Restricted edge connectivity and restricted connectivity of graphs

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Abstract: Let $G = (V, E)$ be a connected graph. Let $G = (V, E)$ be a connected graph. An edge set $F \subset E$ is said to be a k-restricted edge cut, if $G - F$ is disconnected and every component of $G - F$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is the cardinality of a minimum k-restricted edge cut of G. A graph G is λ_k-connected, if G contains a k-restricted edge cut. A λ_k-connected graph G is called λ_k-optimal, if $\lambda_k(G) = \xi_k(G)$, where $\xi_k(G)$ = min$\{|[U, V - U]| : U \subset V, |U| = k$ and $G[U]$ is connected$\}$.An vertex set X is a $k\text{-}restricted\ cut$ of G, if $G - X$ is not connected and every component of $G - X$ has at least k vertices. The $k\text{-}restricted\ connectivity$ $\kappa_k(G)$ (in short κ_k) of G, is the cardinality of a minimum k-restricted cut of G. A λ_k-connected graph G is said to be $super\text{-}\lambda_k$, if G is λ_k-optimal and every minimum k-restricted edge cut isolates a component with exactly k vertices. A κ_k-connected graph G is said to be $super\text{-}\kappa_k$, if $\kappa_3(G) = \xi_3(G)$ and the deletion of each minimum k-restricted cut isolates a component with exactly k vertices. In this paper, we study the restricted edge connectivity and restricted connectivity of graphs, line graphs and a kind of transformation graphs.

Key–Words: 3-Restricted edge connectivity; Super-λ_3; Super-κ_3

1 Introduction

It is well known that graph theory plays a key role in the analysis and design of reliable or invulnerable networks. A network is often modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability. When studying network reliability, we consider the following model [3]. Let $G = (V, E)$ be a graph with the vertices reliable, but the edges may fail independently with the same probability $\rho \in (0, 1)$. One measure of the network reliability is the probability $P(G)$ of G being disconnected:

$$P(G) = \sum_{i=\lambda}^{\epsilon} m_i(G)\rho^i(1 - \rho)^{\epsilon-i},$$

where ϵ is the number of edges in G, $m_i(G)$ is the number of edge cuts of size i, λ is the edge connectivity. $P(G)$ is a polynomial on variable ρ, and is called unreliability polynomial. The smaller $P(G)$ is, the more reliable is the network. In general, to determine $P(G)$ is difficult[3]. When ρ is sufficiently small, the minimum of $P(G)$ can be obtained by maximizing λ first and then minimizing $m_\lambda(G), m_{\lambda+1}(G), \cdots, m_\epsilon(G)$ sequentially [17]. Connectivity is a parameter to measure the reliability of networks.

In this paper, we only consider simple graphs. Let $G = (V, E)$ be a connected graph. For a vertex $v \in V$, $N(v)$ is the set of all vertices adjacent to v. The degree of a vertex v, denoted by $d(v)$, is the size of $N(v)$. If $u, v \in V$, then $d(u, v)$ denotes the length of a shortest (u, v)-path. For $X, Y \subset V$, $d(X, Y)$ denotes the distance between X and Y; more formally, $d(X, Y) = \min\{d(x, y)$: for any $x \in X$ and any $y \in Y\}$. If $v \in V, r \geq 0$ is an integer, then let $N_r(v) = \{w \in V : d(w, v) = r\}$, in particular, $N_1(v) = N(v)$. For $X \subset V$, $N_r(X) = \{w \in V : d(w, X) = r\}$ where $d(w, X) = d(\{w\}, X)$, and $N_1(X) = N(X)$. We denote the diameter and girth by D and g, respectively, and write $G - v$ for $G - \{v\}$. A path is called k-path, if its length is k. For $U \subseteq V$, $G[U]$ is the subgraph of G induced by the vertex subset U, and $[U, V - U]$ is the set of edges with one end in U and the other in $V - U$. And $\xi_k(G)$ = min$\{|[U, V - U]| : U \subset V, |U| = k$ and $G[U]$ is connected$\}$.

Recall that for every graph G we have $\lambda \leq \delta$, where δ is the minimum degree of G. If $\lambda = \delta$, then G is said to be $maximally\ edge\ connected$ or $\lambda\text{-}optimal$. A graph G is super edge connected, or simply super-λ, if every minimum vertex cut is the neighbors of a vertex of G, that is every minimum vertex cut isolates a vertex. In the definitions of $\lambda(G)$, no restrictions are imposed on the components of $G - S$, where S is an edge cut. To compensate for this shortcoming, it would seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G-S$. Following this idea, k-restricted edge connectivity were proposed in [4,5]. An edge set $F \subset E$ is said to be a k-*restricted edge cut*, if $G-F$ is disconnected and every component of $G-F$ has at least k vertices. The k-*restricted edge connectivity* of G, denoted by $\lambda_k(G)$, is the cardinality of a minimum k-restricted edge cut of G. If $|F| = \lambda_k$, then F is called a λ_k-*cut*. Not all connected graphs have λ_k-cuts ($k \geq 2$), for example $K_{1,n-1}$. A graph G is λ_k-connected, if G contains a k-restricted edge cut. A λ_k-connected graph G is called λ_k-*optimal*, if $\lambda_k(G) = \xi_k(G)$.

An vertex set X is a k-*restricted cut* of G, if $G-X$ is not connected and every component of $G-X$ has at least k vertices. The k-*restricted connectivity* $\kappa_k(G)$ (in short κ_k) of G, is the cardinality of a minimum k-restricted cut of G. And X is called a κ_k-cut, if $|X| = \kappa_k$. Not all connected graphs have κ_k-cuts ($k \geq 2$), for example $K_{1,n-1}$. A graph G is κ_k-connected, if a κ_k-cut exists. For $k = 1, 2$ we can see [1, 2, 8-15,18,19]. We will study the case of $k = 3$.

For $X \subset V$, $v \in V \setminus X$ and $u \in N(v)$. Let us introduce the sets

$$\begin{aligned} X_u^+(v) &= \{z \in N(v) - u : d(z, X) = d(v, X) \\ &\quad +1\}; \\ X_u^=(v) &= \{z \in N(v) - u : d(z, X) = d(v, X)\}; \\ X_u^-(v) &= \{z \in N(v) - u : d(z, X) = d(v, X) \\ &\quad -1\}. \end{aligned}$$

Clearly, $X_u^+(v), X_u^=(v)$ and $X_u^-(v)$ form a partition of $N(v) - u$. And $|X_u^+(v)| + |X_u^=(v)| + |X_u^-(v)| = d(v) - 1$. If $d(v) \geq 2$, $u, w \in N(v)$, then

$$\begin{aligned} X_{uw}^+(v) &= \{z \in N(v) - \{u, w\} : d(z, X) = \\ &\quad d(v, X) + 1\}; \\ X_{uw}^=(v) &= \{z \in N(v) - \{u, w\} : d(z, X) = \\ &\quad d(v, X)\}; \\ X_{uw}^-(v) &= \{z \in N(v) - \{u, w\} : d(z, X) = \\ &\quad d(v, X) - 1\}. \end{aligned}$$

Then $X_{uw}^+(v), X_{uw}^=(v)$ and $X_{uw}^-(v)$ form a partition of $N(v) - \{u, w\}$, and $|X_{uw}^+(v)| + |X_{uw}^=(v)| + |X_{uw}^-(v)| = d(v) - 2$.

Wang et al.[16] obtain the following result for $\lambda_3(G)$.

Theorem 1.1. *Let G be a simple connected graph of order $n \geq 6$. If G is not a subgraph of any of the graphs shown in Fig.1, then both $\lambda_3(G)$ is well defined and $\lambda_3(G) \leq \xi_3(G)$.*

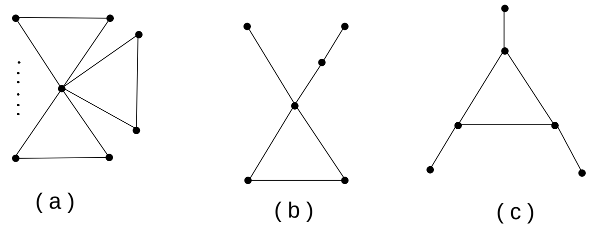


Fig. 1

From this theorem we can see that if G is a connected graph with girth $g \geq 4$ and $\delta \geq 3$, then G has 3-restricted edge cuts.

We also have the following results for $\lambda_3(G)$ and $\kappa_3(G)$.

Theorem 1.2. (1) [6] *Let G be a λ_3-connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and diameter D. If $D \leq g-3$, then G is λ_3-optimal.*

(2) [7] *Let G be a connected graph with girth $g \geq 6$, and minimum degree $\delta \geq 3$. Then G is κ_3-connected and $\kappa_3(G) \leq \xi_3(G)$, if $g \geq 7$ or $\delta \geq 4$.*

(3) [7] *Let G be a κ_3-connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and diameter D. If $D \leq g-4$, then $\kappa_3(G) = \xi_3(G)$.*

In this paper, we investigate super-λ_3 connectivity and super-κ_3 connectivity of graphs with girth $g \geq 4$ and minimum degree $\delta \geq 3$. We also study the connectivity of a kind of transformation graphs. Some sufficient conditions for the graphs to be super-λ_3 (resp. super-κ_3) are given in Theorem 3.1, which depends on diameters of the graphs and their line graphs.

In Section 2 we shall give some properties of 3-restricted edge cuts and 3-restricted cuts of graphs, in Section 3 we prove the sufficient conditions in Theorem 3.1 for graphs to be super-λ_3 (resp. super-κ_3). In Section 4 we study the edge connectivity and super edge connectivity of a kind of transformation graphs.

2 Properties of 3-restricted edge cuts and 3-restricted cuts of graphs

If G is a graph with girth $g \geq 4$, then every connected subgraph of G with three vertices is a path xyz of length two. Thus, $\xi_3(G) = \min\{d(x) + d(y) + d(z) - 4 : xyz$ is a path of length two in $G\}$.

Lemma 2.1. *Let G be a connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and $\xi_3(G)$. Let $X \subseteq V$ be a vertex cut with $|X| \leq \xi_3(G)$ and C be any connected component of $G-X$ with $|V(C)| \geq 3$. Then the following assertions hold:*
(1) *There exists an edge uv in C such that $d(\{u, v\}, X) \geq \lfloor (g-4)/2 \rfloor$.*

(2) *If g is odd and $|V(C)| \geq 4$, then there is a vertex $u \in C$ with $d(u,X) \geq (g-5)/2$ such that $|N_{(g-5)/2}(u) \cap X| \leq 1$.*

Proof. For $g = 4, 5, 6$, both assertions of the lemma hold, since $d(u,X) \geq 1$ for all u in C and $|V(C)| \geq 3$. So suppose that $g \geq 7$ and let $\mu = max\{d(u,X) : u \in V(C)\}$. Note that $\mu \geq 1$. If $\mu \geq \lfloor (g-2)/2 \rfloor$, then both assertions clearly hold. Thus, we assume that $\mu \leq \lfloor (g-4)/2 \rfloor$.

(1) If $\mu = 1$, then the result holds. Thus assume that $\mu \geq 2$.

Claim 1. There is an edge uv in C such that $d(\{u,v\},X) = \mu$.

We argue by contradiction. Suppose that each vertex u in C at $d(u,X) = \mu$ satisfies $d(v,X) = \mu-1$ for all $v \in N(u)$. As $\delta \geq 3$, take $w, v \in N(u)$, then vuw is a 2-path in C. Thus $d(v,X) = d(w,X) = \mu - 1$. Each vertex in $N(X_u^+(w))$ and $N(X_u^+(v))$ is at distance $\mu - 1$ from X. Moreover, we have $|N_{\mu-1}(X_u^=(w)) \cap X| \geq |X_u^=(w)|$. Otherwise, there are two vertices $x_1, x_2 \in X_u^=(w)$ both at distance $\mu-1$ from a vertex $x \in N_{\mu-1}(X_u^=(w)) \cap X$. There is a cycle going through $\{x_1, w, x_2, x\}$ of length at most $2\mu \leq 2\lfloor (g-4)/2 \rfloor \leq g-4$, contrary to the fact that the length of a shortest cycle in G is equal to g. Similarly, we have

$$\begin{aligned}
&|N_{\mu-1}(N(u)-v-w) \cap X| \geq |N(u)-v-w|,\\
&|N_{\mu-1}(X_u^=(v)) \cap X| \geq |X_u^=(v)|,\\
&|N_{\mu-1}(X_u^=(w)) \cap X| \geq |X_u^=(w)|,\\
&|N_{\mu-1}(w) \cap X| \geq |X_u^-(w)|,\\
&|N_{\mu-1}(v) \cap X| \geq |X_u^-(v)|,\\
&|N_{\mu-1}(N(X_u^+(w))-w) \cap X| \geq |X_u^+(w)|,\\
&|N_{\mu-1}(N(X_u^+(v))-v) \cap X| \geq |X_u^+(v)|.
\end{aligned}$$

Likewise, the sets $N_{\mu-1}(X_u^=(w)) \cap X, N_{\mu-1}(N(u)-v-w) \cap X, N_{\mu-1}(X_u^=(v)) \cap X, N_{\mu-1}(w) \cap X, N_{\mu-1}(v) \cap X, N_{\mu-1}(N(X_u^+(w))-w) \cap X$, and $N_{\mu-1}(N(X_u^+(v))-v) \cap X$ are pairwise disjoint. Hence we have

$$\begin{aligned}
\xi_3(G) &\geq |X|\\
&\geq |N_{\mu-1}(X_u^=(w)) \cap X| +\\
&\quad |N_{\mu-1}(w) \cap X| +\\
&\quad |N_{\mu-1}(X_u^=(v)) \cap X| +\\
&\quad |N_{\mu-1}(N(u)-v-w) \cap X| +\\
&\quad |N_{\mu-1}(v) \cap X| +\\
&\quad |N_{\mu-1}(N(X_u^+(w))-w) \cap X| +\\
&\quad |N_{\mu-1}(N(X_u^+(v))-v) \cap X|\\
&\geq |X_u^=(w)| + |X_u^-(w)| +\\
&\quad |X_u^=(v)| + |N(u)-v-w| +\\
&\quad |X_u^-(v)| + |X_u^+(w)|\\
&\quad +|X_u^+(v)|\\
&= d(u) + d(w) + d(v) - 4\\
&\geq \xi_3(G).
\end{aligned}$$

Thus, the above inequalities become equalities, yielding

$$\begin{aligned}
X \;=\; &(N_{\mu-1}(X_u^=(w)) \cap X) \cup\\
&(N_{\mu-1}(N(u)-v-w) \cap X) \cup\\
&(N_{\mu-1}(X_u^=(v)) \cap X)\\
&\cup (N_{\mu-1}(w) \cap X) \cup (N_{\mu-1}(v) \cap X)\\
&\cup (N_{\mu-1}(N(X_u^+(w))-w) \cap X) \cup\\
&(N_{\mu-1}(N(X_u^+(v))-v) \cap X). \qquad (1)
\end{aligned}$$

And

$$\begin{aligned}
&|N_{\mu-1}(N(u)-v-w) \cap X| = |N(u)-v-w|;\\
&|N_{\mu-1}(N(X_u^+(w))-w) \cap X| =\\
&|N(X_u^+(w))-w| = |X_u^+(w)|;\\
&|N_{\mu-1}(N(X_u^+(v))-v) \cap X| = |N(X_u^+(v))-v|\\
&= |X_u^+(v)|. \qquad (2)
\end{aligned}$$

From (2) it follows that if $|X_u^+(w)| > 0$, then every vertex $y \in X_u^+(w)$ has degree 2, which contradicts to the fact that $\delta \geq 3$. Then $X_u^+(w) = \varnothing$. Similarly, $X_u^+(v) = \varnothing$. Furthermore, (2) also implies that each vertex $x \in N(u)-v-w$ has one unique neighbor in X at distance $\mu-1$, that is, $|X_u^-(x)| = 1$. Similarly, for the edge ux we obtain that $X_u^+(x) = \varnothing$, which implies that $X_u^=(x) \neq \varnothing$ because $\delta \geq 3$. Take a vertex $x' \in X_u^=(x)$, from (1) we conclude that there is a cycle passing through $\{x', x, u\}$ and the vertex $y \in N_{\mu-1}(x') \cap X$ of length at most $2(\mu-1)+4 \leq g-1$, then there would be a cycle of length less than g, a contradiction.

Claim 2. $\mu \geq \lfloor (g-4)/2 \rfloor$.

By contradiction, suppose that $\mu \leq \lfloor (g-4)/2 \rfloor - 1$. From Claim 1 we know there is an edge uv in C

such that $d(\{u,v\},X) = \mu$. In this case, $X_u^+(v) = X_v^+(u) = \varnothing$. Then C has a 2-path uvw such that $d(w,X) = \mu$ or $d(w,X) = \mu - 1$.

Firstly, assume that $d(w,X) = \mu$. Thus we have $X_v^+(w) = \varnothing$. Arguing as in Claim 1 we have $|N_\mu(X_{uw}^=(v)) \cap X| \geq |X_{uw}^=(v)|$ and $|N_\mu(v) \cap X| \geq |X_{uw}^-(v)|$. Furthermore, the sets $N_\mu(X_{uw}^=(v)) \cap X, N_\mu(v) \cap X, N_\mu(X_v^=(u)) \cap X, N_\mu(u) \cap X, N_\mu(X_v^=(w)) \cap X$ and $N_\mu(w) \cap X$ are pairwise disjoint. Therefore we have

$$\begin{aligned}
\xi_3(G) &\geq |X| \\
&\geq |N_\mu(X_{uw}^=(v)) \cap X| + |N_\mu(v) \cap X| \\
&+ |N_\mu(X_v^=(u)) \cap X| + |N_\mu(u) \cap X| \\
&+ |N_\mu(X_v^=(w)) \cap X| + |N_\mu(w) \cap X| \\
&\geq |X_{uw}^=(v)| + |X_{uw}^-(v)| + \\
&\quad |X_v^=(u)| + |X_v^-(u)| + \\
&\quad |X_v^=(w)| + |X_v^-(w)| \\
&= d(u) + d(w) + d(v) - 4 \\
&\geq \xi_3(G).
\end{aligned}$$

Thus, the above inequalities become equalities, yielding

$$\begin{aligned}
X &= (N_\mu(X_{uw}^=(v)) \cap X) \cup (N_\mu(v) \cap X) \cup \\
&\quad (N_\mu(X_v^=(u)) \cap X) \cup (N_\mu(u) \cap X) \cup \\
&\quad (N_\mu(X_v^=(w)) \cap X) \cup (N_\mu(w) \cap X) \quad (3)
\end{aligned}$$

and

$$\begin{aligned}
&|N_\mu(X_{uw}^=(v)) \cap X| = |X_{uw}^=(v)|, \\
&|N_\mu(X_v^=(u)) \cap X| = |X_v^=(u)|, \\
&|N_\mu(X_v^=(w)) \cap X| = |X_v^=(w)|. \qquad (4)
\end{aligned}$$

From (4) we know that every vertex $z \in X_{uw}^=(v) \cup X_v^=(u) \cup X_v^=(w)$ has a unique neighbor at distance μ in X. As $\delta \geq 3$, there exists a vertex $z' \in N(z) \cap N_\mu(X)$ and $z' \overline{\in} \{u,v,w\}$, for every $z \in X_{uw}^=(v) \cup X_v^=(u) \cup X_v^=(w)$. From (3) it follows that there is a cycle of length at most $2\mu + 5 \leq g - 1$, contrary to the fact that the length of a shortest cycle in G is equal to g.

Secondly if $d(w,X) = \mu - 1$, then it is analogous to the case of $d(w,X) = \mu$.

As a consequence of both Claim 1 and Claim 2 we conclude that there exists an edge uv in C such that $d(\{u,v\},X) \geq \lfloor (g-4)/2 \rfloor$.

(2) Suppose now that $\mu = (g-5)/2$ otherwise by item (1) we are done. And we denote $C_X = \{u \in V(C) : d(u,X) = (g-5)/2\}$. By item (1) we can take an edge uv in $G[C_X]$.

Firstly, assume $(N(u) - v) \cap C_X \neq \varnothing$ or $(N(v) - u) \cap C_X \neq \varnothing$, say, $(N(v) - u) \cap C_X \neq \varnothing$. Notice that $X_v^+(u) = X_v^+(w) = X_{uw}^+(v) = \varnothing$ and that the sets $X_v^=(u), X_v^-(u), X_v^=(w), X_v^-(w), X_{uw}^=(v)$ and $X_{uw}^-(v)$ are pairwise disjoint. We will prove it by contradiction.

By contradiction, suppose that any vertex u in C_X satisfies $|N_{(g-5)/2}(u) \cap X| \geq 2$. Then we have $|N_{(g-5)/2}(X_v^=(u)) \cap X| \geq 2|X_v^=(u)|, |N_{(g-5)/2}(X_{uw}^=(v)) \cap X| \geq 2|X_{uw}^=(v)|$, and $|N_{(g-5)/2}(X_v^=(w)) \cap X| \geq 2|X_v^=(w)|$. Since the sets $N_{(g-5)/2}(X_v^=(u)) \cap X, N_{(g-7)/2}(X_v^-(u)) \cap X, N_{(g-5)/2}(X_{uw}^=(v)) \cap X, N_{(g-7)/2}(X_{uw}^-(v)) \cap X, N_{(g-5)/2}(X_v^=(w)) \cap X$ and $N_{(g-7)/2}(X_v^-(w)) \cap X$ are pairwise disjoint, it follows that

$$\begin{aligned}
\xi_3(G) &\geq |X| \\
&\geq |N_{(g-5)/2}(X_v^=(u)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_v^-(u)) \cap X| + \\
&\quad |N_{(g-5)/2}(X_{uw}^=(v)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_{uw}^-(v)) \cap X| + \\
&\quad |N_{(g-5)/2}(X_v^=(w)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_v^-(w)) \cap X| \\
&\geq 2|X_v^=(u)| + |X_v^-(u)| + \\
&\quad 2|X_{uw}^=(v)| + |X_{uw}^-(v)| + \\
&\quad 2|X_v^=(w)| + |X_v^-(w)| \\
&\geq \xi_3(G) + |X_v^=(u)| + \\
&\quad |X_{uw}^=(v)| + |X_v^=(w)|.
\end{aligned}$$

Then $X_v^=(u) = X_{uw}^=(v) = X_v^=(w) = \varnothing$ and

$$\begin{aligned}
X &= (N_{(g-5)/2}(u) \cap X) \cup (N_{(g-5)/2}(v) \cap X) \cup \\
&\quad (N_{(g-5)/2}(w) \cap X). \qquad (5)
\end{aligned}$$

Furthermore, we can obtain $|N_{(g-5)/2}(u) \cap X| = |X_v^-(u)|, |N_{(g-5)/2}(v) \cap X| = |X_{uw}^-(v)|$ and $|N_{(g-5)/2}(w) \cap X| = |X_v^-(w)|$. This means that $\mu = (g-5)/2 \geq 2$. As $\delta \geq 3$, we have $|N(z) \cap (C_X - u)| \geq d(z) - 2 \geq 1$ for all $z \in X_v^-(u)$ (Otherwise a cycle of length at most $g-2$ would appear). Take a vertex $z \in X_v^-(u)$ and consider a vertex $z' \in N(z) \cap (C_X - u)$. Then from (5) a cycle of length at most $g-1$ would appear, a contradiction.

Secondly, if $(N(u) - v) \cap C_X = \varnothing$ and $(N(v) - u) \cap C_X = \varnothing$, then take a vertex w in $N(v)$ with $d(w,X) = (g-7)/2$. Hence uvw is a 2-path in C, it is analogous to the above case. $\square$

Let $G = (V,E)$ be a λ_3-connected graph. An arbitrary λ_3-cut F can be denoted by $[V(C), V(\overline{C})]$, where C and $\overline{C}$ are the only two components of $G - F$. There are $X \subseteq V(C)$ and $Y \subseteq V(\overline{C})$ such that $X \cup Y$ is the set of the end vertices of $[V(C), V(\overline{C})]$, and so $[V(C), V(\overline{C})] = [X,Y]$.

A λ_3-connected graph G is said to be $super$-λ_3, if G is λ_3-optimal and every minimum 3-restricted edge cut isolates a component with exactly three vertices. A κ_3-connected graph G is said to be $super$-κ_3, if $\kappa_3(G) = \xi_3(G)$ and the deletion of each minimum 3-restricted cut isolates a component with exactly three vertices.

Lemma 2.2. *Let G be a connected graph with girth $g \geq 6$, and minimum degree $\delta \geq 3$. Let $[V(C), V(\overline{C})]$ = $[X, Y]$ be a λ_3-cut. Then the following assertions hold:*
(1) *If $V(C) = X$, then G is super-λ_3.*
(2) *If G is not super-λ_3, then $C - X$ has a component with at least three vertices.*

Proof. Since $g \geq 6$ and $\delta \geq 3$, by Theorem 1.1 G is λ_3-connected.

(1) Suppose that $V(C) = X$, then each vertex of C is incident with some edges of $[X, Y]$. If $|V(C)| = 3$, then we are done. So assume that $|V(C)| \geq 4$. Let uvw be a 2-path of C. Because $\delta \geq 3$, we assume that $|X_v^{=}(u)| \geq 1$. Since girth $g \geq 6$, thus arguing as before, we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G)\\ &= |[X, Y]|\\ &\geq |[u, Y]| + |[v, Y]| +\\ &\quad |[w, Y]| + |[X_v^{=}(u), Y]| +\\ &\quad |[X_{uw}^{=}(v), Y]| + |[X_v^{=}(w), Y]|\\ &\geq |[u, Y]| + |[v, Y]| +\\ &\quad |[w, Y]| + |X_v^{=}(u)| +\\ &\quad |X_{uw}^{=}(v)| + |X_v^{=}(w)|\\ &\geq 3 + d(u) - 1 + d(v) - 2 +\\ &\quad d(w) - 1\\ &> \xi_3(G),\end{aligned}$$

which is a contradiction.

(2) By item (1) we have $C - X \neq \varnothing$. Suppose that any component of $C-X$ has at most two vertices. Let $C_1, C_2, \cdots, C_k$ be the components of $C - X$.

Case 1. Each component C_i satisfies $|C_i| = 1$.

Take C_1 from $C_1, C_2, \cdots, C_k$. Let $C_1 = \{v\}$. Then $N(v) \subseteq X$. And $\delta \geq 3$, we pick $u, w \in N(v)$, and thus uvw is a 2-path in C. Arguing as item (1), we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G)\\ &= |[X, Y]|\\ &\geq |[N(u) - v, Y]| + |[N(w) - v, Y]| +\\ &\quad |[N(v) - u - w, Y]|\\ &\geq |N(u) - v| + |N(w) - v| +\\ &\quad |N(v) - u - w|\\ &= d(u) + d(v) + d(w) - 4\\ &\geq \xi_3(G).\end{aligned}$$

It follows that $|[N(u)-v, Y]| = |N(u)-v|, |[N(v)-u-w, Y]| = |N(v)-u-w|, |[N(w)-v, Y]| = |N(w)-v|$ and $X = (N(u)-v) \cup (N(v)-u-w) \cup (N(w)-v)$. Hence $[\{u, w\}, Y] = \varnothing$, which is a contradiction.

Case 2. There is a component C_1 with $|C_1| = 2$.

Assume that $V(C_1) = \{u, v\}$. Then $C_1 = K_2$, and $N(u) - v \subseteq X, N(v) - u \subseteq X$. Take $w \in X \cap (N(v) - u)$. Then uvw is a 2-path in C. As $g \geq 6$, arguing as in (1), we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G)\\ &= |[X, Y]|\\ &\geq |[N(u) - v, Y]| + |[N(v) - u - w, Y]| +\\ &\quad |[(N(w) - v) \cap X, Y]| + |[w, Y]|\\ &= d(u) + d(v) + d(w) - 4\\ &\geq \xi_3(G).\end{aligned}$$

It follows that $|[N(u)-v, Y]| = |N(u)-v|, |[N(v)-u-w, Y]| = |N(v)-u-w|, |[(N(w)-v)\cap X, Y]| = |(N(w)-v)\cap X|$ and $X = (N(u)-v)\cup(N(v)-u-w)\cup((N(w)-v)\cap X)\cup\{w\}$. Therefore, for any $x \in (N(u)-v)\cup(N(v)-u-w)\cup((N(w)-v)\cap X)$, we have $|[x, Y]| = 1$. Since $g \geq 6$ and $\delta \geq 3$, it follows that $N(x) \cap (X - x) = \varnothing$. So x is adjacent to some C_i's ($2 \leq i \leq k$). If there is a $C_i = \{y\}$ such that $y \in N(x)$, then $N(y) \subseteq X$. As $g \geq 6$ and $\delta \geq 3$, we have $|N(y) \cap (N(u) - v)| \leq 1, |N(y) \cap (N(v) - u)| \leq 1$ and $|N(y) \cap (N(w) \cap X)| \leq 1$.

Without loss of generality, we assume that $|N(y)\cap(N(w)\cap X)| = 1$, then $N(y)\cap(N(v)-u) = \varnothing, \{u, v\} \nsubseteq N(y)$, and we have $|N(y) \cap (N(u) - v)| \geq 2$. There is a cycle with length smaller than g, a contradiction. If $|N(y) \cap (N(w) \cap X)| = 0$, then $|N(y)\cap(N(u)-v)| \geq 2$ or $|N(y)\cap(N(v)-u)| \geq 2$. There is also a cycle of length smaller than g, which is impossible.

If there is a $|C_j| = 2$ which x is adjacent to, then it is analogous to the case of $|C_i| = 1$. We discuss the neighbors of each vertex in C_j, we can obtain the required result. ☐

Recall that in the line graph $L(G)$ of a graph G, each vertex represents an edge of G, and two vertices in a line graph are adjacent if and only if the corresponding edges of G are adjacent. Let us consider the edges $x_1y_1, x_2y_2 \in E(G)$. The distance between the corresponding vertices of $L(G)$ satisfies

$$d_{L(G)}(x_1y_1, x_2y_2) = d_G(\{x_1, y_1\}, \{x_2, y_2\})+1, \quad (6)$$

which is useful to prove that $D(G)-1 \leq D(L(G)) \leq D(G) + 1$.

3 Some sufficient conditions for graphs to be super-λ_3 (resp. super-κ_3)

Now, we will show Theorem 3.1 by contradiction.

Theorem 3.1. *Let G be a connected graph with girth $g \geq 4$ and minimum degree $\delta \geq 3$. The following assertions hold:*

(1) *If $D(G) \leq g - 4$, then G is super-λ_3.*

(2) *If $D(G) \leq g - 5$, then G is super-κ_3.*

(3) *If the diameter of the line graph $D(L(G)) \leq g - 4$, then G is super-λ_3.*

(4) *If the diameter of the line graph $D(L(G)) \leq g - 5$, then G is super-κ_3.*

Proof. Since $g \geq 4$, clearly G is different from the graphs in Fig.1. Thus, by Theorem 1.1, G is λ_3-connected. Moreover, if $g \in \{4, 5, 6\}$, then theorem clearly holds. So we assume that $g \geq 7$. By part (2) of Theorem 1.2, G is κ_3-connected.

(1) From Theorem 1.2 it follows that $\lambda_3 = \xi_3$. Assume that G is not super-λ_3. Let $[V(C), V(\overline{C})]$ = $[X, Y]$ be a λ_3-cut with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C - X$ and $\overline{C} - Y$ contain a connected component say H and K, respectively, of cardinality at least three vertices. Hence both X and Y are cutsets with $|X|, |Y| \leq \xi_3(G)$. From Lemma 2.1 there exist two vertices $u \in V(H)$ and $\overline{u} \in V(K)$ such that $g - 4 \geq D(G) \geq d(u, \overline{u}) \geq d(u, X) + 1 + d(\overline{u}, Y) \geq 2\lfloor(g - 4)/2\rfloor + 1$, which is a contradiction if g is even.

And for g odd all the inequalities become equalities. This means that $\max\{d(u, X) : u \in V(H)\} = (g - 5)/2$ and $\max\{d(\overline{u}, Y) : \overline{u} \in V(K)\} = (g - 5)/2$. Thus by Lemma 2.1, we can find $u \in V(H)$ with $d(u, X) = (g - 5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\overline{u} \in V(K)$ with $d(\overline{u}, Y) = (g - 5)/2$ such that $N_{(g-5)/2}(\overline{u}) \cap Y = \{\overline{x}\}$ for some $\overline{x} \in Y$. As $d(u, \overline{u}) = g - 4$, it follows that $x\overline{x} \in [X, Y]$. Clearly we can find a vertex $v \in N(u)$ with $d(v, X) = (g - 5)/2$, because otherwise $|N_{(g-5)/2}(u) \cap X| \geq |N(u)| \geq 2$. Since $d(v, \overline{u}) = g - 4$ we must have $x \in N_{(g-5)/2}(v)$ or $\overline{x} \in N_{(g-3)/2}(v)$. As a consequence, the path from u to $\overline{x}$ together with the path from v to $\overline{x}$ and the edge uv form a cycle of length at most $g - 2$, which is a contradiction.

(2) From Theorem 1.2 it follows that $\kappa_3 = \xi_3$. Assume that G is not super-κ_3. Let X be an any κ_3-cut and consider two connected components $C, \overline{C}$ of $G - X$ with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. From Lemma 2.1 there exist two vertices $u \in V(C)$ and $\overline{u} \in V(\overline{C})$ such that $g - 5 \geq D(G) \geq d(u, \overline{u}) \geq d(u, X) + d(\overline{u}, X) \geq 2\lfloor(g - 4)/2\rfloor$, which is a contradiction if g is even.

And for g odd all the inequalities become equalities. This means that $\max\{d(u, X) : u \in V(C)\} = (g-5)/2$ and $\max\{d(\overline{u}, Y) : \overline{u} \in V(\overline{C})\} = (g-5)/2$. Thus by Lemma 2.1, we can find $u \in V(C)$ with $d(u, X) = (g - 5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\overline{u} \in V(\overline{C})$ with $d(\overline{u}, Y) = (g-5)/2$ such that $N_{(g-5)/2}(\overline{u}) \cap Y = \{\overline{x}\}$ for some $\overline{x} \in Y$. As $d(u, \overline{u}) = g - 5$, it follows that $x = \overline{x}$. Clearly we can find a vertex $v \in N(u)$ with $d(v, X) = (g - 5)/2$. Since $d(v, \overline{u}) = g - 5$ we must have $x \in N_{(g-5)/2}(v)$. As a consequence, the path from u to x together with the path from v to x and the edge uv form a cycle of length at most $g - 4$, which is a contradiction.

(3) Since $D(L(G)) \leq g - 4$, then the diameter $D(G) \leq g - 3$, which means that $\lambda_3 = \xi_3$ by Theorem 1.2. Assume that G is not super-λ_3. Let $[V(C), V(\overline{C})]$ = $[X, Y]$ be a λ_3-cut with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C - X$ and $\overline{C} - Y$ contain a connected component say H and K, respectively, of cardinality at least three. Hence both X and Y are cutsets with $|X|, |Y| \leq \xi_3(G)$. From Lemma 2.1 there exists an edge uv in $C - X$ and there exist an edge $\overline{u}\,\overline{v}$ in $\overline{C} - Y$ satisfying $d(\{u, v\}, X) \geq \lfloor(g - 4)/2\rfloor$ and $d(\{\overline{u}, \overline{v}\}, Y) \geq \lfloor(g - 4)/2\rfloor$. Then by using (6) we have

$$\begin{aligned} g - 4 &\geq D(L(G)) \\ &\geq d_{L(G)}(uv, \overline{u}\,\overline{v}) \\ &= d_G(\{u, v\}, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq d_G(\{u, v\}, X) + 1 + \\ &\quad\ d_G(Y, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq 2\lfloor(g - 4)/2\rfloor + 2, \end{aligned}$$

which is impossible.

(4) Now $D(L(G)) \leq g - 5$. Thus the diameter $D(G) \leq g-4$, which means that $\kappa_3 = \xi_3$ by Theorem 1.2. Assume that G is not super-κ_3. Let X be an any κ_3-cut and consider two connected components

$C, \overline{C}$ of $G - X$ with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. From Lemma 2.1 there exists an edge uv in $C - X$ and there exists an edge $\overline{u}\,\overline{v}$ in $\overline{C} - X$ satisfying $d(\{u, v\}, X) \geq \lfloor (g-4)/2 \rfloor$ and $d(\{\overline{u}, \overline{v}\}, X) \geq \lfloor (g-4)/2 \rfloor$. Then by using (6) we have

$$\begin{aligned} g - 5 &\geq D(L(G)) \\ &\geq d_{L(G)}(uv, \overline{u}\,\overline{v}) \\ &= d_G(\{u, v\}, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq d_G(\{u, v\}, X) + d_G(X, \{\overline{u}, \overline{v}\}) \\ &\quad +1 \\ &\geq 2\lfloor (g-4)/2 \rfloor + 1, \end{aligned}$$

which is impossible. □

4 Connectivity of transformation graphs

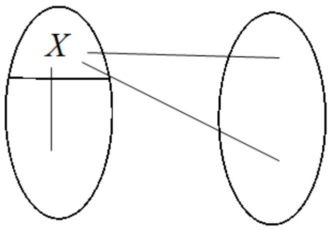
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. We suppose $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_n\}$. We define $G = G_1 \oplus G_2 : V(G) = V_1 \cup V_2$, $E(G) = E_1 \cup E_2 \cup \{x_i y_i : i = 1, \cdots, n\}$. We have $\delta(G) = min\{\delta(G_1) + 1, \delta(G_2) + 1\}$.

Theorem 4.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected graphs. And $\lambda(G_1) = \delta(G_1)$, $\lambda(G_2) = \delta(G_2)$. Then $\lambda(G) = \delta(G)$.*

Proof. We assume $\lambda(G) < \delta(G)$. There exists an edge cut F such that $|F| = \lambda(G)$ and $F = [X, \bar{X}]$, where $X \subseteq V(G)$ and $\bar{X} = V(G) - X$.

Case 1. $X \subseteq V_1$ or $X \subseteq V_2$.

We can see Fig.2 for illustration.



G_1 G_2

Fig.2

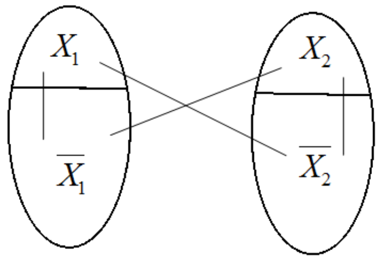
We assume $X \subseteq V_1$. Then

$$\begin{aligned} \delta(G) &> |[X, \bar{X}]| \\ &= |[X, V_1 - X]| + |[X, V_2]| \\ &\geq \lambda(G_1) + 1 \\ &= \delta(G_1) + 1, \end{aligned}$$

a contradiction.

Case 2. $X_1 = X \cap V_1 \neq \varnothing$ and $X_2 = X \cap V_2 \neq \varnothing$.

Set $\bar{X}_1 = V_1 - X_1$ and $\bar{X}_2 = V_2 - X_2$. We can see Fig.3 for illustration.



G_1 Fig.3 G_2

We have

$$\begin{aligned} \delta(G) &> |[X, \bar{X}]| \\ &= |[X_1, \bar{X}_1]| + |[X_2, \bar{X}_2]| \\ &\quad + |[X_1, \bar{X}_2]| + |[X_2, \bar{X}_1]| \\ &\geq \lambda(G_1) + \lambda(G_2) \\ &\geq \delta(G_1) + 1, \end{aligned}$$

a contradiction.

Both of two cases we are done. □

The hypercube $Q_n = (V, E)$ with $|V| = 2^n$ and $|E| = n2^{n-1}$. Every vertex can be represent by an n-bit binary string. Two vertices are adjacent if and only if their binary string representation differs in only one bit position. The hypercube $Q_n = Q_{n-1} \oplus Q_{n-1}$.

By Theorem 4.1 we have the following result.

Corollary 4.2. $\lambda(Q_n) = \delta(Q_n) = n$.

Theorem 4.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected graphs. And G_1, G_2 are super-λ, $\delta(G_1) \geq 2, \delta(G_2) \geq 2$. Then $G = G_1 \oplus G_2$ is super-λ.*

Proof. By contradiction. We assume $\delta(G_1) \leq \delta(G_2)$. And $\delta(G) = \lambda(G)$ by Theorem 4.1. Suppose that $G = G_1 \oplus G_2$ is not super-λ. Then there is an edge cut F with $|F| = \delta(G) = \lambda(G)$ such that $G - F$ is not connected but has no isolated vertex. Thus each component of $G - F$ has at least two vertices.

We assume $F = [X, \bar{X}]$, where $X \subseteq V(G)$ and $\bar{X} = V(G) - X$.

Case 1. $X \subseteq V_1$ or $X \subseteq V_2$.

We assume $X \subseteq V_1$. We can see Fig.4 for illustration.

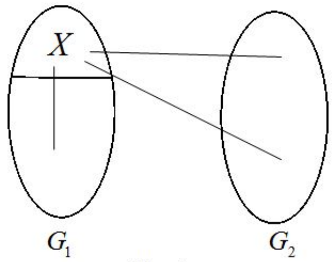


Fig.4

Then

$$
\begin{aligned}
\delta(G) &= |[X, \bar{X}]| \\
&= |[X, V_1 - X]| + |[X, V_2]| \\
&\geq \lambda(G_1) + \lambda(G_2) \\
&= \delta(G_1) + \delta(G_2) \\
&\geq \delta(G_1) + 2,
\end{aligned}
$$

a contradiction.

Case 2. $X_1 = X \cap V_1 \neq \varnothing$ and $X_2 = X \cap V_2 \neq \varnothing$.

Set $\bar{X}_1 = V_1 - X_1$ and $\bar{X}_2 = V_2 - X_2$. We can see Fig.5 for illustration.

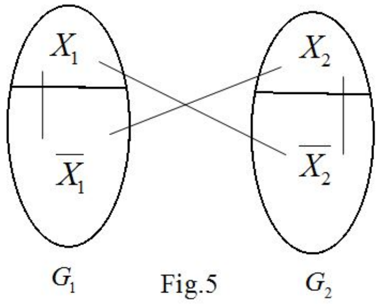


Fig.5

We have

$$
\begin{aligned}
\delta(G) &= |[X, \bar{X}]| \\
&= |[X_1, \bar{X}_1]| + |[X_2, \bar{X}_2]| \\
&\quad + |[X_1, \bar{X}_2]| + |[X_2, \bar{X}_1]| \\
&\geq \lambda(G_1) + \lambda(G_2) \\
&= \delta(G_1) + \delta(G_2) \\
&\geq \delta(G_1) + 2,
\end{aligned}
$$

a contradiction.

Both of two cases we are done.

□

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