The Gutenberg - Richter Law deviations due to random distribution of block sizes

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Abstract: This paper studies properties of a continuum with structure. The characteristic size of the structure governs the fact that difference relations do not automatically transform into differential ones. It is impossible to consider an infinitesimal volume of a body, to which we could apply the major conservation laws, because the minimal representative volume of the body must contain at least a few elementary microstructures. The corresponding equations of motions are the equations of infinite order, solutions of which include, along with sound waves, the unusual waves propagating with abnormal low velocities, not bounded below. It is shown that in such media weak perturbations can increase or decrease outside the limits. The number of complex roots of the corresponding dispersion equation, which can be interpreted as the number of unstable solutions, depends on the specific surface of cracks and is an almost linear dependence on a logarithmic scale, as in the seismological law of Gutenberg-Richter. If the distance between one pore (crack) to another one is the random value with some distribution, we must write another dispersion equation and examine different scenarios depending on statistical characteristics of the random distribution. In this case, there are sufficient deviations from Gutenberg-Richter law, and this theoretical result corresponds to some field and laboratory observations.

Key words: specific surface, operator of continuity, equation of motion, structured media, catastrophes.

1. Introduction

The idea of creation of new model of space is the following. We consider some finite volume of body. In this case, the surface forces apply to sphere of radius l_0 while the inertial forces apply into center of structure. There is no possibility to tend an elementary volume into zero and coincides points of surface and the point of inertial forces like in classical continuum. We must consider a finite volume like representative volume of body and we have a problem of different positions of surface and inertial forces.

There is urgent necessary to translate surface forces into the center of structure by special translation operator. Obvious consequence of this action will be a possibility to apply the conservation law of ordinary mechanics into some image of structural continuum like in usual classical model of continuous media. Some results of new model of structured continuum published earlier [1], but it is necessary to repeated some formulas from [1] in order to present new idea about effect of the random distribution. This idea may be present more clearly by using previous results, because there is closed relation between them.

2. Problem formulation

The one-dimensional operator of field translation from point x into point $x \pm l_0$ gives by symbolic formula [2]:

$$u(x \pm l_0) = u(x)e^{\pm l_0 D_x} \qquad (1)$$

In this formula: $D_x = \partial / \partial x$ (2)

There is a relation between the average distance between one crack to another one (or one pore to another one) l_0 and the specific surface σ_0, namely

$$\sigma_0 l_0 = 4(1 - f) \qquad (3)$$

Here f is the porosity.

The analogous operator of translation for some sphere gives by expression:

$$P\left(D_x, D_y, D_z; l_0\right) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{l_0\omega}\sin\theta d\theta d\varphi = \frac{sinh\left(l_0\sqrt{\Delta}\right)}{l_0\sqrt{\Delta}} = E + \frac{l_0^2}{3!}\Delta + \frac{l_0^4}{5!}\Delta\Delta + \cdots \quad (4)$$

Here E is the unit operator. And
$\omega = D_x sin\theta cos\varphi + D_y sin\theta sin\varphi + D_z cos\theta$
According to Poisson formula, we have [3]:

$$\int_0^{2\pi}\int_0^{\pi} f([\alpha sin\theta cos\varphi + \beta sin\theta sin\varphi + \gamma cos\theta]sin\theta d\theta d\varphi) = 2\pi\int_0^{\pi} f\left(\sqrt{\alpha^2+\beta^2+\gamma^2}cosp\right)sinpdp \quad (5)$$

In addition, P operator rewrites as follows:

$$P\left(D_x, D_y, D_z; l_0\right) = \frac{1}{2}\int_{-1}^{1} e^{l_0\sqrt{\Delta}t}dt = \int_0^1 cosh\left(l_0\sqrt{\Delta}t\right)dt = \frac{sinh\left(l_0\sqrt{\Delta}\right)}{l_0\sqrt{\Delta}} = E + \frac{l_0^2}{3!}\Delta + \frac{l_0^4}{5!}\Delta\Delta + \cdots \quad (6)$$

The P operator represents the operator of continuity. Classical Cauchy and Poisson continuous model of space means that $P=E$. In structured medium, the last relation is not valid.

3. Operator of continuity and equations of motion in blocked media

The equation of motion for blocked media with constant value l_0 published earlier [1] and gives by expression

$$\frac{\partial}{\partial x}[P(\sigma_{ik})] = \rho\ddot{u}_i \quad (7)$$

The more detailed form is following

$$\frac{\partial}{\partial x}\left[(E + \frac{l_0^2}{3!}\Delta + \frac{l_0^4}{5!}\Delta\Delta + \cdots)\sigma_{ik}\right] = \rho\ddot{u}_i \quad (8)$$

In one-dimensional situation and for case of stationary vibration this equation takes a form

$$\frac{\partial}{\partial x}\left[(E + \frac{l_0^2}{3!}\Delta + \frac{l_0^4}{5!}\Delta\Delta + \cdots)\sigma\right] = \rho\ddot{u} \quad (9)$$

The substitution $u = e^{ikx}$ gives the dispersion equation for unknown wave number k, or for unknown wave velocity, which depends on range of structure l_0 or specific surface of sample σ_0.

$$\frac{sin(kl_0)}{kl_0} = \frac{k_S^2}{k^2} \quad (10)$$

The value k_S is the usual wave number of P or S waves. It is evident that by $l_0 \to 0$ the wave number $k \to k_S$ i.e. the wave velocity is equal to V_P or V_S elastic wave velocity. However if l_0 is not very small value, the wave velocity decreases up to zero by $kl_0 \to m\pi$, if m is integer number. In case, $sin(kl_0) < 0$, an equation (10) has complex roots, which correspond to unstable solutions. They describe damping processes and the opposite ones, i.e. catastrophes.

4. Gamma distribution of the structure sizes and the types of continuity operators

The gamma distribution for random value x takes a form [1]

$$F(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x z^{\alpha-1}e^{-\beta z}dz \quad (11)$$

$F(\infty) = 1; \alpha > 1$

This integral has a sense at $\alpha>0$, the physical sense requires the zero value of probability for infinite small cracks and we need to use more hard require (11). The average value of some quantity z takes a form

$$< z > = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x z^\alpha e^{-\beta z}dz =$$

$$= \frac{\Gamma(1+\alpha)}{\beta\Gamma(\alpha)} = 1; \beta = \alpha \quad (12)$$

The requirement of average distance between one crack to another leads the equality of two parameters in the formula (12). The deviation at mentioned average distance equal to unit has a form

$$\sigma^2 = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x ([\![z-1)]\!]^2 z^{\alpha-1}e^{-\beta z}dz = \frac{\beta^\alpha}{\Gamma(\alpha)}\left[\frac{\Gamma(2+\alpha)}{\beta^{2+\alpha}} - 2\frac{\Gamma(1+\alpha)}{\beta^{1+\alpha}} + \frac{\Gamma(\alpha)}{\beta^\alpha}\right] = \frac{1}{\alpha} \quad (13)$$

Hence in this case a parameter α is the inverse deviation. A structure of operator P with random variable distance $(l_0\xi)$ gives by expression [1].

$$P(\omega; l_0\xi) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{l_0\xi\omega} sin\theta d\theta d\varphi \qquad (14)$$

Here ξ is the random variable with Gamma distribution and unit average value. Changing the integrand to the average volume of exponent, according to the formula (4) we can write the new form of the operator *P*, corresponding to random value of distance from one crack to another with an average value equal to l_0. In this case, the role of a value ω plays the symbolic expression

$$\omega = D_x sin\theta cos\varphi + D_y sin\theta sin\varphi + D_z cos\theta \qquad (15)$$

The mathematical expectation of the value

$< e^{\xi\omega} >$ is

$$< e^{\xi\omega} \geq \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{\xi\omega}\xi^{\alpha-1}e^{-\alpha\xi}d\xi = \left(\frac{\alpha}{\alpha-\omega}\right)^\alpha \qquad (16)$$

The application of Poisson formula [3] gives an expression

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f\left(an_x + bn_y + cn_z\right) sin\theta d\theta d\varphi = \frac{1}{2}\int_{-1}^{1} f\left(l_0\sqrt{\Delta}t\right)dt \qquad (17)$$

If we use relation (17), the Poisson formula determines the form of operator of continuity P, namely

$$P(\omega; l_0\xi, \alpha) = \frac{1}{2}\int_{-1}^{1}\left(\frac{\alpha}{\alpha - l_0\sqrt{\Delta}t}\right)^\alpha dt \quad (18)$$

Thus for blocked media with random distance between cracks and pores with gamma distribution we have dispersion equation for stationary vibrations

$$\frac{1}{2}\int_{-1}^{1}\left(\frac{\alpha}{\alpha - l_0\sqrt{\Delta}t}\right)^\alpha dt = \frac{k_s^2}{k^2} \qquad (19)$$

At $\alpha\to\infty$, equation (22) takes a form (10), which published earlier [1]. The dispersion equation (22) describes the discrete set of solutions, corresponding to set of it roots. For positive values $sin(kl_0) \geq 0$ there are real roots only, while for negative values $sin(kl_0) < 0$, there are complex values, corresponding to damping or infinitely growing solutions (catastrophes).

5. The different forms of dispersion equations

The approach we can make, calculating the average value of the integral (5), not integrand of it. It means we calculate the average value the left hand of (23), and the value l_0 has a Gamma distribution. The mathematical expectation of it takes a form

$$M\left(\frac{sin(kl_0)}{kl_0}\right) = \frac{\alpha^\alpha}{\Gamma(\alpha)}\int_0^\infty \frac{sin(kl_0x)}{kl_0x} x^{\alpha-1} e^{-\alpha x}dx \ (20)$$

The integral (20) rewrites by following

$$\frac{\alpha^\alpha}{\Gamma(\alpha)}\int_0^\infty \frac{sin(kl_0x)}{kl_0x} x^{\alpha-1} e^{-\alpha x}dx = \frac{1}{2ikl_0}\frac{\propto^\propto}{\Gamma(\alpha)} \times \int_0^\infty\left[e^{x(\alpha-ikl_0)} - e^{x(\alpha+ikl_0)}\right]dx = \frac{1}{2ikl_0}\frac{\propto^\propto}{\Gamma(\alpha)}\left[\left(\frac{1}{\alpha-ikl_0}\right)^\alpha - \left(\frac{1}{\alpha+ikl_0}\right)^{\alpha-1}\right] \ (21)$$

For large value α, (small variance σ^2) there is an asymptotic representation by expression

$$\frac{\alpha^\alpha}{\Gamma(\alpha)}\int_0^\infty \frac{sin(kl_0x)}{kl_0x} x^{\alpha-1} e^{-\alpha x}dx = \frac{1}{2ikl_0}\frac{\propto^\propto}{\Gamma(\alpha)} \times \int_0^\infty\left[e^{x(\alpha-ikl_0)} - e^{x(\alpha+ikl_0)}\right]dx = \frac{1}{2ikl_0}\frac{\propto^\propto}{\Gamma(\alpha)}\left[\left(\frac{1}{\alpha-ikl_0}\right)^\alpha - \left(\frac{1}{\alpha+ikl_0}\right)^{\alpha-1}\right] \ (22$$

In the case $\alpha = 1$ (exponential distribution)

$$M_1\left(\frac{sin(kl_0)}{kl_0}\right) = \int_0^\infty \frac{sin(kl_0x)}{kl_0x} e^{-x}dx = \frac{arctan(kl_0)}{kl_0} \qquad (23)$$

Thus (23) for Gamma distribution of the average distance from one pore to another one there are dispersion equation

$$\frac{1}{2ikl_0}\frac{\propto^\propto}{\Gamma(\alpha)}\left[\left(\frac{1}{\alpha-ikl_0}\right)^\alpha - \left(\frac{1}{\alpha+ikl_0}\right)^{\alpha-1}\right] = \frac{k_s^2}{k^2} \qquad (24)$$

The asymptotic dispersion equation for small variance $\sigma^2$is

$$\frac{\alpha}{\alpha-1}\left[\frac{sin(kl_0)}{kl_0}-\frac{cos(kl_0)}{\alpha}\right]=\frac{k_S^2}{k^2} \qquad (25)$$

In the special case, $\alpha=1$ the dispersion equation is very simple, namely

$$\frac{arctan(kl_0)}{kl_0}=\frac{k_S^2}{k^2} \qquad (26)$$

At $\alpha\to\infty$ (variance $\sigma^2\to 0$) the equation (25) takes his usual form (10).

As to equation (26), this behavior at the small sizes of structures is analogous like equation (10), but at increasing of structure, sizes there are no complex roots in (26). It means that in blocked media there are some distributions of structure sizes, which exclude catastrophes.

Some words about statics. Equations of equilibrium do not contain a factor k_S^2 in formulas (24-26). Thus for zero frequencies the equations (24, 25, 26) take

$$\frac{sin(kl_0)}{kl_0}=0;\ \frac{arctan(kl_0)}{kl_0}=0;$$

$$\frac{sin(kl_0)}{kl_0}-\frac{cos(kl_0)}{\alpha}=0 \qquad (27)$$

The first equation (27) is satisfied at $kl_0=m\pi$ (m is the integer). The second one in (27) is satisfied asymptotically only (at $kl_0\to\infty$). The third equation takes a form

$$\frac{sin(kl_0)}{kl_0}=\frac{cos(kl_0)}{\alpha} \qquad (28)$$

The first equation (27) has not complex roots, and the second one (27) too. The third equation (27), (28) has complex roots. It means that in this case can be catastrophes in statics.

The integral in (21) calculates analytically, namely

$$\frac{1}{2}\int_{-1}^{1}(1-i\beta xt)^{-\frac{1}{\beta}}dt=-\frac{1}{2ix(1-\beta)}\cdot\left[(1+i\beta x)^{1-\frac{1}{\beta}}-(1-i\beta x)^{1-\frac{1}{\beta}}\right]\approx\frac{1}{1-\beta}\left(\frac{sinx}{x}-\beta cosx\right) \qquad (29)$$

In the last formula $\beta=1/\alpha=\sigma^2$ and we suppose, that $(1+i\beta x)^{-1/\beta}\approx e^{-ix}; x=kl_0$.

The equation (29) for small values β may change of approximate dispersion equation with accuracy of β^2 by following expression

$$\frac{1}{2}\int_{-1}^{1}(1-i\beta xt)^{-1/\beta}dt=\frac{1}{1-\beta}\left(\frac{sinx}{x}-\beta cosx\right) \qquad (30)$$

Using another representation of the integral (30), there is a possibility to have more accurate dispersion equation.

$$\frac{1}{2}\int_{-1}^{1}(1-i\beta xt)^{-1/\beta}dt=(1-i\beta x)^2(1-i\beta x)^{-1-\frac{1}{\beta}}-(1+i\beta x)^2(1+i\beta x)^{-1-\frac{1}{\beta}} \qquad (31)$$

In this case, there is dispersion equation with accuracy of β^3, i.e.

$$\frac{1}{1-\beta}\times\left(\frac{sin(1+\beta)x}{x}-2\beta\, cos(1+\beta)\,x-\beta^2\, xsin(1+\beta)\,x\right)=\frac{\varepsilon^2}{x^2} \qquad (32)$$

Here is $\varepsilon=k_S l_0$. Analogously, using the same method, we can get the expansion with accuracy to β^4, i.e.

$$\frac{1}{1-\beta}\times\left(\frac{sin(1+\beta)x}{x}-2\beta\, cos(1+\beta)\,x-\beta^2\, x\, sin(1+\beta)\,x\right)=\frac{\varepsilon^2}{x^2} \qquad (33)$$

It gives the dispersion equation in the form

$$\frac{1}{1-\beta}\times\left(\frac{sin(1+2\beta)x}{x}-3\beta\, cos(1+2\beta)\,x-6\beta^2\, xsin(1+2\beta)\,x+\beta^3x^2\, cos(1+2\beta)\,x\right)=\frac{\varepsilon^2}{x^2} \qquad (34)$$

6. The Gutenberg- Richter Law deviations

The numerical investigations of dispersion equations in different approximations (35), (37), (39) and the case of zero variance (31) give on the Fig.1. The all graphs represent a relation between numbers of complex roots (unstable states) of dispersion equation (22) depend on the specific surface of pores and cracks. The energy of seismic waves is proportional to specific surface of cracks. It gives a possibility to compare theoretical graphs the number of complex roots versus specific surface and practical observation of analogous relations between the number of earthquakes and their energy. This theoretical relation in logarithmic scale in the case zero variance closed to linear function. The tangent of this line closed to ½, like in real observation in seismology. However, the variance causes the deviations from mentioned relation. The rough approximation with accuracy to σ^2 corresponds to the Fig 1. a) the formula (30). The more accurate approximation with accuracy to σ^4 corresponds to the Fig1.b). Formula (32). The final approximation with accuracy to σ^6 corresponds to the Fig1.c.) Formula (34). More exactly on the Fig.1 on the horizontal exes is not energy itself, but dimensionless value $\lambda\sigma_0 / 8\pi(1-f)$. This parameter contains the specific surface of cracks σ_0, which is proportional to deficit of potential energy due to cracks. This energy is forming the kinetic energy of seismic waves. Figure1. The number of complex roots N versus energy E for different variance σ^2 Log-log scales a) Linear in σ^2 approximation; b) addition of square in σ^2 terms; c) addition of cubic in σ^2 terms.

On the Fig.2 there are experimental relations between the number of acoustic events and their energy (by Vinogradov S. D., 1989)

Another observation considered to seismological examples. In the Altay region (Russia), there are some set observations, which give the changing of a slope of the diagram (NE), i.e. the number of events versus energy. This material obtained by Emanov at all in 2005 [5]. On the Fig.3, the horizontal exes represent classes of energy K_p, which are proportional to the energy. In seismology, there are discussions about a nature of deviations from the Gutenberg-Richter law. Some authors explain these effects due to not good statistics of them. However, this paper represents another viewpoint. There are physical phenomena, which causes the mentioned deviations.

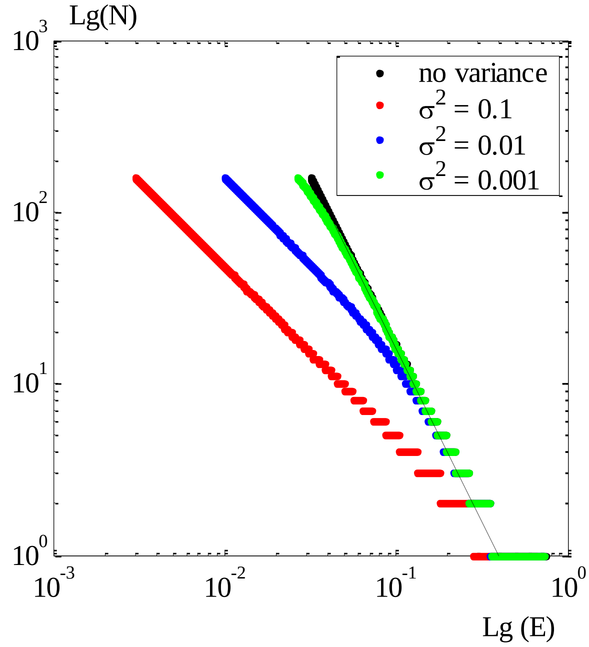

Fig.1a
Relation between energy versus numbers of unstable solutions (the first approximation, formula (30)).

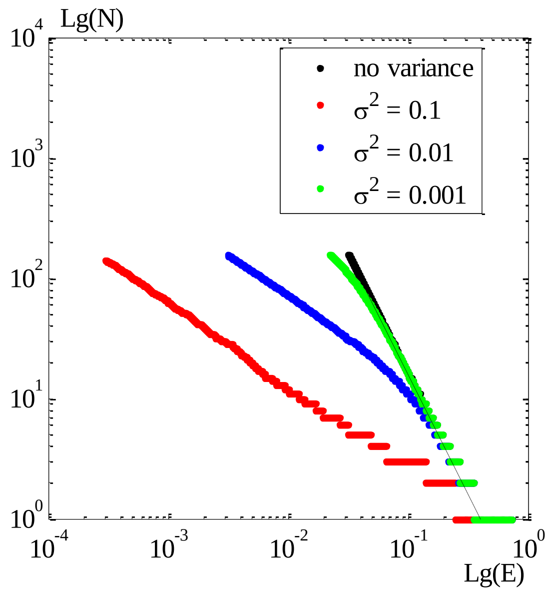

Fig.1b Relation between energy versus numbers of unstable solutions (the second approximation, formula (32)).

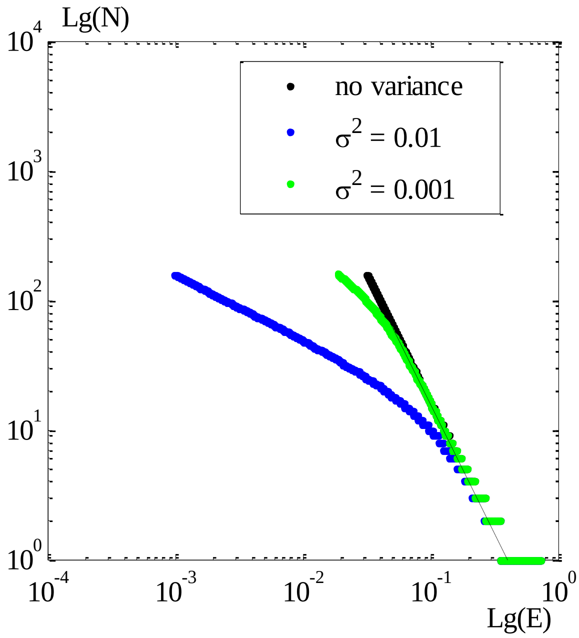

Fig.1c

Relation between energy versus numbers of unstable solutions (the third approximation, formula (34)).

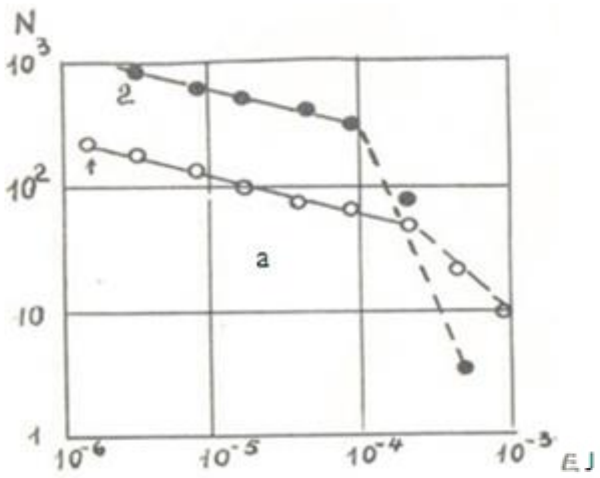

Fig. 2 a

Experimental diagrams of N (E, Joules). (The number of seismic events versus energy) dependence for cement with silica sand samples. These results obtained in the experiments with different sample and press plate contact conditions. (Vinogradov S.D. 1989)

1. Experiments without spacers, 2 experiments with spacers, a– coarse break stone samples.

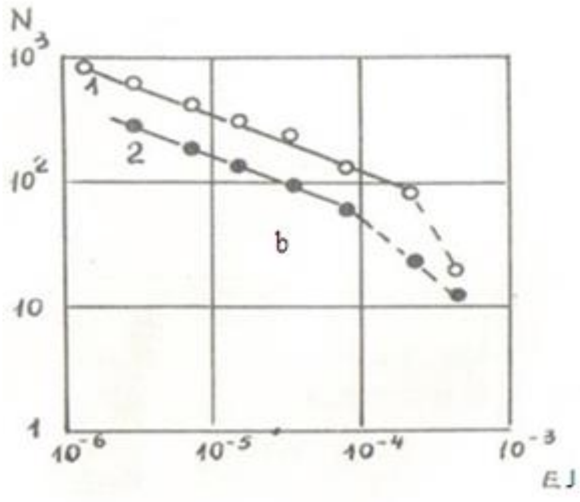

Fig.2b

Experimental diagrams of N (E, Joules). (The number of seismic events versus energy) dependence for cement with silica sand samples. 1 Experiments without spacers, 2 – experiments with spacers, b - fine break stone samples.

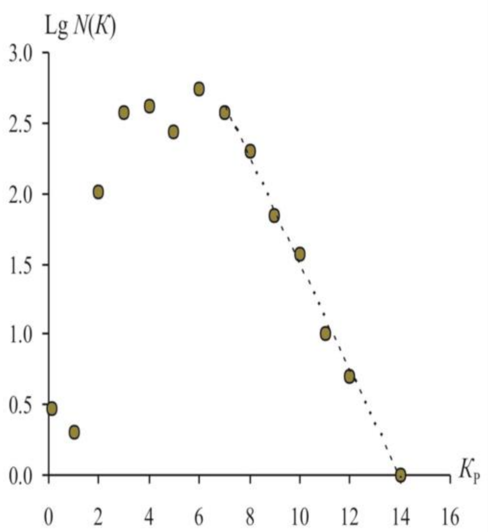

Fig. 3

Experimental dependence seismic events number N versus energy K_p. (Emanov A. F. at all, 2005).

7. Conclusions

1. The model of structured continuum describes by partial differential equations of the infinite order due to large degrees of freedom in blocked medium. Besides, of usual *P* and *S* waves this model predicts many waves with abnormally small velocities, which not bound below.

2. Dynamics equation of structural continuum gives unstable solutions, which correspond to complex roots of dispersion equation. The number of complex roots depending of specific surface of cracks represents as a straight line in the logarithmic scale if the specific surface and the distance between two cracks is a constant. This straight line corresponds to well-known Gutenberg-Richter Law.

3. The variance of block sizes causes the deviations from the Gutenberg-Richter Law. For large energy, this effect is small, while for small energy there is sufficient. Instead of linear relation, there is a curve with positive curvature.

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