dual concept of monetary units. The analysis used the axiomatic method, set cardinality and a bijective function to test this equivalence, which led to the conclusion that assets and claims on assets are not equivalent.

This research is the second test of the assets-claims on assets relationship using axiomatic theory. The first one was based on the equality of sets and showed that these sets were not equal. In the same manner, this test based on the equivalence of the assets-claims on assets sets led to the conclusion that their cardinalities were not equal, and so these sets are not equivalent.

These tests are about assets-claims on assets sets, and not about their mathematical expression, the accounting equation, which, as previously commented, must be analyzed by other mathematical methods.

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