Non-linear mathematical models for the jets penetrating liquid pool of different density under diverse physical conditions and their simulation

IVAN V. KAZACHKOV [(1), (2)] and OLEXANDER V. KONOVAL [(2)]
[(1)] Department of Energy Technology, Division of Heat and Power
Royal Institute of Technology
Brinellvägen, 68, Stockholm, 10044 - SWEDEN
Ivan.Kazachkov@energy.kth.se http://www.kth.se/itm/inst?l=en_UK
[(2)] Faculty of Physics and Mathematics, Division of Applied Mathematics and Informatics
Nizhyn Mykola Gogol State University
Kropyv'yans'koho, 2, Nizhyn, Chernigiv region, 16600, UKRAINE
suiciderash@gmail.com http://www.ndu.edu.ua/

Abstract: - The problem of modelling and simulation for the jets penetrating pool of other liquids is considered under different physical conditions and situations. It may happen for example in severe accidents at the nuclear power plants in touch with development and operation of the passive protection systems against severe accidents, as well as in many other problems. The specific peculiarities of the penetrating jets are discussed and mathematical modelling of the problem is considered. The non-linear second-order differential equation and the Cauchy problem is analyzed and solved analytically using the simultaneous transformation for both dependent and independent variables. The result obtained may be useful for theoretical and practical applications, where the liquid jet or solid rod is penetrating the pool of other liquid.

Key-Words: - Non-Linear, Second-Order, Differential, Equation, Analytical Solution, Jet, Penetration, Pool, Mathematical Modelling, Transformation.

1 Introduction to a problem

The peculiarities of penetration dynamics for a liquid jet into the other liquid pool of different densities have been studied in a number of papers in the metal, nuclear and other industries, e.g. [1-10].

An objective for the present paper is determining the model for penetration behaviours of a thick jet into a fluid pool, analytical solution of the Cauchy problem and analysis of the solution obtained.

Phenomenon of a jet penetration into a pool of other liquid is considered as follows. According to the phenomenon studied in [6, 9, 10] jet is penetrating a pool with nearly constant radius up to some point in a pool, then at the point of "bifurcation" it is abruptly changing its radius substantially (jet switches its one constant radius to the another one) as illustrated by experimental data borrowed from [6] shown in Fig. 1.

Both data, experimental and numerical, for the support of the above-mentioned peculiarities of a jet penetration into a pool are presented in the Fig. 1 for the corresponding moments of time (in ms). This phenomenon contradicts to the classic jet scheme when jet is considering as gradually changing its radius due to the losses of the velocity [1, 2, 7, 8].

Widening of a jet is linear with a distance, and all cross-jet velocity profiles, except those very near the orifice, are similar to one another, after suitable averaging over turbulent fluctuations. Similar schematic representations were considered for the laminar jets as well. This corresponded very well to a huge number of experimental data but not for the cases considered in [6, 9, 10] and here when for the thick penetrating jet the main forces are buoyancy, hydrodynamic drag and inertia.

2 Non-linear mathematical model for a jet penetrating a pool of other liquid

2.1 Phenomena of a jet penetration into pool of other liquid at diverse physical conditions

In short, the general behaviours of the plunging jet consists of surface cavity of a pool liquid by the initial impact of the jet, air pocket formation during the penetration, radial bottom spreading of the jet and entrained air and interfacial instability between the pool liquid and entrained air.

It must be underlined that analytical solutions for continuous and finite jets have reasonably described the characteristics of the penetration behaviours.

And numerical model simulated these general behaviours of the plunging jet and provided reasonable match on the penetration velocities [6, 9-11].

The multiple experimental results have clearly shown that penetrating jet is going first with approximately stable radius and then is abruptly changing its radius to the other one, bigger. This bifurcation point may be explained from the analytical solution obtained.

Thus, the jet penetrating the pool is supposed as a body of a variable mass assuming that the jet is moving under an inertia force acting against the drag and buoyancy forces. Here the surface forces are supposed to be negligible comparing to the three above-mentioned forces.

Radius of a jet is assumed to be approximately constant during a jet penetration or at least during some part of the length of penetration. This allows calculating the jet penetration process step-by-step in a general case taking for the beginning some constant radius of a jet, and then change it to the other one constant jet radius, and so on.

With account of the above-mentioned, the equation of a jet momentum considering a jet as a body of variable mass is written as follows [11]:

$$\pi {r_0}^2 \rho_1 \frac{d(hv_1)}{dt} = \pi {r_0}^2 h(\rho_1 - \rho_2)g - \frac{1}{2}\rho_2 v_1^2 \pi {r_0}^2 \quad (1)$$

for the case of the isothermal jet with a radius r_0. Here ρ_2 and ρ_1 are densities of the pool and jet, respectively, $v_1 = dh/dt$ - velocity of a jet, t – time, h- length of a jet penetration into a pool (from its free surface), g is acceleration due to gravity, $\pi {r_0}^2$ is a jet cross-section area, $0.5\rho_2 v_1^2 \pi {r_0}^2$ is a drag force acting on a head of the jet from the pool, which was in a rest before jet penetration.

The multiplayer 0.5 is taken in the above equation as a maximal possible value. In a reality some part of the kinetic energy of the disturbed pool is wasted.

Then the equation of a jet momentum for the non-isothermal conditions may be written for the unit cross-section in a following form [11]:

$$\rho_1 \frac{d(hv_1)}{dt} = h(\rho_1 - \rho_2)g - \frac{1}{2}\rho_2 v_1^2 - \rho_v RT_1\,, \quad (2)$$

where the vapour pressure action (the last term in an equation) on a jet is taken maximal possible, ρ_v is a vapour density, the last term in eq. (2) represents pressure due to vaporization of the coolant in a pool,

Fig.1. Jet penetration experimental data: initial velocity of a jet - 4m/s, 6m/s, 9m/s, respectively

R is the universal gas constant and T_1 is a jet temperature. The energy conservation equations may be presented in the following simplified form

$$\frac{\partial T_1}{\partial t} + v_1 \frac{\partial T_1}{\partial x} = \kappa_1^{\,2} \frac{\partial^2 T_1}{\partial x^2}\,,$$
$$\frac{dT_2}{dt} = \kappa_2^{\,2} \Delta T_2\,. \quad (3)$$

Here κ is a thermal diffusivity coefficient of the medium (jet or pool). The corresponding initial and boundary conditions must be stated for the equation array (3).

2.2 Mathematical model in a dimensionless form

The non-linear equation (1) may be transformed to the dimensionless form:

$$\frac{d^2\bar{h}}{d\bar{t}^2} + \frac{2+\rho_{21}}{2\bar{h}}\left(\frac{d\bar{h}}{d\bar{t}}\right)^2 + \frac{\rho_{21}-1}{Fr} = 0\,, \quad (4)$$

where $\rho_{21}=\rho_2/\rho_1$, $Fr=u_0^2/(gr_0)$ is the Froude number, which characterizes the ratio of the inertia and buoyancy forces. Here the following scales were implemented for the velocity and for the time, respectively: u_0 and r_0/u_0. Here $\bar{h}$ and $\bar{t}$ are dimensionless values, u_0 is the initial velocity of a jet before its penetration into the pool.

The initial conditions for the jet momentum equation (4) are stated as:

$$\bar{t}=0, \quad \bar{h}=0\,, \quad d\bar{h}/d\bar{t}=1\,, \quad (5)$$

So that it yields to the Cauchy problem (4), (5).

The non-linear second-order Cauchy problem (4), (5) thus obtained is now solved analytically using the simultaneous transformations for the both dependent as well as independent variables according to the methodology presented in [12].

2.3 Penetration of the solid rod into a pool

If a solid rod is penetrating a pool of liquid, the equation of momentum (1) changes to the following one

$$\pi r_0^{\,2}\rho_1 H\frac{dv_1}{dt} = \pi r_0^{\,2}(H\rho_1 - h\rho_2)g - \frac{1}{2}\rho_2 v_1^2 \pi r_0^{\,2} \quad (6)$$

for the penetration depth $h \le H$, where H is the length of the rod. And

$$\pi r_0^{\,2}\rho_1 H\frac{dv_1}{dt} = \pi r_0^{\,2}H(\rho_1 - \rho_2)g - \frac{1}{2}\rho_2 v_1^2 \pi r_0^{\,2} \text{ - } (7)$$

for the penetration depth $h > H$, when the solid rod is moving totally inside the pool.

The equations thus obtained for two cases of the solid rod penetration into a pool are transformed to the more general dimensionless forms. The equation (6) for the first case, which corresponds to an initial stage of penetration, is written as the following dimensionless array:

$$\bar{H}\frac{d^2\bar{h}}{d\bar{t}^2} = \frac{\bar{H} - \bar{h}\rho_{21}}{Fr} - \frac{1}{2}\rho_{21}\left(\frac{d\bar{h}}{d\bar{t}}\right)^2,$$
$$\bar{v}_1 = \frac{d\bar{h}}{d\bar{t}}\,. \quad (8)$$

Equation (7) in a dimensionless form is as follows

$$\bar{H}\frac{d\bar{v}_1}{d\bar{t}} = \frac{\bar{H}(1-\rho_{21})}{Fr} - \frac{1}{2}\rho_{21}\bar{v}_1^2\,. \quad (9)$$

Both equations, (8) and (9), show that solid rod penetration substantially differ from penetration of the jet having the same radius as solid rod.

The last equation (9) describes movement of the rod when it is totally in a pool. It is easily integrated with the initial condition

$$\bar{t}=0\,, \quad \bar{v}_1 = 1\,, \quad \bar{h}=0\,, \quad (10)$$

where $\bar{v}_1 = 1$ must be replaced by $\bar{v}_1 = \bar{v}_1^0$ in case of equation (9) as far as velocity of the rod decreased by its total immerse into a pool.

For the three different cases depending on the density ratio (solid rod to a pool) the solution of (9) is got as shown below:

1) $\rho_1 > \rho_2$ ($\rho_{12} > 1$, rod is denser than a pool):

$$\left|\frac{\bar{v}_1 - \sqrt{2\bar{H}\frac{\rho_{12}-1}{Fr}}}{\bar{v}_1 + \sqrt{2\bar{H}\frac{\rho_{12}-1}{Fr}}}\right| = \left|\frac{\bar{v}_1^0 - \sqrt{2\bar{H}\frac{\rho_{12}-1}{Fr}}}{\bar{v}_1^0 + \sqrt{2\bar{H}\frac{\rho_{12}-1}{Fr}}}\right| e^{-\frac{\rho_{21}}{2\bar{H}}\bar{t}}\,, \quad (11)$$

where from for $\bar v_1 \le \bar v_1{}^0 < \sqrt{2\overline{H}\dfrac{\rho_{12}-1}{Fr}}$ one can get formula for velocity of the rod penetration:

$$\bar v_1 = \sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}}\left[\sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}} + \bar v_1^0 + (\bar v_1^0 + -\sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}})e^{-\frac{\rho_{21}}{2\overline{H}}\bar t}\right]\left[\sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}} + \bar v_1^0 + (\sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}} - \bar v_1^0)e^{-\frac{\rho_{21}}{2\overline{H}}\bar t}\right]^{-1}$$

with the modified (velocity $\bar v_1^0$ instead of 1) initial condition (10). Remarkably it is the same relation in case of

$$\bar v_1 \ge \bar v_1{}^0 \ge \sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}} .$$

For $\rho_{12} >> 1$ the equation above gives for low density pool and long rod ($\rho_{21}/\overline{H} << 1$) the following simplification with account of

$$e^{-\frac{\rho_{21}}{2\overline{H}}\bar t} \approx 1 - \frac{\rho_{21}}{2\overline{H}}\bar t \ \text{ by } \ \frac{\rho_{21}}{2\overline{H}}\bar t << 1:$$

$$\bar v_1 = \sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}}\left[2\bar v_1^0 + \frac{\rho_{21}\bar t}{2\overline{H}}(\sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}} - \bar v_1^0)\right]\cdot$$

$$\cdot\left[2\sqrt{2\overline{H}\frac{\rho_{21}-1}{Fr}} - \frac{\rho_{21}\bar t}{2\overline{H}}(\sqrt{2\overline{H}\frac{\rho_{12}-1}{Fr}} - \bar v_1^0)\right]^{-1},$$

which shows slow decrease of velocity.

2) $\rho_1 = \rho_2$ ($\rho_{12} = 1$, the rod and the pool are of the same density):

$$\bar v_1 = \frac{\bar v_1^0}{1 + \dfrac{\bar t}{2\overline{H}}}, \qquad (12)$$

In case of the same densities, as shown by (12), velocity of the rod penetration is asymptotically approaching zero. And the longer is rod, the slower is this process. Solution (12) shows slow hyperbolic decreasing the velocity of the rod with time.

3) $\rho_1 < \rho_2$ ($\rho_{12} < 1$, pool is denser than a solid rod):

$$\bar v_1 = \sqrt{2\overline{H}\frac{1-\rho_{12}}{Fr}}tg\left[arctg\frac{\bar v_1^0}{\sqrt{2\overline{H}\frac{1-\rho_{12}}{Fr}}} + -\sqrt{\frac{(\rho_{21}-1)\rho_{21}}{2Fr\overline{H}}}\bar t\right] \text{-} \qquad (13)$$

for the initial conditions:

$$\bar t = 0, \quad \bar v_1 = \bar v_1^0, \quad \bar h = h^0, \qquad (14)$$

and

$$\bar v_1 = -\sqrt{2\overline{H}\frac{1-\rho_{12}}{Fr}}tg\left[\sqrt{\frac{(\rho_{21}-1)\rho_{21}}{2Fr\overline{H}}}\bar t\right] \text{-} \qquad (15)$$

for the initial conditions:

$$\bar t = 0, \quad \bar v_1 = 0, \quad \bar h = h_0^0,$$

which are particular case of (14) when rod (jet) stops at some depth of penetration (velocity is zero) and afterwards further movement is going backwards, to the pool surface due to buoyancy force.

The solutions (13), (14) and (15), (16) thus obtained describe the case when jet or rod is penetrating a pool and the other case when jet/rod is moving back to a pool surface due to the buoyancy forces, correspondingly.

Analysis of the formula (13) shows that by penetration of a gas jet into dense pool ($\rho_1 << \rho_2$) the velocity of penetration is approximately

$$\bar v_1 = \sqrt{2\overline{H}\frac{1}{Fr}}tg\left[arctg\frac{\bar v_1^0}{\sqrt{2\overline{H}\frac{1}{Fr}}} - \sqrt{\frac{1}{2Fr\overline{H}}}\rho_{21}\bar t\right],$$

where from by $Fr >> 1$ (high-speed jet) one can get approximation

$$\bar v_1 = \bar v_1^0 - \sqrt{2\overline{H}\frac{1}{Fr}}tg\left[\sqrt{\frac{1}{2Fr\overline{H}}}\rho_{21}\bar t\right]$$

so that velocity decreases fast with the time.

3 Analytical solution of the non-linear second-order differential equation

The equation (4) is substantially non-linear due to the second term, which contains product of the function $1/\bar{h}$ and $(d\bar{h}/d\bar{t})^2$. In this section the transformation for analytical solution of this equation is found and then the equation is solved.

3.1 Transformation of the equation to a standard form

First the equation (4) is presented in the following form with the new function y according to [12]:

$$y'' + \frac{a}{y} y'^2 + \frac{b}{x} y' + \frac{c}{x^2} + dx^r \cdot g^s = 0 . \qquad (16)$$

Here are: $r \neq -2, s \neq 1$, $b = c = r = s = 0$, $a = \frac{2+\rho_{21}}{2}$, $d = \frac{(\rho_{21}-1)}{Fr}$.

Then for $y = e^{\psi}$ the standard equation (16) is transformed to

$$\psi'' + (a+1)\psi'^2 + \frac{b}{x}\psi' + \frac{c}{x^2} + dx^r e^{((s-1)\psi)} = 0 . \qquad (17)$$

with the next new function ψ. The independent variable here is x.

3.2 Simultaneous transformation of the dependent and independent variables

Now the function ψ (dependent variable) and independent variable x are transformed as follows

$$x = f \cdot \left(\frac{dx}{df}\right), \quad y = \frac{df}{d\xi}\left(\frac{x}{f}\right)^{(2+r)(1-s)}$$

where from the equation (7) yields to the third-order differential equation of the form

$$f_3 + \frac{a-1}{f_1} f_2^2 + \frac{b}{f} f_1 f_2 + \frac{c}{f^2} f_1^3 + df^r f_1^{s+2} = 0 , \quad (18)$$

Here are: $f_n = \frac{d^n f}{d\xi^n}$, $n = 1,2,3$.

The can be seen the equation (18) is still substantially non-linear and needs further transformation.

Afterwards the equation (18) may be replaced by the following equation array

$$\frac{dy}{dx} = x^{(2+r)/(1-s)-1}\left[z^{-a} w(z) + \frac{2+r}{1-s} z\right], \qquad (19)$$

$$\frac{d^2 y}{dx^2} = x^{(2+r)/(1-s)-2}\begin{pmatrix} -az^{-2a-1}w + z^{-2a}w' + \\ \frac{2+r}{1-s} z^{-a} \end{pmatrix} w(z) + \qquad (20)$$

$$+\left(\frac{2+r}{1-s} - 1\right) x^{(2+r)/(1-s)-2}\left[z^{-a} w(z) + \frac{2+r}{1-s} z\right]$$

Now substituting the equations (19) and (20) into the outgoing standard equation (16) one can get after division on the value $x^{(2+r)/(1-s)-2} w$:

$$\frac{dw}{dz} = F(z) + \frac{G_1(z)}{w}, \qquad (21)$$

where are

$$F(z) = \left[a - b - 2(a+1)\frac{2+r}{1-s}\right] z^2 ,$$

$$G_1(z) = \left[1 - b - \frac{2+r}{1-s} - (a+1)\left[\frac{2+r}{1-s}\right]^2 - c\right] \cdot z^{2a+1} - dz^{2a+s}$$

3.3 Transformation to the first-order differential equation

Now the equation (21) can be transformed to the following form with the assignments $A = 1 - b - 2(a+1)\frac{2+r}{1-s} \neq 0$ and $\alpha \neq 0$:

$$Z = \left(\frac{A^2}{d}\right)^{1/(1-s)} z , \qquad (1212)$$

$$W = d^{\frac{(a+1)}{(s-1)}} A^{\frac{-1-2(a+1)}{(s-1)}} w ,$$

$$\frac{dW}{dZ} = Z^a + \frac{1}{W}\left(KZ^{2a+1} - Z^{2a+s}\right), \qquad (23)$$

where are:

$$K(b,r)=\frac{\begin{Bmatrix}(1-s)(2+r)-(a+1)(2+r)^2\\-c(1-s)^2\end{Bmatrix}-(1-s)(2+r)b}{\left[\{1-s-2(a+1)(2+r)\}-(1-s)b\right]^2}=$$

$$=\frac{\begin{aligned}&\{2(1-s)(1-b)-4(a+1)-c(1-s)^2\}\\&+\{(1-s)(1-b)-4(a+1)\}r-(a+1)r^2\end{aligned}}{\left[(1-s)(1-b)-4(a+1)-2(a+1)r\right]^2}.$$

The first-order differential equation (23) is of the type as $\frac{dz}{dx}=\frac{1}{x}\cdot\frac{1}{z^a}w(z)$, which is in this case as follows:

$$\frac{dW}{dZ}=Z^a+\frac{1}{W}\left(KZ^{2a+1}-Z^{2a}\right), \qquad (24)$$

with the assignments

$$K=\frac{2-4(a+1)+[1-4(a+1)]\cdot 0-(a+1)\cdot 0}{[1-4(a+1)]^2}= =\frac{-2\cdot(\rho_{12}+1)}{(2\cdot\rho_{12}+3)^2},$$

and

$Z^{-1}=KV^2$, ($Z^{s-1}=KV^2$ for $K=-2\frac{2a+s+1}{(4a+s+3)^2}\neq 0$ or ∞),

$$v=\frac{1}{V}+kWV^{(2a+s+1)/(1-s)}= =\frac{1}{V}+kWV^{(\rho_{21}+1)/(2a+1)},$$

$$k=-\frac{1}{2}(4a+s+3)K^{(1+a)/(1-s)}= =-\frac{1}{2}(2\rho_{21}+3)K^{a+1}=$$

$$=-\frac{1}{2}(2\rho_{21}+3)\left[\frac{-2(\rho_{21}-1)}{(2\rho_{21}+3)^2}\right]^{1+\frac{\rho_{21}}{2}}.$$

Substituting all these above equations into (24) yields the following first-order equation

$$\frac{dV}{dv}=\frac{1}{\rho_{21}+1}\cdot\frac{Vv-1}{v^2-1},$$

or

$$\frac{dV}{dv}-\frac{1}{2}\cdot\frac{1}{\rho_{21}+1}\cdot\frac{2v}{v^2}V=\frac{-1}{\rho_{21}+1}\cdot\frac{1}{v^2-1}. \qquad (25)$$

General solution of the equation (25) thus obtained is written as follows

$$\left(v^2-1\right)^{\delta}V=\frac{-1}{\rho_{21}+1}\int\left(v^2-1\right)^{-\theta}dv+c, \qquad (26)$$

where are:

$$\delta=\frac{1}{2}\cdot\frac{-1}{\rho_{21}+1};\quad \theta=\frac{1}{2}\cdot\frac{2\rho_{21}+3}{\rho_{21}+1};$$

$$c=const;\quad \delta=\frac{-1}{2(\rho_{21}+1)};\quad \theta=\frac{\rho_{21}+3/2}{\rho_{21}+1}.$$

3.4 Analysis of the analytical solution obtained

The following integral is considered

$$\int\left(v^2-1\right)^{-1}dv=\frac{1}{2}\ln\left|v^2-1\right|-\ln|v+1|=\ln\sqrt{\left|\frac{v-1}{v+1}\right|},$$

The integral $\int\left(v^2-1\right)^{-\theta}dv$ exists for the integer and rational values of the parameter θ.

Assuming that $\rho_{21}=1$, $\theta=5/4$; $\rho_{21}\ll 1$, $\theta=3/2$, one can get

$$\int\left(v^2-1\right)^{-3/2}dv=\frac{v}{\left(v^2-1\right)^{1/2}}.$$

Then for the case of $\rho_{21}\ll 1$ the values of the parameters are:

$$\delta\approx-\frac{1}{2},\ k\approx\frac{1}{3},\ K\approx-\frac{2}{9},$$

$$d\approx-\frac{1}{Fr},\ a\approx\frac{2+\rho_{21}}{2},\ A\approx-3$$

and it yields to:

$$v = \frac{1}{v} + \kappa WV \,, \quad v = \frac{1}{v} + \frac{2}{9} WV \,,$$

where from follows as far as $W = -\frac{6}{Fr} \cdot \frac{y}{x^2}$, $Z = -9Fr \cdot \frac{y}{x^2}$, then there is $V = \frac{1}{KZ} = \frac{x^2}{2Fry}$.

Finally,

$$V = -v + c\left(v^2 - 1\right)^{1/2}, \quad V^2 = \frac{x^2}{2Fry},$$

$$v = \frac{1}{V} + \frac{4\mathrm{V}}{3\mathrm{Fr}} \frac{y}{x^2}.$$

Now it is got the following:

$$v \cdot V = 1 + \frac{4\mathrm{V}}{3\mathrm{Fr}} \frac{y}{x^2}, \rightarrow$$

$$v \cdot V = 1 + \frac{4\mathrm{x}^2}{3\mathrm{Fr}2\mathrm{Fr}} \frac{y}{x^2} = 1 + \frac{2}{3\mathrm{Fr}^2},$$

which results in

$$V^2 = \frac{x^2}{2\mathrm{Fr}y}, \quad V = -v + c\left(v^2 - 1\right)^{1/2}.$$

Excluding from the equations value v and substituting the value V instead of x^2 / y yields:

$$\left(\frac{x^2}{y}\right)^2 + 2\left[2\left(Fr + \frac{2}{3\mathrm{Fr}}\right) + c^2 Fr\right]\left(\frac{x^2}{y}\right) + \\ + 4\left(1 - c^2\right)\left(Fr + \frac{2}{3\mathrm{Fr}}\right) = 0 \,,$$

where from follows

$$\frac{y}{x^2} = 1 / \left\{ \begin{aligned} & -\left[2\left(Fr + \frac{2}{3\mathrm{Fr}}\right) + c^2 Fr\right] + \\ & \pm \sqrt{\left[2\left(Fr + \frac{2}{3\mathrm{Fr}}\right) + c^2 Fr\right]^2 + \\ + 4\left(c^2 - 1\right)\left(Fr + \frac{2}{3\mathrm{Fr}}\right)^2} \end{aligned} \right\}. \quad (27)$$

In the solution (27) square root is always positive for all values of the constant *c*, therefore with account of $\rho_{21} \ll 1$ one can get the physically right conclusion that gravitational forces accelerates the jet while the drag force decelerates it. From this point, one of the solutions is physically wrong and finally the solution in the variables $\bar{t}$, $\bar{h}$ is as follows

$$\frac{\bar{h}}{\bar{t}^2} = \quad (28)$$

$$= \frac{1}{\sqrt{\left[2\left(Fr + \frac{2}{3\mathrm{Fr}}\right) + c^2 Fr\right]^2 + +4\left(c^2 - 1\right)\left(Fr + \frac{2}{3\mathrm{Fr}}\right)^2} + \\ -\left[2\left(Fr + \frac{2}{3\mathrm{Fr}}\right) + c^2 Fr\right]}$$

The solution obtained (28) is trivial and has restricted applications for the case when jet is penetrating a pool starting with zero velocity at the pool free surface.

3.5 Analytical solution for general case

The Cauchy problem (4), (5) contains singularity due to a contact of moving jet with a pool in a rest because at the contact area of a jet and pool velocity will change abruptly (shock). Jet is loosing some velocity while liquid in a pool is getting moving.

To avoid singularity of the initial conditions (5), the following initial conditions might be considered instead of the above-mentioned initial conditions [11]:

$$\bar{t} = 0 \,, \quad \bar{h} = \bar{h}_0 \,, \quad d\bar{h} / d\bar{t} = \bar{u}_p \,, \quad (29)$$

where $\bar{h}_0$ and $\bar{u}_p$ are the initial length and velocity of a jet penetration (after a first contact of a jet with a pool). The equation (4) is solved with the initial conditions (29), where h_0 and u_p are computed in a dimension form using the Bernoulli equation and the jet momentum equation as follows:

$$0.5\rho_1 u_0^2 = 0.5\rho_1 u_p^2 + (\rho_1 - \rho_2) g h_0 / u_0^2 - 0.5\rho_2 u_p^2 \,, \quad (30)$$

$$\rho_1 H u_0 = \rho_1 H u_p + \rho_2 h_0 u_p \,,$$

where H is the initial length for the finite jet falling into the pool. In case of a jet spreading from a nozzle (not a jet of the finite length), this value is determined by a pressure at the nozzle.

Solution of the equation array (30) is presented in a dimensionless form as follows

$$\bar{h}_0 = \frac{\bar{H}}{\rho_{21}}\left(\frac{\sqrt{1+\rho_{21}}}{\sqrt{1+2(1-\rho_{21})h_0/Fr}}-1\right),$$
$$\bar{u}_p = \sqrt{\frac{1+2\bar{h}_0(1-\rho_{21})/Fr}{1+\rho_{21}}}. \tag{31}$$

The Cauchy problem (4), (29) was solved with the special simultaneous transformations of the dependent and independent variables with account of (31):

$$\bar{h} = \left(\frac{2A+1}{2}\right)^{\frac{2}{2A+1}} X^{\frac{2}{2A+1}},$$
$$d\bar{t} = \left(\frac{1}{2A+1}\right)^{\frac{1}{2A+1}} X^{\frac{1}{2A+1}} d\tau, \tag{32}$$

where $A=1+\rho_{21}/2$. Then using the equation (32) with a few further simple transformations yields the following linear second-order equation in the new variables:

$$\frac{d^2 y}{d\tau^2} + \frac{\rho_{21}^2 - 1}{2Fr^2} = 0.$$

Here y is a new variable by the equation $X = e^y$. Finally the solution is $y = c_1 e^{k\tau} + c_2 e^{-k\tau}$, where c_1, c_2 are the constants computed from the initial conditions (29).

The eigen values k are computed as

$$k = \sqrt{(1-\rho_{21})[1+0.5(1+\rho_{21})]/Fr}.$$

In case a pool is denser than a jet ($\rho_{21}>1$), k is an imaginary value, and the solution is

$$y = c_1^r \cos k\tau + c_2^r \sin k\tau.$$

The exact analytical solution thus obtained was based on the assumption about the constant jet radius; therefore it is strict for a solid rod penetration into the pool and for some initial part of a jet penetration before the remarkable growing of its radius. It might be used as approximate step-by-step solution for a jet penetration into a pool for small temporal intervals correcting a jet radius from step to step.

3.6 Bifurcation of the jet radius during its penetration into a pool

One could estimate an evolution of the jet's radius to get support of the assumptions made or obtain an idea on how to correct solution in a good correspondence to the existing experimental data. For this the Bernoulli equation and the mass conservation equation were considered [9-11] for a jet penetrating a pool. In a dimension form they are:

$$S_1((\rho_1-\rho_2)hg + 0.5\rho_1 v_1^2) = 0.5\rho_1 u_0^2 S_0,$$
$$\rho_1 v_1 S_1 = \rho_1 u_0 S_0. \tag{33}$$

S is a cross-section area of the jet. Indexes 0 and 1 denote the initial state and the current state of the jet, respectively.

In a dimensionless form the equation array (33) yields:

$$\bar{S}_1\left[2\bar{h}(1-\rho_{21})/Fr + \bar{v}_1^2\right] = 1,$$
$$\bar{S}_1\bar{v}_1 = 1. \tag{34}$$

Equation array (34) has the following solution:

$$\bar{S}_1 = \frac{Fr}{4\bar{h}(1-\rho_{21})}\left[1 \pm \sqrt{1-8\bar{h}(1-\rho_{21})/Fr}\right],$$
$$\bar{v}_1 = 1/\bar{S}_1 \tag{35}$$

The solution (35) clearly shows that there are two possible values for the radius of the jet: one is the initial radius of the jet while the other value means that the jet may loose its stability and change abruptly its radius to a bigger one. This corresponds very well to the experimental data, some of that shown in Fig. 1 above.

Thus, we have two available solutions for a jet radius, with the point of bifurcation, which depends on the Fourier number and density ratio as follows:

$$\bar{h} = \frac{Fr}{8(1-\rho_{21})}. \tag{36}$$

The equation (36) was got from requirement of positive value under square root in (35). At this point the value under square root equals zero. Let us call it the point of bifurcation [9-11]. After this point the solution (35) does not exist in real numbers, therefore the jet switches its radius abruptly between these two available stable states.

Starting penetration into the pool with the initial cross-sections, e.g. $\overline{S}_1 = 1$, after some small penetration depth or, more generally, in case of $8\overline{h}(1-\rho_{2/1}) << Fr$, the solution (35) gives the following pair of the available jet radiuses to switch between them:

$$\overline{S}_1 \approx 1, \quad \overline{S}_1 \approx \frac{Fr}{2\overline{h}(1-\rho_{21})} >> 1. \qquad (37)$$

Obviously, there are no physical reasons for a jet to grow abruptly from a section area 1 to a bigger one at the beginning of the jet penetration into a pool because the jet momentum directs mainly along its axis. But then, with a jet further penetration into a pool, due to instability of a jet causing by its free surface perturbations and by a loss of momentum, the jet area may change at any moment.

3.7 Examples illustrating the peculiarities of the jet bifurcation phenomenon

Jet is starting penetration into a pool from $\overline{S}_1 = 1$ and then it grows to a cross-section $\overline{S}_1 = 2$ at the point

$$\overline{h}_1 = \frac{Fr}{8(1-\rho_{21})} = \frac{1}{8Ri},$$

when further existence of the two possible jet's radiuses is impossible. Here $Ri = (1-\rho_{21})/Fr$ is the Richardson's number (the ratio of the momentum to the buoyancy forces).

The jet penetrates into a pool at the distance $h = h_0$ determined by the initial length of a jet, Froude number and density ratio. In case of a long jet as well as the jet, which is permanently spreading from a nozzle, the initial penetration length is determined by the Froude number and the density ratio.

The jet is going with increase of its radius untill the bifurcation point $h = h_1$. After bifurcation, the jet is abruptly switching to another nearly constant radius. Applying the solution obtained to those parts with their own initial data, the whole jet might be computed based on the analytical solution obtained. correspondence to the above experimental pictures.

3.8 Dimensionless time for the non-linear analytical solution

Dimensionless time $\bar{t}$ in the transformation (32) is determined through the variable τ as follows:

Jet is denser than pool (ρ_{21}<1)-

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \int e^{\frac{c_1 e^{k\tau} + c_2 e^{-k\tau}}{\rho_{21}+3}} d\tau + c_3; \qquad (38)$$

Pool is denser than jet (ρ_{21}>1)-

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \int e^{\frac{c_1' \cos k\tau + c_2' \sin k\tau}{\rho_{21}+3}} d\tau + c_3, \qquad (39)$$

where the constants c_3 are calculated later on. For τ<<1, the following linear approximations by $k\tau$ are satisfied: $e^{\pm k\tau} \approx 1 \pm k\tau$, $\cos k\tau \approx 1$, $\sin k\tau \approx k\tau$. Therefore the equations (38) and (39) yield:

for ρ_{21}<1,

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{k(c_1-c_2)} e^{\frac{1}{3+\rho_{2/}}[(c_1+c_2)+k(c_1-c_2)\tau]} + c_3.$$

for ρ_{21}>1,

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{c_2' k} e^{\frac{c_1' + c_2' k\tau}{3+\rho_{21}}} + c_3.$$

The constants c_3 are got from these equations requiring $\bar{t} = 0$, which leads to $\tau = 0$. Consequently, the real dimensionless time $\bar{t}$ is computed through the introduced variable τ as follows:

ρ_{21}<1,

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{k(c_1-c_2)} \cdot \left\{ e^{\frac{1}{3+\rho_{21}}[(c_1+c_2)+k(c_1-c_2)\tau]} - e^{\frac{c_1+c_2}{3+\rho_{21}}} \right\}; \qquad (40)$$

ρ_{21}>1,

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{c_2^{'}k}\,(e^{\frac{c_1^{'}+c_2^{'}k\tau}{3+\rho_{21}}} - e^{\frac{c_1^{'}}{3+\rho_{21}}}). \quad (41)$$

The equations (40), (41) are satisfied in a vicinity of $\tau = 0$. In general case, one needs computing the integrals in the expressions (38), (39) numerically. But for $\rho_{2/1}$~1 and *Fr*>>1, the multiplayer of τ must be small value, which is available for computing with the approximations (40), (41) in a wider region of τ, and even if τ is not small but the condition $k\tau$ <<1 is still satisfied.

Further transformation of the expression (40) is presented in the form:

$$\bar{t} = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{k(c_1-c_2)} e^{\frac{c_1 e^{k\tau}+c_2 e^{-k\tau}}{3+\rho_{21}}} - \bar{t}_0, \quad (42)$$

$$\bar{t}_0 = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{k(c_1-c_2)} e^{\frac{c_1+c_2}{3+\rho_{21}}}. \quad (43)$$

The corresponding expressions for ρ_{21}>1 are obtained from (41) similarly.

3.9 Calculation of the constants for solution

Using the initial conditions (29) and correlations (31), substituting the equation (34) into (29) one can get the constants. For ρ_{21}<1 (jet is denser than pool) the constants are computer from the equations:

$$c_1 + c_2 = \ln\left[\left(\frac{2}{3+\rho_{21}}\right)\bar{h}_0^{-\frac{3+\rho_{21}}{2}}\right],$$

$$c_1 - c_2 = \frac{\bar{u}_p}{\sqrt{\bar{h}_0}}\sqrt{\frac{Fr}{1-\rho_{21}}}\left(\frac{2}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}-\frac{1}{2}}(3+\rho_{21})^{-\frac{1}{3+\rho_{11}}}$$

where from yields:

$$c_1 = \frac{1}{2}\left[\ln\left(\frac{2}{3+\rho_{21}}\bar{h}_0^{-\frac{3+\rho_{21}}{2}}\right) + \frac{\bar{u}_p}{\sqrt{\bar{h}_0}}\sqrt{\frac{Fr}{1-\rho_{21}}}\cdot\right. \quad (44)$$

$$\left.\cdot\left(\frac{2}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}-\frac{1}{2}}(3+\rho_{21})^{-\frac{1}{3+\rho_{21}}}\right],$$

$$c_2 = \frac{1}{2}\left[\ln\left(\frac{2}{3+\rho_{21}}\bar{h}_0^{-\frac{3+\rho_{21}}{2}}\right) - \frac{\bar{u}_p}{\sqrt{\bar{h}_0}}\sqrt{\frac{Fr}{1-\rho_{21}}}\cdot\right.$$

$$\left.\cdot\left(\frac{2}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}-\frac{1}{2}}(3+\rho_{21})^{-\frac{1}{3+\rho_{21}}}\right].$$

For ρ_{21}>1 (pool is denser than jet), from (34), (29), accounting (37), yield the constants $c'_{1,2}$:

$$c_1^{'} = \ln\left[\left(\frac{2}{3+\rho_{21}}\right)\bar{h}_0^{-\frac{3+\rho_{21}}{2}}\right],$$

$$c_2^{'} = \frac{\bar{u}_p}{\sqrt{\bar{h}_0}}\sqrt{\frac{Fr}{1-\rho_{21}}}\left(\frac{2}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}-\frac{1}{2}}(3+\rho_{21})^{-\frac{1}{3+\rho_{21}}}. \quad (45)$$

4 Transformation of the non-linear solution to an explicit form

4.1 Explicit form of the solution obtained

The solution (14) can be transformed to an explicit form as the function of *t* (exclude the artificial time τ). For this purpose, from (23), (24) yields

$$\bar{t} + \bar{t}_0 = \left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}} \frac{3+\rho_{21}}{k(c_1-c_2)}\cdot$$

$$\cdot e^{\frac{1}{3+\rho_{21}}[(c_1+c_2)+k(c_1-c_2)\tau]},$$

and further it goes to

$$e^{k\tau} = \left[(\bar{t}+\bar{t}_0)k(c_1-c_2)\cdot\right.$$

$$\left.\cdot\left(\frac{1}{3+\rho_{21}}\right)^{\frac{1}{3+\rho_{21}}-1} e^{-\frac{c_1+c_2}{3+\rho_{21}}}\right]^{\frac{3+\rho_{21}}{c_1-c_2}}, \quad (46)$$

or

$$e^{k\tau} = \left[(\bar{t}+\bar{t}_0)\frac{\bar{u}_p}{2\bar{h}_0}\right]^{\alpha}. \quad (47)$$

With account of (46) yields:

$$\alpha = \frac{\sqrt{1-\rho_{21}}}{\bar{u}_p}\sqrt{\frac{\bar{h}_0}{Fr}}\left(\frac{2}{3+\rho_{21}}\right)^{\frac{\rho_{21}+1}{2(3+\rho_{21})}} \cdot \cdot (3+\rho_{21})^{\frac{1}{3+\rho_{21}}+1} \quad (48)$$

The accounting $e^{c_1 e^{k\tau}+c_2 e^{-k\tau}} = e^{c_1 e^{k\tau}} e^{c_2 e^{-k\tau}}$ and using the equations (47), (48), (42)-(43), one can come to a solution (34) for the penetration depth h as a function of the real temporal variable t (for $k\tau$ <<1):

$$\bar{h} = \left(\frac{3+\rho_{21}}{2}\right)^{\frac{2}{3+\rho_{21}}(1-chk\tau)} \bar{h}_0^{chk\tau} e^{\frac{2}{\alpha}shk\tau}, \quad (49)$$

where ch, sh denote the hyperbolic cosine and sine, respectively, $e^{k\tau}$ is expressed through t by (47). The velocity of a jet penetration into a pool is determined from (34) or (49) using the relations $\bar{v}_1 = d\bar{h}/d\bar{t} = (d\bar{h}/d\tau)d\tau/d\bar{t}$. In the cases of $\rho_{2/1}$<1 and $\rho_{2/1}$>1, it results in

$$\frac{d\bar{h}}{d\bar{t}} = \left(\frac{3+\rho_{21}}{2}\right)^{\frac{2}{3+\rho_{21}}-1} \left(3+\rho_{21}\right)^{\frac{1}{3+\rho_{21}}} \cdot \cdot k(c_1 e^{k\tau} - c_2 e^{-.k\tau}) e^{\frac{c_1 e^{k\tau}+c_2 e^{-.k\tau}}{3+\rho_{21}}} \quad (50)$$

and

$$\frac{d\bar{h}}{d\bar{t}} = \left(\frac{3+\rho_{21}}{2}\right)^{\frac{2}{3+\rho_{21}}-1} \left(3+\rho_{21}\right)^{\frac{1}{3+\rho_{21}}} \cdot \cdot k(c_2' \cos k\tau - c_1' \sin k\tau) e^{\frac{c_1' \cos k\tau + c_2' \sin k\tau}{3+\rho_{21}}} \quad (51)$$

respectively.

4.2 Basic parameters of jet penetration into pool

The equations obtained for the non-linear model of a jet, e.g. (49)-(51), allow computing the parameters of a jet penetrating a pool. For example, the penetration depth h_* is determined by condition $dh/dt=0$, therefore h_* and the correspondent penetration time t_*, for ρ_{21}<1 and ρ_{21}>1 are computed, correspondingly, as

$$\rho_{21}<1, \quad \bar{h}_* = \left(\frac{3+\rho_{21}}{2}\right)^{\frac{2}{3+\rho_{21}}} e^{\frac{4}{3+\rho_{21}}\sqrt{c_1 c_2}},$$
$$\bar{t}_* = \frac{\left(3+\rho_{21}\right)^{1-\frac{1}{3+\rho_{21}}}}{k(c_1-c_2)} e^{\frac{2\sqrt{c_1 c_2}}{3+\rho_{/1}}} - \bar{t}_0; \quad (52)$$

$$\rho_{21}>1, \quad \bar{h}_* = \left(\frac{3+\rho_{21}}{2}\right)^{\frac{2}{3+\rho_{21}}} e^{2\frac{c_1'+(c_2')^2/c_1'}{3+\rho_{21}}},$$
$$\bar{t}_* = \frac{\left(3+\rho_{21}\right)^{1-\frac{1}{3+\rho_{21}}}}{kc_2'} e^{\frac{c_1'+(c_2')^2/c_1'}{3+\rho_{21}}} - \bar{t}_0. \quad (53)$$

The solution (52) and (53) for two different cases (for the pool, which is denser than a jet and for the inverse situation) are evidently absolutely different.

4.3 Analysis of the solution for limit cases

Analysis of the analytical solution obtained is easier performed for the limit cases when the solution is substantially simplified. For example, if ρ_{21}<<1, (1-ρ_{21})h_0<<Fr, then (32), (43), (46), (48) result in:

$$\bar{h}_0 \approx \frac{\bar{H}}{2}, \quad \bar{t}_0 \approx \bar{H},$$
$$\alpha \approx 2.85\sqrt{\frac{\bar{H}}{Fr}}, \quad e^{k\tau} \approx \left(\frac{\bar{t}}{\bar{H}}+1\right)^{\alpha},$$

which can be easily analysed. Here should be noted that this approximation satisfies a wide range of parameters because many practical situations correspond to the large Froude numbers.

Accounting (26), (44), from (49), (50) yields the following approximate solution for the depth of a jet penetration, as well as for its velocity and acceleration:

$$\bar{h} = \left(\frac{3}{2}\right)^{2/3(1-chk\tau)} \left(\frac{\bar{H}}{2}\right)^{chk\tau} e^{\frac{1}{1,43}\sqrt{\frac{Fr}{\bar{H}}}shk\tau}, \quad (54)$$

$$\bar{v}_1 = \frac{d\bar{h}}{d\bar{t}} = 2.85\frac{\bar{h}}{\bar{H}}\sqrt{\frac{\bar{H}}{Fr}}\left(\frac{\bar{t}}{\bar{H}}+1\right)^{-1} \cdot \cdot \{\ln\left[\frac{\bar{H}}{2}\left(\frac{2}{3}\right)^{2/3}\right]shk\tau + \frac{1}{1.43}\sqrt{\frac{Fr}{\bar{H}}}chk\tau\}, \quad (55)$$

or, with explicit expression for dh/dt, (55) is

$$\bar{v}_1 = \left(\frac{3}{2}\right)^{2/3}\left[\frac{\bar{H}}{2}\left(\frac{2}{3}\right)^{2/3}\right]^{chk\tau} e^{\frac{1}{1.43}\sqrt{\frac{Fr}{\bar{H}}}shk\tau}\frac{2.85}{\bar{H}}\sqrt{\frac{\bar{H}}{Fr}}\left(\frac{\bar{t}}{\bar{H}}+1\right)^{-1}\cdot$$

$$\cdot\{\ln\left[\frac{\bar{H}}{2}\left(\frac{2}{3}\right)^{2/3}\right]shk\tau+\frac{1}{1.43}\sqrt{\frac{Fr}{\bar{H}}}chk\tau\}$$

where are:

$$chk\tau \approx \frac{1}{2}\left[\left(\frac{\bar{t}}{\bar{H}}+1\right)^{2.85\sqrt{\bar{H}/Fr}}+\left(\frac{\bar{t}}{\bar{H}}+1\right)^{-2.85\sqrt{\bar{H}/Fr}}\right],$$

$$shk\tau \approx \frac{1}{2}\left[\left(\frac{\bar{t}}{\bar{H}}+1\right)^{2.85\sqrt{\bar{H}/Fr}}-\left(\frac{\bar{t}}{\bar{H}}+1\right)^{-2.85\sqrt{\bar{H}/Fr}}\right]. \quad (56)$$

Analysis of the expressions (55), (56) thus obtained shows dependence of the jet characteristics on the parameters $\sqrt{\bar{H}/Fr}$, $\bar{t}/\bar{H}$. A key feature of a jet penetration is determined by the Froude number and initial jet length, e.g. for $\bar{H}/Fr$<<1:

$$\left(\frac{\bar{t}}{\bar{H}}+1\right)^{2.85\sqrt{\frac{\bar{H}}{Fr}}} \approx 1+2.85\sqrt{\frac{\bar{H}}{Fr}}\ln\left(\frac{\bar{t}}{\bar{H}}+1\right),$$

$$shk\tau \approx 2.85\sqrt{\frac{\bar{H}}{Fr}}\ln\left(\frac{\bar{t}}{\bar{H}}+1\right), \qquad chk\tau \approx 1,$$

up to a limit $\bar{t}/\bar{H}$~1 and even higher. For example,

$$10^{0,1} \approx 1{,}23, \;\; 1000^{0,1} \approx 2,$$

therefore the approximations used here satisfy a wide range of the varying parameters. By such assumptions, linearization of the solution (35) by the parameter $\bar{H}/Fr$ yields

$$\bar{h} \approx \frac{\bar{H}}{2}\left(\frac{\bar{t}}{\bar{H}}+1\right)^2,$$

$$\bar{v}_1 \approx \{4{,}06\frac{\bar{H}}{Fr}\ln\left[\frac{\bar{H}}{2}\left(\frac{2}{3}\right)^{2/3}\right]\ln\left(\frac{\bar{t}}{\bar{H}}+1\right)+1\}\left(\frac{\bar{t}}{\bar{H}}+1\right) \quad (57)$$

With an order of the term $\ln\left[\bar{H}/2(2/3)^{2/3}\right]\ln(\bar{t}/\bar{H}+1)$ restricted by 1, further simplification is as follows

$$\bar{v}_1 \approx \frac{\bar{t}}{\bar{H}}+1,$$

$$\frac{d^2\bar{h}}{d\bar{t}^2} \approx \frac{1}{\bar{H}}+\frac{2}{Fr}\ln\left[\frac{\bar{H}}{2}\left(\frac{2}{3}\right)^{2/3}\left(\frac{\bar{t}}{\bar{H}}+1\right)^2\right] \approx \frac{1}{\bar{H}}.$$

Thus, in this case velocity of a jet is linear function of time and hyperbolic by the jet's length. Here $\bar{H}$~1 or $\bar{H}$>>1 were considered because by $\bar{H}$<<1 there is actually no jet (a length of a jet supposed to be at least larger than its diameter). But this case might be also considered using the solution obtained.

4.4 Parameters of the jet in a general case

With account of the above mentioned, substitution of (43) into (46)-(49) results for ρ_{21}<<1 in the following:

$$\bar{h} \approx \left(\frac{3}{2}\right)^{2/3}\left[\left(\frac{2}{3}\right)^{2/3}\frac{\rho_{21}}{2}Fr\right]^{chk\tau} e^{\frac{1}{\sqrt{\rho_{21}}}shk\tau},$$

$$\frac{dh}{dt} \approx \frac{2h}{\sqrt{\rho_{2/1}}Fr}\left(\frac{t}{\rho_{2/1}Fr}+1\right)^{-1}\left\{\ln\left[\left(\frac{2}{3}\right)^{\frac{2}{3}}\frac{\rho_{2/1}}{2}Fr\right]shk\tau+\frac{chk\tau}{\sqrt{\rho_{2/1}}}\right\} \quad (58)$$

where are:

$$\bar{t}_0 = 2\bar{h}_0 = \rho_{21}Fr, \quad \bar{h}_0 \approx \frac{\rho_{21}}{2}Fr,$$

$$e^{k\tau} \approx \left(\frac{\bar{t}}{\rho_{21}Fr}+1\right)^{2\sqrt{\rho_{21}}}. \quad (59)$$

The equations (59) yield for $\bar{t} << \rho_{21}Fr$ the following approximations:

$$shk\tau \approx 2\bar{t}/(\rho_{21}^{3/2}/Fr^2), \quad chk\tau \approx 1,$$

therefore solution of the problem in a form (58) goes to the following simplified expressions:

$$\bar{h} \approx \bar{h}_0+\frac{\bar{t}}{\rho_{21}Fr}, \quad \bar{v}_1 \approx \frac{1}{\rho_{21}Fr}. \quad (60)$$

Analysis of the simple limit solution (60) shows that at the beginning of the jet penetration the depth of penetration is a linear function of time, and the velocity of penetration is nearly constant being inversely proportional to the density ratio and to the Froude number.

4.5 Estimation of the solution for long time

Similar approximation for the extended time $t >> \rho_{21} Fr$ is the following:

$$\bar{h} \approx \left(\frac{3}{2}\right)^{2/3} \left[\left(\frac{2}{3}\right)^{2/3} \frac{\rho_{21}}{2} Fr\right]^{\frac{1}{2}\left(\frac{\bar{t}}{\rho_{21}Fr}\right)^{2\sqrt{\rho_{21}}}} \cdot e^{\frac{1}{\sqrt{\rho_{21}}}\left(\frac{\bar{t}}{\rho_{21}Fr}\right)2\sqrt{\rho_{21}}}, \quad (61)$$

the length of a jet penetration is growing with time.

A full penetration length is determined by the condition of $\bar{v}_1 = 0$, so that

$$shk\tau_* \ln\left[\left(\frac{2}{3}\right)^{2/3} \frac{\rho_{21}}{2} Fr\right] = -\frac{chk\tau_*}{\sqrt{\rho_{21}}},$$

with a time and a depth of penetration, $\tau_*, \bar{h}_*$, respectively. Solving this equation with (61) yields

$$\bar{h}_* = \left(\frac{3}{2}\right)^{2/3} \left[\left(\frac{2}{3}\right)^{2/3} \frac{\rho_{21}}{2} Fr\right]^{chk\tau_*} e^{-\frac{chk\tau_*}{\rho_{2/1} \ln\left[(2/3)^{2/3} \frac{\rho_{2/1}}{2} Fr\right]}},$$

$$\gamma = -\sqrt{\rho_{21}} \ln^{-1}\left[\left(\frac{2}{3}\right)^{2/3} \frac{\rho_{21}}{2} Fr\right], \quad (62)$$

$$\left(\frac{\bar{t}_*}{\rho_{21}Fr}+1\right)^{2\sqrt{\rho_{21}}} - \left(\frac{\bar{t}_*}{\rho_{21}Fr}+1\right)^{-2\sqrt{\rho_{21}}} = \gamma\left[\left(\frac{\bar{t}_*}{\rho_{21}Fr}+1\right)^{2\sqrt{\rho_{21}}} + \left(\frac{\bar{t}_*}{\rho_{21}Fr}+1\right)^{-2\sqrt{\rho_{21}}}\right],$$

and further goes for the penetration time:

$$\bar{t}_* = \left[\left(\frac{1+\gamma}{1-\gamma}\right)^{0.25/\sqrt{\rho_{/1}}} - 1\right]\rho_{21} Fr, \quad (63)$$

where

$$chk\tau_* = \frac{1}{2}\left\{\left[\frac{\sqrt{\rho_{21}} \ln\left[(2/3)^{2/3} 0.5\rho_{21}Fr\right] - 1}{\sqrt{\rho_{21}} \ln\left[(2/3)^{2/3} 0.5\rho_{21}Fr\right] + 1}\right]^{0.5} + \left[\frac{\sqrt{\rho_{21}} \ln\left[(2/3)^{2/3} 0.5\rho_{21}Fr\right] - 1}{\sqrt{\rho_{21}} \ln\left[(2/3)^{2/3} 0.5\rho_{21}Fr\right] + 1}\right]^{-0.5}\right\}.$$

If

$(2/3)^{2/3} 0.5\rho_{21}Fr << 1 \Rightarrow -\ln\left[(2/3)^{2/3} 0.5\rho_{21}Fr\right] >> 1$, $chk\tau_* \approx 1$, then from (62) yields:

$$h_* \approx 0.5 Fr \rho_{21} e^{-\frac{\rho_{12}}{\ln\left[(2/3)^{2/3} 0.5\rho_{21}Fr\right]}}. \quad (64)$$

The formula (64) is clear concerning the Froude number (the higher velocity, the deeper jet's penetration). But concerning the density ration influence is more complex function.

5 Correspondence of the model to experimental data

To validate the model developed and the analytical solution obtained, the computed penetration lepth of a jet had been compared to experimental data from the literature and the correspondence was got good for a number of different cases [9-11].

6 Conclusion

Non-linear analytical models for the continuous and finite jets to predict the maximum penetration of the plunging jet were developed and reasonably described the characteristics of the penetration behaviours.

The non-linear second-order differential equation and the Cauchy problem was analyzed and solved analytically using the simultaneous transformations for the both dependent, as well as independent variables in the differential equation.

An analytical solution obtained is accurate for the solid rod penetration into a liquid pool and is approximate for the jet penetration into a pool of other liquid. The jet penetrating a pool of other liquid was investigated for different conditions.

Problem is of interest for modelling and simulation of diverse practical applications, e.g. severe NPP accidents in touch with development of the passive protection systems.

Analyses on the penetration phenomena of a jet into another liquid at the isothermal and non-isothermal conditions were performed. The non-linear analytical models for the jet to predict the maximum penetration into a pool were developed and reasonably described the characteristics of the penetration behaviours.

The analytical solution for the non-linear second-order differential equation may be of interest as a mathematical result too.

References:

[1] M.I. Gurevich, Theory of jets in ideal fluids, Translated from the Russian edition by: Robert l. Street and Konstantin Zagustin, Academic press, New York/London, 1965.

[2] J.S. Turner, Jets and plumes with negative or reversing buoyancy, Journal of Fluid Mechanics, Vol.26, No.4, 1966, pp. 779-792.

[3] M.A. Lavrent'ev and B.V. Shabat, The problems of hydrodynamics and their mathematical models, Moscow, Nauka, 1973 (In Russian).

[4] M.L. Corradini, B.J. Kim, M.D. Oh, Vapor explosions in light water reactors: A review of theory and modeling, Progress in Nuclear Energy, Vol.22, No.1, 1988, pp. 1-117.

[5] M. Saito, et al. Experimental study on penetration behaviors of water jet into Freon-11 and Liquid Nitrogen, ANS Proceedings, Natl. Heat Transfer Conference, Houston, Texas, USA, July 24-27, 1988.

[6] H.S. Park, I.V. Kazachkov, B.R. Sehgal, Y. Maruyama and J. Sugimoto, Analysis of Plunging Jet Penetration into Liquid Pool in Isothermal Conditions, ICMF 2001: Fourth International Conference on Multiphase Flow, New Orleans, Louisiana, U.S.A., May 27 - June 1, 2001.

[7] G.N. Abramovich, L. Schindel, The Theory of Turbulent Jets, MIT Press, 1963.

[8] F. Bonetto, D. Drew and R.T. Lahey, Jr., The analysis of a plunging liquid jet-air entrainment process, Chemical Engineering Communications, Vol.130, 1994, pp. 11-29.

[9] I.V. Kazachkov, A.H. Moghaddam, Modeling of thermal hydraulic processes during severe accidents at nuclear power plants, National Technical University of Ukraine "KPI", Kyiv, 2008 (in Russian).

[10] I.V. Kazachkov and V.H. Moghaddam. Specific peculiarities of the jets penetrating the liquid pool of different density under severe accidents at the NPP conditions and their modeling and simulation// WSEAS Transactions on Applied and Theoretical Mechanics.- 2012.- Vol. 7.- Issue 4.- P. 276-287.

[11] I.V. Kazachkov. The Mathematical Models for Penetration of a Liquid Jets into a Pool// WSEAS Transactions on fluid mechanics.- Issue 2, Volume 6, April 2011.- P. 71-91.

[12] P.L. Sachdev, Non-linear ordinary differential equations and their applications, Marcel Dekker, Inc., 1991.