Phenomenological Analysis of the Additive Combinational Internal Resonance in Nonlinear Vibrations of Fractionally Damped Thin Plates

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Abstract: The dynamic behaviour of a nonlinear plate embedded into a fractional derivative viscoelastic medium is studied by the method of multiple time scales under the conditions of the combinational internal resonances of additive-difference type using a newly developed approach resulting in uncoupling the linear parts of equations of motion of the plate. The influence of viscosity on the energy exchange mechanism between interacting nonlinear modes has been analyzed. The phenomenological analysis carried out for the combinational internal resonances of the additive type with the help of the phase portraits constructed for different magnitudes of the plate parameters reveals the great variety of vibrational motions: stationary vibrations, two-sided energy exchange between two subsystems under consideration, and one-sided energy interchange resulting in the complete one-sided energy transfer.

Key–Words: Nonlinear elastic plate, Free nonlinear damped vibrations, Combinational internal resonances, Fractional derivative, Method of multiple time scales

1 Introduction

It is well known that the nonlinear vibrations of plates are an important area of applied mechanics, since plates are used as structural elements inmany fields of industry and technology [1]. Extensive review of recent research developments in the field could be found in Amabili [2, 3] and Sathyamoorthy [4].

Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in multimode response with a strong interaction of the modes involved [5] accompanied by the energy exchange phenomenon.

Nonlinear free vibrations of a thin plate embedded into a fractional derivative viscoelastic medium have been considered recently [6, 7] for the case when the plate motion is described by three coupled nonlinear differential equations. It has been shown that the occurrence of the internal resonance results in the interaction of modes corresponding to the mutually orthogonal displacements. As this takes place, the displacement functions are determined in terms of eigenfunctions of linear vibrations. The procedure resulting in decoupling linear parts of equations has been proposed with the further utilization of the method of multiple scales for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales.

It has been shown that the phenomenon of internal resonance could be very critical, since in the thin plate under consideration the internal resonance is always present. Moreover, its type depends on the order of smallness of the viscosity involved into consideration. Thus, at the ε-order, damped vibrations occur within the two-to-one and one-to-one-to-two internal resonance [6]. Other types of the internal resonance, such as one-to-one, one-to-one-to-one, and combinational resonances of the additive and difference types could be found at ε^2-order [7, 8], i.e., the type of the resonance depends on the order of smallness of the fractional derivative entering in the equations of motion of the plate.

The phenomenological analysis has been carried out in [9] for the one-to-one internal resonance coupling two interacting modes using the hydrodynamic analogy suggested in [10].

In the present paper, the qualitative analysis of the combinational internal resonance of the additive type, resulting in coupling of three different modes, is carried out with the help of the phase portraits constructed for different magnitudes of the plate parameters, what allows us to study the great variety of vibrational motions: stationary vibrations, periodic energy exchange between three subsystems under consideration, and one-sided energy interchange resulting in the complete one-sided energy transfer.

2 Governing Equations Describing the Additive Combinational Resonance $\omega_1 + \omega_2 = 2\omega_3$

In the recent paper by Rossikhin et al. [7] it has been shown that the following three combinational resonances could occur during vibrations of a free supported non-linear thin rectangular plate (Fig. 1):

$$\omega_1 + \omega_2 = 2\omega_3, \tag{1}$$
$$\omega_1 - \omega_2 = 2\omega_3, \tag{2}$$
$$\omega_2 - \omega_1 = 2\omega_3, \tag{3}$$

where ω_1 and ω_2 are some particular natural frequencies of in-plane vibrations, and ω_3 is one of the natural frequencies of the out-of-plane modes.

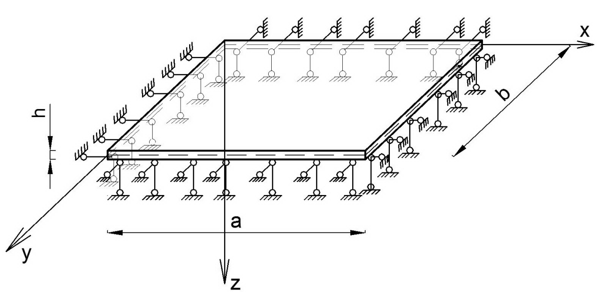

Figure 1: Scheme of a freely supported rectangular plate

Reference to (1)-(3) shows that the combinational resonance (1) is of the additive type, while combinational resonances (2) and (3) are of the difference type. In the present paper, we will focus our attention on the qualitative analysis of the case of the additive $\omega_1 + \omega_2 = 2\omega_3$ combinational internal resonance, when two different modes of in-plane vibrations are coupled with a certain mode of out-of-plane vibrations.

Using the approach suggested in [7], it could be shown that in the case of the additive combinational resonance (1) the set of equations describing the modulations of amplitudes a_i and phases φ_i $(i = 1, 2, 3)$ has the following form:

$$\left(a_1^2\right)^{\cdot} + s_1 a_1^2 = -2\omega_1^{-1}\zeta_1 k_8 a_1 a_2 a_3^2 \sin\delta, \tag{4}$$

$$\left(a_2^2\right)^{\cdot} + s_2 a_2^2 = -2\omega_2^{-1}\zeta_2 k_7 a_1 a_2 a_3^2 \sin\delta, \tag{5}$$

$$\left(a_3^2\right)^{\cdot} + s_3 a_3^2 = \omega_3^{-1}(\zeta_{13}k_8 + \zeta_{23}k_7) a_1 a_2 a_3^2 \sin\delta, \tag{6}$$

$$\begin{aligned}\dot\varphi_1 &= \frac{1}{2}\,\sigma_1 + \omega_1^{-1}\zeta_1 k_5\, a_3^2 \\ &+ \omega_1^{-1}\zeta_1 k_8 a_1^{-1} a_2 a_3^2 \cos\delta,\end{aligned} \tag{7}$$

$$\begin{aligned}\dot\varphi_2 &= \frac{1}{2}\,\sigma_2 + \omega_2^{-1}\zeta_2 k_6\, a_3^2 \\ &+ \omega_2^{-1}\zeta_2 k_7 a_1 a_2^{-1} a_3^2 \cos\delta,\end{aligned} \tag{8}$$

$$\begin{aligned}\dot\varphi_3 &= \frac{1}{2}\,\sigma_3 + \frac{1}{2}\,\omega_3{}^{-1}\zeta_{13}k_7 a_1^2 + \frac{1}{2}\,\omega_3{}^{-1}\zeta_{23}k_8 a_2^2 \\ &+ \frac{1}{2}\,\omega_3{}^{-1}\left(\zeta_{13}k_2 + \zeta_{23}k_4\right) a_3^2 \\ &+ \frac{1}{2}\,\omega_3{}^{-1}\left(\zeta_{13}k_8 + \zeta_{23}k_7\right) a_1 a_2 \cos\delta,\end{aligned} \tag{9}$$

where the phase difference has the form $\delta = 2\varphi_3 - \varphi_2 - \varphi_1$, an overdot denotes the differentiation with respect to T_2, ζ_m, ζ_{mn} and k_j $(m = 1, 2,\ n = 3,\ j = 1, 2, ...8)$ are some coefficients depending on the interacting modes [7], $s_i = \mu_i \tau_i^\gamma \omega_i^{\gamma-1}\sin\psi$, $\sigma_i = \mu_i \tau_i^\gamma \omega_i^{\gamma-1}\cos\psi$ $(i = 1, 2, 3)$, $\psi = \frac{1}{2}\ \pi\gamma$, τ_i is the relaxation time of the ith generalized displacement, μ_i is the viscosity coefficient of the ith mode, and $0 < \gamma \le 1$ is the fractional parameter [7].

Introducing new functions $\xi_1(T_2)$, $\xi_2(T_2)$, and $\xi_3(T_2)$ such that

$$\begin{aligned}a_1^2 &= \frac{\zeta_1 k_8}{\omega_1}\,\xi_1 \exp(-s_1 T_2), \\ a_2^2 &= \frac{\zeta_2 k_7}{\omega_2}\,\xi_2 \exp(-s_2 T_2), \\ a_3^2 &= \frac{\zeta_{13}k_8 + \zeta_{23}k_7}{\omega_3}\,\xi_3 \exp(-s_3 T_2),\end{aligned} \tag{10}$$

and adding Eqs. (4)-(6) with due account for (10) yield

$$\dot\xi_1 e^{-s_1 T_2} + \dot\xi_2 e^{-s_2 T_2} + 4\dot\xi_3 e^{-s_3 T_2} = 0, \tag{11}$$

while subtracting (7) and (8) from the doubled (9) we obtain

$$\begin{aligned}\dot\delta &= 2\dot\varphi_3 - \dot\varphi_1 - \dot\varphi_2 \\ &= \Sigma + \left[\omega_3{}^{-1}\left(\zeta_{13}k_2 + \zeta_{23}k_4\right)\right. \\ &- \left.\omega_1{}^{-1}\zeta_1 k_5 - \omega_2{}^{-1}\zeta_2 k_6\right] a_3^2 \\ &+ \omega_3{}^{-1}\zeta_{13}k_7 a_1^2 + \omega_3{}^{-1}\zeta_{23}k_8 a_2^2 \\ &+ \left[\omega_3{}^{-1}\left(\zeta_{13}k_8 + \zeta_{23}k_7\right) a_1 a_2\right. \\ &\left. - \omega_1{}^{-1}\zeta_1 k_8\ \frac{a_2 a_3^2}{a_1} - \omega_2{}^{-1}\zeta_2 k_7\ \frac{a_1 a_3^2}{a_2}\right]\cos\delta,\end{aligned} \tag{12}$$

where $2\Sigma = 2\sigma_3 - \sigma_1 - \sigma_2$.

Considering (10), equations (4), (6) and (12) could be rewritten in the following form:

$$\dot{\xi}_1 = -2b\sqrt{\xi_1\xi_2}\,\xi_3 e^{-(s_3+1/2s_2-1/2s_1)T_2}\sin\delta, \quad (13)$$

$$\dot{\xi}_3 = b\sqrt{\xi_1\xi_2}\,\xi_3 e^{-1/2(s_2+s_1)T_2}\sin\delta, \quad (14)$$

$$\begin{aligned}\dot{\delta} - \Sigma &= K_1\xi_1 e^{-s_1T_2} + K_2\xi_2 e^{-s_2T_2} + K_3\xi_3 e^{-s_3T_2} \\ &+ \left(\frac{1}{2}\frac{\dot{\xi}_1}{\xi_1} + \frac{1}{2}\frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3}\right)\cot\delta \\ &= K_1\xi_1 e^{-s_1T_2} + K_2\xi_2 e^{-s_2T_2} + K_3\xi_3 e^{-s_3T_2} \\ &+ b\left(\sqrt{\xi_1\xi_2}e^{-1/2(s_1+s_2)T_2}\right. \\ &- \frac{\xi_3\sqrt{\xi_2}}{\sqrt{\xi_1}}e^{-(s_3+1/2s_2-1/2s_1)T_2} \\ &- \left.\frac{\xi_3\sqrt{\xi_1}}{\sqrt{\xi_2}}e^{-(s_3+1/2s_1-1/2s_2)T_2}\right)\cos\delta, \quad (15)\end{aligned}$$

where

$$b = \frac{\zeta_{13}k_8 + \zeta_{23}k_7}{\omega_3}\sqrt{\frac{\zeta_1\zeta_2k_7k_8}{\omega_1\omega_2}},$$

$$K_1 = \frac{\zeta_1\zeta_{13}k_7k_8}{\omega_1\omega_3}, \quad K_2 = \frac{\zeta_2\zeta_{23}k_7k_8}{\omega_2\omega_3},$$

$$K_3 = \frac{\zeta_{13}k_8 + \zeta_{23}k_7}{2\omega_3}\left(\frac{\zeta_{13}k_2 + \zeta_{23}k_4}{\omega_3} - \frac{\zeta_1k_5}{\omega_1} - \frac{\zeta_2k_6}{\omega_2}\right)$$

The non-linear set of Eqs. (11), (13), (14), and (15), with the initial conditions

$$\xi_1\Big|_{T_2=0} = \xi_{10}, \quad \xi_2\Big|_{T_2=0} = \xi_{20}, \quad \xi_3\Big|_{T_2=0} = \xi_{30}, \quad \delta\Big|_{T_2=0} = \delta_0 \quad (16)$$

completely describe the vibrational process of the mechanical system being investigated under the condition of the additive combinational internal resonance $2\omega_3 = \omega_1 + \omega_2$, and could be solved numerically.

In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, Eq. (11) has the form

$$\dot{\xi}_1 + \dot{\xi}_2 + 4\dot{\xi}_3 = 0, \quad (17)$$

whence it follows that

$$\xi_1 + \xi_2 + 4\xi_3 = E_0, \quad (18)$$

and

$$\xi_1 = 2E_0(c_1-\xi), \quad \xi_2 = 2E_0(c_2-\xi), \quad \xi_3 = E_0(c_3+\xi), \quad (19)$$

where c_1, c_2, and c_3 are constants of integration such that

$$2c_1 + 2c_2 + 4c_3 = 1.$$

Considering (19), Eqs. (13) and (15) are reduced to

$$\dot{\xi} = 2bE_0\sqrt{(c_1-\xi)(c_2-\xi)}\,(c_3+\xi)e^{-sT_2}\sin\delta, \quad (20)$$

$$\begin{aligned}\dot{\delta} &= 2E_0\left[K_1(c_1-\xi)+K_2(c_2-\xi)+K_3(c_3+\xi)\right]e^{-sT_2} \\ &+ bE_0\left[2\sqrt{(c_1-\xi)(c_2-\xi)} - (c_3+\xi)\sqrt{\frac{c_2-\xi}{c_1-\xi}}\right. \\ &- \left.(c_3+\xi)\sqrt{\frac{c_1-\xi}{c_2-\xi}}\right]e^{-sT_2}\cos\delta. \quad (21)\end{aligned}$$

The set of equations (20) and (21) could be integrated, resulting in its first integral

$$\begin{aligned}G(\xi,\delta) &= (c_3+\xi)\sqrt{(c_1-\xi)(c_2-\xi)}\cos\delta \\ &- \frac{1}{2}K_1b^{-1}(c_1-\xi)^2 - \frac{1}{2}K_2b^{-1}(c_2-\xi)^2 \\ &+ \frac{1}{2}K_3b^{-1}(c_3+\xi)^2 = G_0(\xi_0,\delta_0). \quad (22)\end{aligned}$$

This first integral (22) defines the stream function $G(\xi,\delta)$ such that

$$v_\xi = \dot{\xi} = -2bE_0\frac{\partial G}{\partial\delta}e^{-sT_2}, \quad v_\delta = \dot{\delta} = 2bE_0\frac{\partial G}{\partial\xi}e^{-sT_2},$$

which describes steady-state vibrations of an elastic plate decaying with time.

3 Numerical Investigations

Now let us carry out the qualitative analysis of the case of the additive $\omega_1 + \omega_2 = 2\omega_3$ combinational internal resonance, when two different modes of in-plane vibrations are coupled with a certain mode of out-of-plane vibrations.

For this case, the stream-function $G(\xi,\delta)$ is defined by relationship (22), and the phase portrait to be constructed according to (22) depends essentially on the magnitudes of the coefficients K_1, K_2, and K_3.

3.1 The Case $K_1 = K_2 = K_3 = 0$

For this case, the stream-function $G(\xi,\delta)$ (22) is reduced to

$$G(\xi,\delta) = (c_3+\xi)\sqrt{(c_1-\xi)(c_2-\xi)}\cos\delta = G_0(\xi_0,\delta_0), \quad (23)$$

whence it follows that the stream-function also depends on the constants of integration c_1, c_2, and c_3.

Eliminating δ from (20) and (21) and integrating over the time the resulting equation, we obtain

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{(c_3+\xi)^2(c_1-\xi)(c_2-\xi)-G_0^2}} = \frac{2bE_0}{s}\left(1-e^{-sT_2}\right). \qquad (24)$$

The solution of (24) allows one to find the value $\xi(T_2)$, and thus, to solve the problem under consideration.

3.1.1 The Subcase $c_1 = c_2 = 0$ and $c_3 = \frac{1}{4}$

The stream function (23) takes the form

$$G(\xi,\delta) = (\frac{1}{4}+\xi)(-\xi)\cos\delta = G_0(\xi_0,\delta_0).$$

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Fig. 2. Magnitudes of G are indicated by digits near the curves which correspond to the stream-lines; the flow direction of the phase fluid elements are shown by arrows on the stream-lines.

In the case under consideration, the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = -2bE_0\left(\frac{1}{4}+\xi\right)\xi\sin\delta\; e^{-sT_2},$$

$$v_\delta = \dot{\delta} = -2bE_0\left(\frac{1}{4}+2\xi\right)\cos\delta\; e^{-sT_2}.$$

Reference to Figure 2 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$. As this takes place, the flow in each such rectangle becomes isolated. On three sides of the rectangle, namely: $\xi = 0$ and $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$, $G = 0$ and inside each rectangle the value G preserves its sign. Along the side $\xi = 1$ the stream function G changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 2).

The lines with $G = 0$ are separatrixes which are connected with each other at the stationary saddle-like points with coordinates $\xi = \xi_0 = 0$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$.

All stream-lines inside the rectangle are nonclosed, in so doing their initial and terminal points locate on the line $\xi = 1$. The distribution of the velocities of the phase fluid points is shown in Fig. 2 along the lines $\xi = 0, 0.3, 0.5, 0.7, 1$ and $\delta = 0$ and $-\frac{\pi}{2}$, wherein $v^*_{\delta \text{ or } \xi} = \frac{v_{\delta \text{ or } \xi}}{2bE_0}$.

Along the line $\xi = 0$ the phase modulated regime decaying with time is realized

$$\xi(T_2) = \xi_0 = 0$$
$$\ln\left|\tan\left(\frac{1}{2}\delta + \frac{\pi}{4}\right)\right|\Big|_{\delta_0}^{\delta} = -\frac{bE_0}{2s}\left(1-e^{-sT_2}\right).$$

Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ in the presence of conventional viscosity the solution could be written for the amplitude modulated regimes decaying with time

$$\ln\left|\frac{\xi}{\frac{1}{4}+\xi}\right|_{\xi_0}^{\xi} = \mp\frac{bE_0}{2s}\left(1-e^{-sT_2}\right),$$
$$\delta(T_2) = \delta_0 = \pm\frac{\pi}{2} \pm 2\pi n, \quad n = 0, 1, 2, ...$$

3.1.2 The Subcase $c_1 = c_2 = c_3 = \frac{1}{8}$

The stream function (23) takes the form

$$G(\xi,\delta) = (\frac{1}{64}-\xi^2)\cos\delta = G_0(\xi_0,\delta_0).$$

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Figure 3, from which it is evident that the infinite channel $(-\infty < \delta < \infty)$ bounded by the lines $\xi = 0$ and $\xi = 1$ is divided into a set of rectangles by the lines $\xi = \frac{1}{8}$ and $\delta = \pm\frac{\pi}{2} \pm 2\pi n$ $(n = 0, 1, 2, ...)$. Within each rectangle, the value of G preserves its sign and all stream-lines are nonclosed with initial and terminal points locating on the boundary lines $\xi = 0$ and $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ and $\xi = 0$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 3).

In the case under consideration, the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0\left(\frac{1}{64}-\xi^2\right)\sin\delta\; e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0\,(-2\xi)\cos\delta\; e^{-sT_2}$$

whence it follows that along the separatrixes with $G(\xi,\delta) = 0$, which are the boundaries of rectangles, the following analytical solutions could be found:
the phase modulated regime along the line $\xi = 1/8$

$$\xi(T_2) = \xi_0 = \frac{1}{8}$$
$$\ln\left|\tan\left(\frac{1}{2}\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = -\frac{bE_0}{2s}\left(1-e^{-sT_2}\right),$$

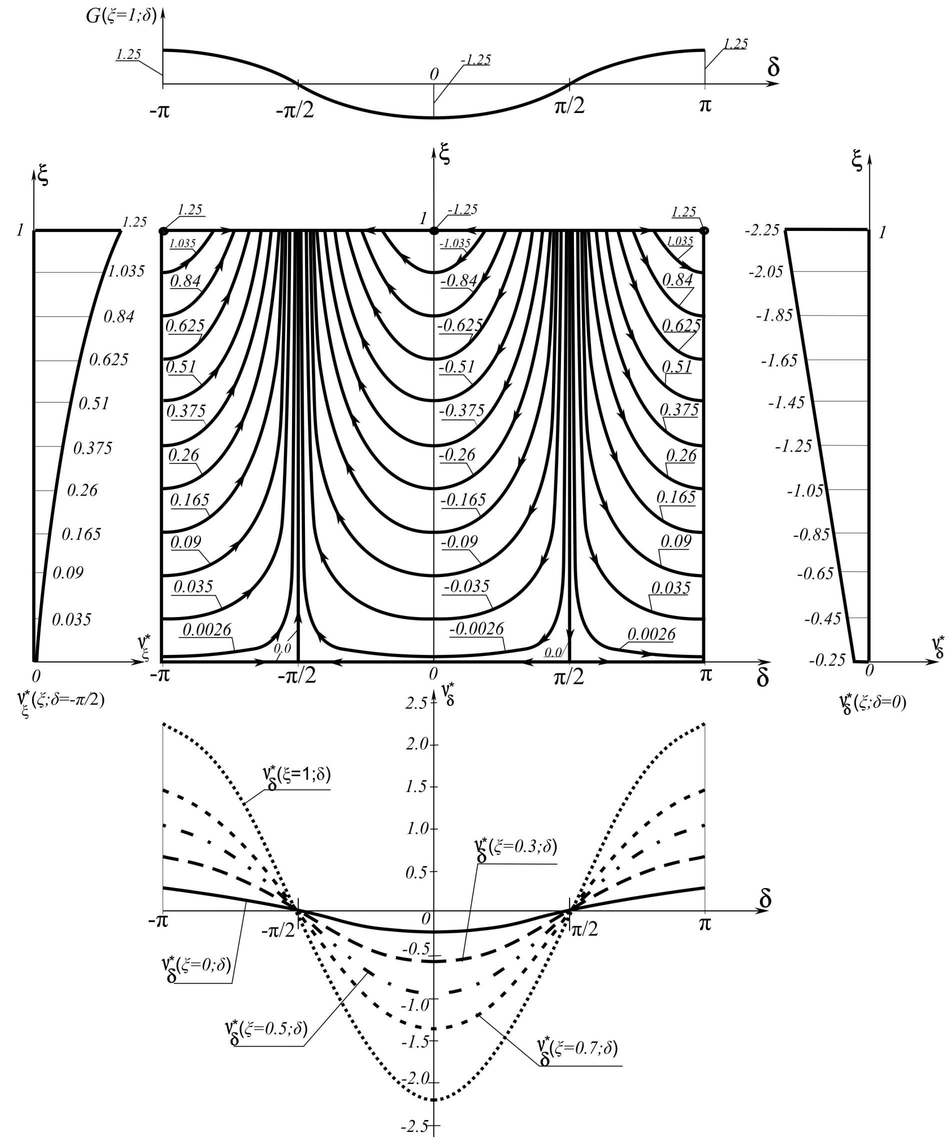

Figure 2: Phase portrait for the case of the additive combinational internal resonance at $K_1 = K_2 = K_3 = 0$ and $c_1 = c_2 = 0$, $c_3 = \frac{1}{4}$

and along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ the amplitude modulated regimes decaying with time

$$\left|\ln \frac{\xi - \frac{1}{8}}{\xi + \frac{1}{8}}\right|_{\xi_0}^{\xi} = \mp \frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

$$\delta(T_2) = \delta_0 = \pm\frac{\pi}{2} \pm 2\pi n, \quad n = 0, 1, 2, ...$$

in so doing the separatrixes are connected with each other at the saddle-like points with coordinates $\xi = \xi_0 = 1/8$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ corresponding to unstable stationary regimes.

On the low boundary $\xi = 0$ at any magnitude of the phase difference δ the velocity $v_\delta = \dot{\delta}(T_2) = 0$, while the velocity $v_\xi = \dot{\xi}(\delta, T_2)$ vanishes to zero at $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$. Therefore, the center-like points with coordinates $\xi = \xi_0 = 0$, $\delta = \delta_0 = 2\pi n$, $G = 1/64$ and $\xi = \xi_0 = 0$, $\delta = \delta_0 = \pi \pm 2\pi n$, $G = -1/64$ correspond to stable stationary regimes.

3.1.3 The Subcase $c_1 = c_2 = \frac{1}{4}$, $c_3 = 0$

The stream function (23) takes the form

$$G(\xi, \delta) = (\frac{1}{4} - \xi)\xi \cos\delta = G_0(\xi_0, \delta_0).$$

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Fig. 4. Reference to Figure 4 shows that the infinite channel $(-\infty < \delta < \infty)$

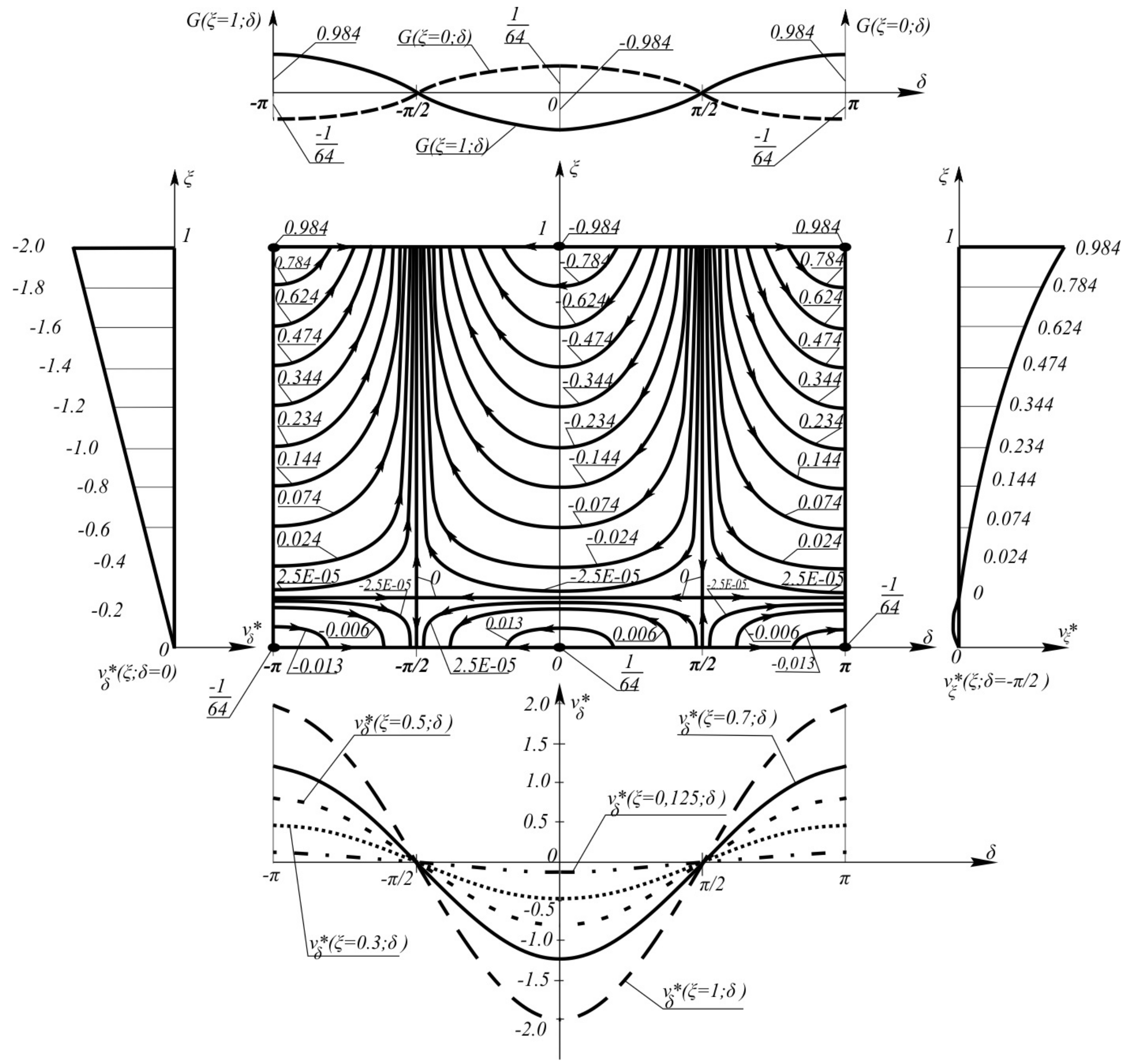

Figure 3: Phase portrait for the case of the additive combinational internal resonance at $K_1 = K_2 = K_3 = 0$ and $c_1 = c_2 = c_3 = \frac{1}{8}$

bounded by the lines $\xi = 0$ and $\xi = 1$ is divided into a set of rectangles by the lines $\xi = \frac{1}{4}$ and $\delta = \pm\frac{\pi}{2} \pm 2\pi n \quad (n = 0, 1, 2, ...)$. Within each rectangle, the value of G preserves its sign. As this takes place, in the upper rectangles all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1, \delta = \pm\pi n \, (n = 0, 1, 2, ...)$ (Fig. 4), while along the line $\xi = 0$ the magnitude of the stream function is constant and equal to $G = 0$, and in the bottom rectangles all stream-lines are closed. The function G attains its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n \, (n = 0, 1, 2, ...)$ and $\xi = \frac{1}{8}$, $\delta = \pm\pi n \, (n = 0, 1, 2, ...)$ within the upper and bottom rectangles, respectively, in so doing the points with the coordinates $\xi = \xi_0 = \frac{1}{8}$, $\delta = \delta_0 = \pm\pi n$ are centers corresponding to stationary motions.

In the case under consideration, the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0 \left(\frac{1}{4} - \xi\right) \xi \sin\delta \, e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0 \left(\frac{1}{4} - 2\xi\right) \cos\delta \, e^{-sT_2},$$

whence it follows that along the separatrixes with $G(\xi, \delta) = 0$, which are the boundaries of rectangles, the following analytical solutions could be found:

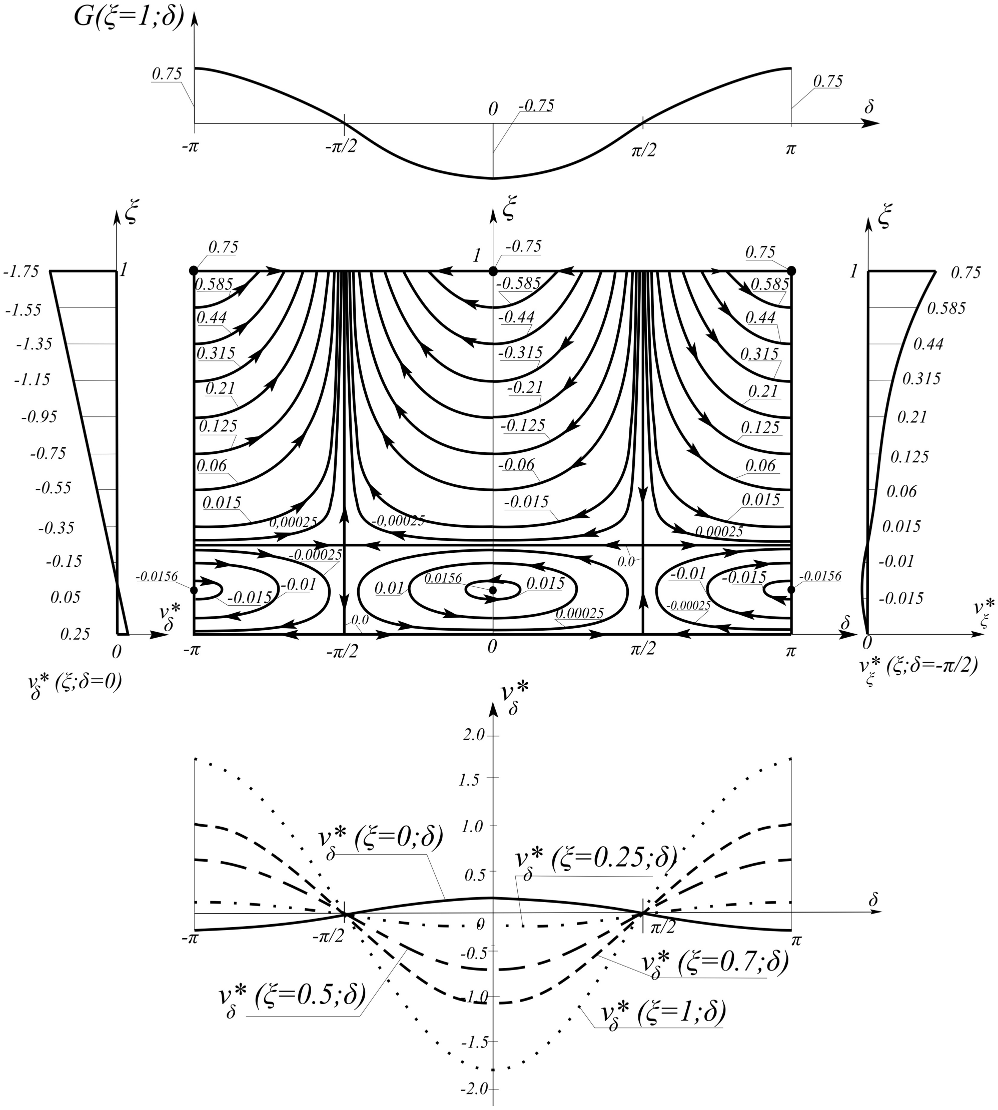

Figure 4: Phase portrait for the case of the additive combinational internal resonance at $K_1 = K_2 = K_3 = 0$ and $c_1 = c_2 = \frac{1}{4}$, $c_3 = 0$

the phase modulated regime along the line $\xi = 0$

$$\xi(T_2) = \xi_0 = 0$$
$$\ln\left|\tan\left(\frac{1}{2}\,\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = \frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

the phase modulated regime along the line $\xi = 1/4$

$$\xi(T_2) = \xi_0 = \frac{1}{4}$$
$$\ln\left|\tan\left(\frac{1}{2}\,\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = \frac{3bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

and along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ the amplitude modulated regimes decaying with time

$$\left|\ln\frac{\xi}{\xi - \frac{1}{4}}\right|_{\xi_0}^{\xi} = \mp\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$
$$\delta(T_2) = \delta_0 = \pm\frac{\pi}{2} \pm 2\pi n, \quad n = 0, 1, 2, ...$$

in so doing the separatrixes are connected with each other at the saddle-like points with coordinates $\xi = \xi_0 = 1/4$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ and $\xi = \xi_0 = 0$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ corresponding to unstable stationary regimes.

Comparison of Figures 3 and 4 shows that center-like points with $G = \pm 1/64$ shifted from the line $\xi = 0$ to the line $\xi = 1/8$.

3.1.4 The Subcase $c_1 = c_2 = \frac{1}{2}$, $c_3 = -\frac{1}{4}$

The stream function (23) takes the form

$$G(\xi,\delta) = \left(\xi - \frac{1}{4}\right)\left(\frac{1}{2} - \xi\right)\cos\delta = G_0(\xi_0,\delta_0),$$

and in the case under consideration the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0\left(\xi - \frac{1}{4}\right)\left(\frac{1}{2} - \xi\right)\sin\delta\; e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0\left(\frac{3}{4} - 2\xi\right)\cos\delta\; e^{-sT_2}.$$

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Fig. 5. Reference to Figure 5 shows that the infinite channel ($-\infty < \delta < \infty$) bounded by the lines $\xi = 0$ and $\xi = 1$ is divided into a set of rectangles by the lines $\xi = \frac{1}{4}$, $\xi = \frac{1}{2}$, and $\delta = \pm\frac{\pi}{2} \pm 2\pi n \;\; (n = 0, 1, 2, ...)$, along which $G(\xi,\delta) = 0$. Within each rectangle, the value of G preserves its sign. As this takes place, in the upper rectangles all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 5). Within the bottom rectangles, all stream-lines are also nonclosed with initial and terminal points locating on the boundary line $\xi = 0$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 0$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$. Within the middle rectangles all stream-lines are closed corresponding to periodic changes in amplitudes and phase difference, in so doing the function G attains its extreme magnitudes at the points with the coordinates $\xi = \frac{3}{8}$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ which are the center-like points corresponding to stationary motions.

Comparison of Figures 4 and 5 shows that the zone of the width of 1/4 containing closed stream-lines is shifted upwards.

Along the separatrixes with $G(\xi,\delta) = 0$, which are the boundaries of rectangles, the following analytical solutions could be found:
the phase modulated regime along the line $\xi = 1/4$

$$\xi(T_2) = \xi_0 = \frac{1}{4}$$
$$\ln\left|\tan\left(\frac{1}{2}\delta + \frac{\pi}{4}\right)\right|\Big|_{\delta_0}^{\delta} = \frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

the phase modulated regime along the line $\xi = 1/2$

$$\xi(T_2) = \xi_0 = \frac{1}{2}$$
$$\ln\left|\tan\left(\frac{1}{2}\delta + \frac{\pi}{4}\right)\right|\Big|_{\delta_0}^{\delta} = -\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

and along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ the amplitude modulated regimes decaying with time

$$\left|\ln\frac{\xi - \frac{1}{2}}{\xi - \frac{1}{4}}\right|_{\xi_0}^{\xi} = \mp\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$
$$\delta(T_2) = \delta_0 = \pm\frac{\pi}{2} \pm 2\pi n, \;\; n = 0, 1, 2, ...$$

in so doing the separatrixes are connected with each other at the saddle-like points with coordinates $\xi = \xi_0 = 1/4$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ and $\xi = \xi_0 = 1/2$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ corresponding to unstable stationary regimes.

3.1.5 The Subcase $c_1 = c_2 = 1$, $c_3 = -\frac{3}{4}$

The stream function (23) takes the form

$$G(\xi,\delta) = \left(\xi - \frac{3}{4}\right)(1 - \xi)\cos\delta = G_0(\xi_0,\delta_0),$$

and in the case under consideration the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0\left(\xi - \frac{1}{4}\right)\left(\frac{1}{2} - \xi\right)\sin\delta\; e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0\left(\frac{3}{4} - 2\xi\right)\cos\delta\; e^{-sT_2}.$$

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Fig. 6. Reference to Figure 6 shows that the infinite channel ($-\infty < \delta < \infty$) bounded by the lines $\xi = 0$ and $\xi = 1$ is divided into a set of rectangles by the lines $\xi = \frac{3}{4}$ and $\delta = \pm\frac{\pi}{2} \pm 2\pi n \;\; (n = 0, 1, 2, ...)$, along which $G(\xi,\delta) = 0$. Within each rectangle, the value of G preserves its sign.

As this takes place, within the bottom rectangles, all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 0$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 0$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$. Within the upper rectangles all stream-lines are closed corresponding to periodic changes in amplitudes and phase difference, in so doing the function G attains its extreme magnitudes at the points with the coordinates $\xi = \frac{3}{8}$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ which are

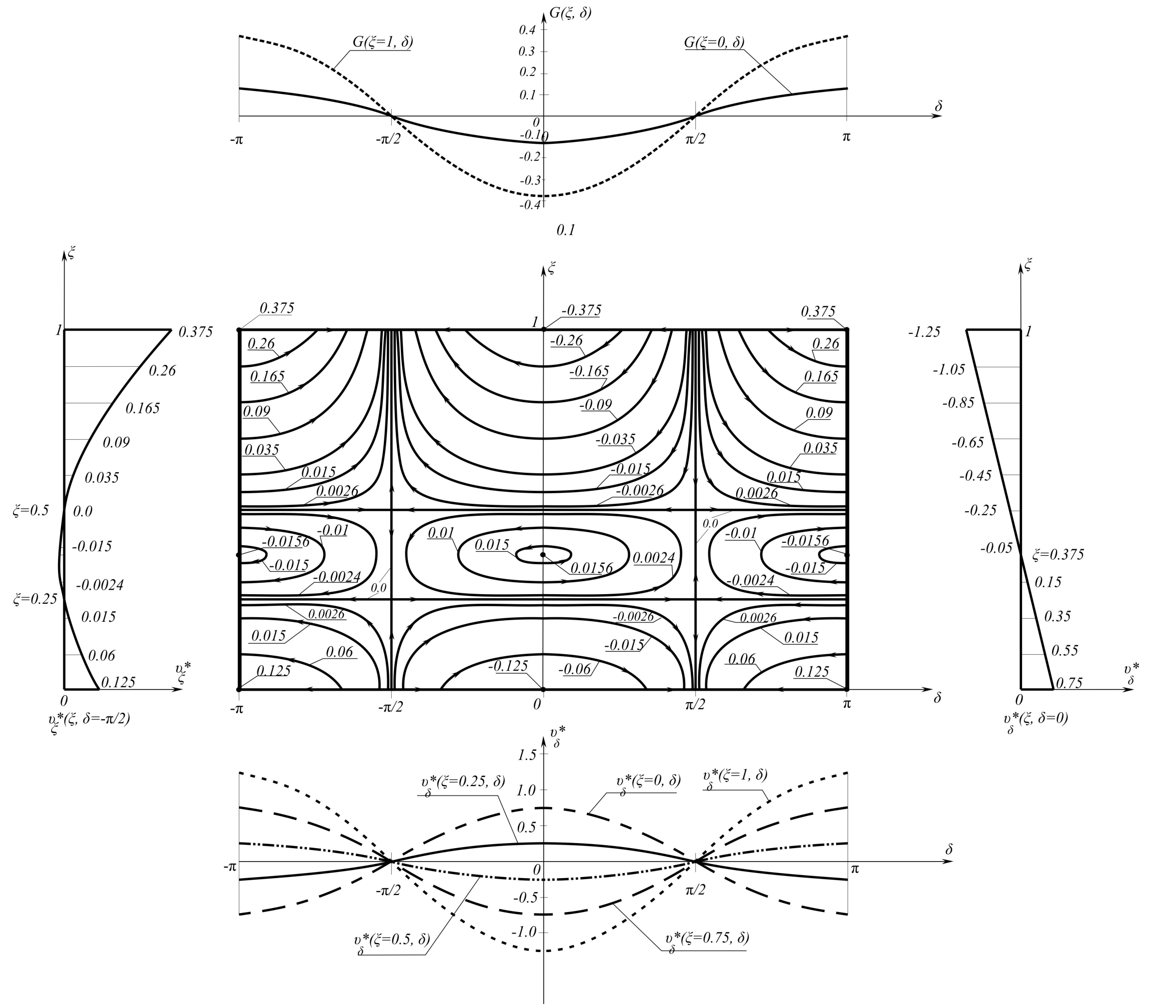

Figure 5: Phase portrait for the case of the additive combinational internal resonance at $K_1 = K_2 = K_3 = 0$ and $c_1 = c_2 = \frac{1}{2}$, $c_3 = -\frac{1}{4}$

the center-like points corresponding to stationary motions.

Comparison of Figures 5 and 6 shows that the zone of the width of 1/4 containing closed streamlines is shifted upwards once again, and its lower and upper boundaries $\xi = 3/4$ and $\xi = 1$ correspond to the following phase modulated regimes with $G_0 = 0$:

$$\xi(T_2) = \xi_0 = 1$$
$$\ln\left|\tan\left(\frac{1}{2}\,\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = -\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

and

$$\xi(T_2) = \xi_0 = \frac{3}{4}$$
$$\ln\left|\tan\left(\frac{1}{2}\,\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = \frac{bE_0}{2s}\left(1 - e^{-sT_2}\right).$$

Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ with $G_0 = 0$ the amplitude modulated regimes decaying with time are realized

$$\left|\ln\frac{\xi - 1}{\xi - \frac{3}{4}}\right|_{\xi_0}^{\xi} = \mp\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$
$$\delta(T_2) = \delta_0 = \pm\frac{\pi}{2} \pm 2\pi n, \quad n = 0, 1, 2, ...$$

in so doing the separatrixes are connected with each other at the saddle-like points with coordinates $\xi = \xi_0 = 3/4$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ and $\xi = \xi_0 = 1$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ corresponding to unstable stationary regimes.

3.1.6 The Subcase $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, $c_3 = -\frac{1}{4}$

In this case the stream function (23) takes the form

$$G(\xi, \delta) = \left(\xi - \frac{1}{4}\right)\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)}\cos\delta = G_0(\xi_0, \delta_0),$$

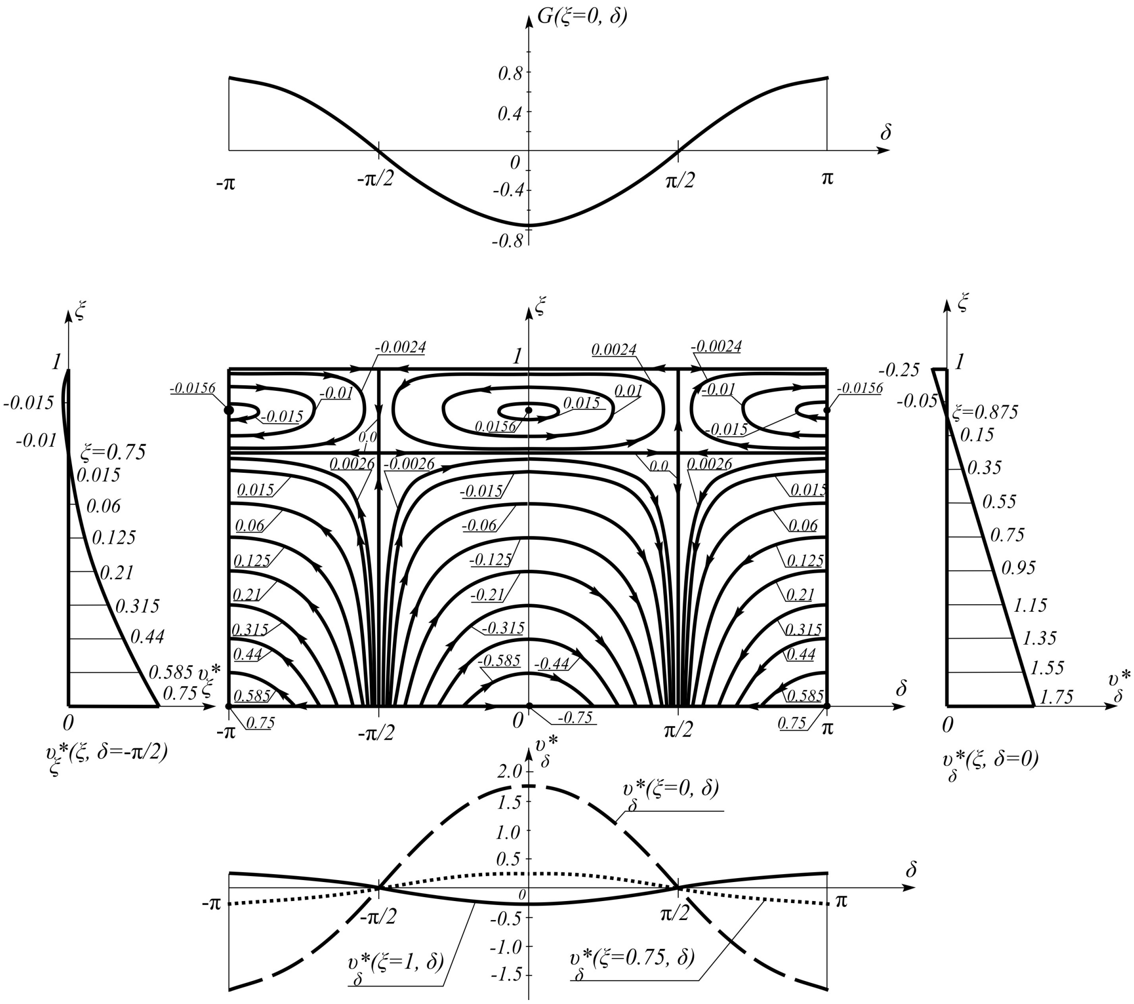

Figure 6: Phase portrait for the case of the additive combinational internal resonance at $K_1 = K_2 = K_3 = 0$ and $c_1 = c_2 = 1$, $c_3 = -\frac{3}{4}$

and the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0 \left(\xi - \frac{1}{4}\right) \sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)} \times \sin\delta\, e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0 \left[\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)} - \frac{\left(\xi - \frac{1}{4}\right)(1 - 2\xi)}{2\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)}}\right] \cos\delta\, e^{-sT_2}.$$

The stream-lines are presented in Figure 7, from which it is evident that if the constants c_1 and c_2 are not equal then along the channel there exists a zone free from stream-lines.

As this takes place, the upper zone of the infinite channel ($-\infty < \delta < \infty$) is bounded by the lines $\xi = 3/5$ and $\xi = 1$ and it is divided into a set of rectangles by the lines $\delta = \pm\frac{\pi}{2} \pm 2\pi n \ \ (n = 0, 1, 2, ...)$, along which $G(\xi, \delta) = 0$. Within each rectangle, the value of G preserves its sign, and all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 7).

The lower zone is bounded by the lines $\xi = 0$ and $\xi = 2/5$. It is subdivided into two subzones by the line $\xi = 1/4$ and is divided into a set of rectangles by the lines $\delta = \pm\frac{\pi}{2} \pm 2\pi n \ \ (n = 0, 1, 2, ...)$, along which $G(\xi, \delta) = 0$. Within the bottom rectangles, all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 0$, along which the stream-function changes periodically

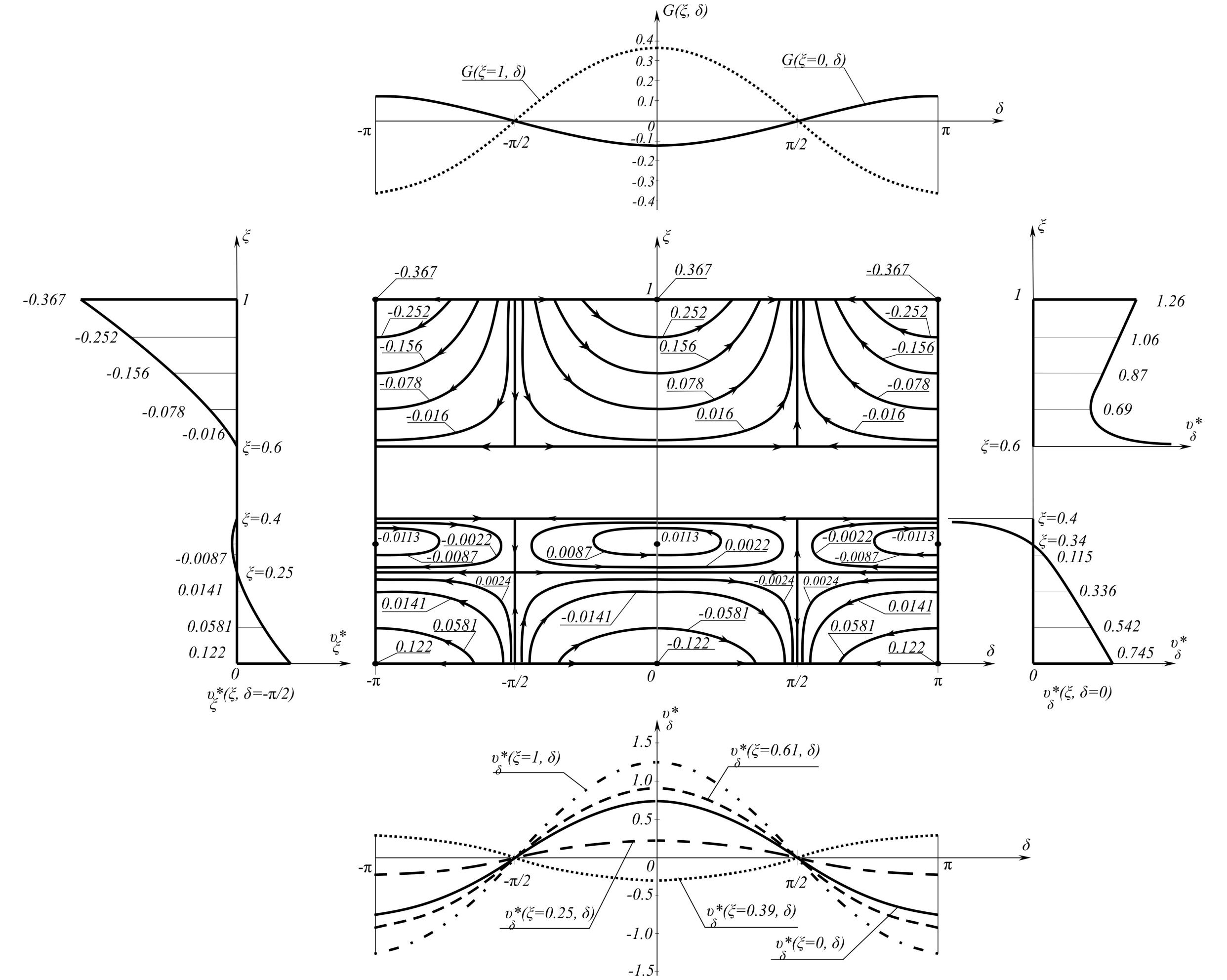

Figure 7: Phase portrait for the case of the additive combinational internal resonance at $K_1 = K_2 = K_3 = 0$ and $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, $c_3 = -\frac{1}{4}$

attaining its extreme magnitudes at the points with the coordinates $\xi = 0$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$. Within the middle rectangles all stream-lines are closed corresponding to periodic changes in amplitudes and phase difference, in so doing the function G attains its extreme magnitudes at the points with the coordinates $\xi = 0.348$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ which are the center-like points corresponding to stationary motions.

Along the boundary $\xi = 1/4$ with $G = 0$ the phase modulated regime takes place

$$\xi(T_2) = \xi_0 = \frac{1}{4}$$
$$\ln\left|\tan\left(\frac{1}{2}\,\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = \frac{\sqrt{21}\,bE_0}{10s}\left(1 - e^{-sT_2}\right),$$

in so doing the points with coordinates $\xi = \xi_0 = 1/4$, $\delta = \delta_0 = \pm(\pi/2) \pm 2\pi n$ are the saddle-like points connecting the separatrixes with each other and corresponding to unstable stationary regimes.

From Figure 7 it is seen that on the lines $\xi = 2/5$ and $\xi = 3/5$ bounding the empty zone the velocity of the phase fluid particles tends to infinity, that is why the transition of the particles from the points with coordinates $\xi = \xi_0 = 3/5$, $\delta = -\pi/2 \pm 2\pi n$ to the points $\xi = \xi_0 = 3/5$, $\delta = \pi/2 \pm 2\pi n$, as well as from the points $\xi = \xi_0 = 3/5$, $\delta = \pi/2 \pm 2\pi n$ to the points $\xi = \xi_0 = 3/5$, $\delta = -\pi/2 \pm 2\pi n$ occurs instantaneously.

3.2 The Influence of Coefficients K_1, K_2, and K_3 on the Character of the Phase Portraits

Now we will trace the influence of parameters K_1, K_2, and K_3 on the character of the phase portraits at the fixed magnitudes of the coefficients c_1, c_2, and c_3.

3.2.1 The Case $c_1 = c_2 = \frac{1}{4}$ and $c_3 = 0$

First we will study the subcase when $K_1 b^{-1} = 1$, while $K_2 = K_3 = 0$. Then the stream function (22) takes the form

$$G(\xi,\delta) = (\frac{1}{4} - \xi)\xi\cos\delta - \frac{1}{2}(\frac{1}{4} - \xi)^2 = G_0(\xi_0, \delta_0).$$

In the case under consideration, the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0 \left(\frac{1}{4} - \xi\right) \xi \sin\delta \, e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0 \left[\left(\frac{1}{4} - 2\xi\right)\cos\delta - \xi + \frac{1}{4}\right] e^{-sT_2},$$

whence it follows that along the separatrixes with $G(\xi,\delta) = 0$ the following analytical solutions could be found:
the phase modulated regime along the line $\xi = 1/4$

$$\xi(T_2) = \xi_0 = \frac{1}{4}$$
$$\ln\left|\tan\left(\frac{1}{2}\delta + \frac{\pi}{4}\right)\right|_{\delta_0}^{\delta} = -\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

and along the curved parts of the separatrixes

$$\left|\ln\frac{\xi}{\xi - \frac{1}{4}}\right|_{\xi_0}^{\xi} = -\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$
$$\dot{\delta} = 2bE_0 \left[\left(\frac{1}{4} - 2\xi\right)\cos\delta - \xi + \frac{1}{4}\right] e^{-sT_2},$$

wherein ξ varies from $\xi_{\min} = 1/12$ at $\delta = \pm 2\pi n$ to $\xi_{\max} = 1$ at $\delta = \pm\arccos(-3/8) \pm 2\pi n$.

The line $\xi = \xi_0 = 1/4$ divides the channel into two rectangles. Within the upper rectangle, all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 8), in so doing the stream-lines with $G_0 = 0$ separate the zones where the value of G preserves its sign.

The curvilinear and rectilinear parts of the separatrixes intersect at the saddle-like stationary points with coordinates $\xi_0 = 1/4$ and $\delta_0 = \pm\pi/2 \pm 2\pi n$ (see Fig. 8) bounding the low rectangle into the zones within which all stream-lines either are closed or non-closed. Within the zones with closed stream-lines the function G attains its extreme magnitudes equal to 1/96 at the center-like points with coordinates $\xi = \xi_0 = \frac{1}{6}$, $\delta = \delta_0 = \pm 2\pi n$ corresponding to stable stationary motions. Along the non-closed stream-lines amplitudes are changed periodically, while the phases vary aperiodically and phase fluid points move along the positive direction.

Along the low boundary $\xi = 0$ with $G_0 = -1/32$ the phase modulated regime is realized

$$\xi(T_2) = \xi_0 = 0,$$
$$\tan\left(\frac{1}{2}\delta\right)\Big|_{\delta_0}^{\delta} = \frac{bE_0}{2s}\left(1 - e^{-sT_2}\right),$$

in so doing the phase velocity v_δ varies periodically from 0 at the points $\delta = \pm\pi \pm 2\pi n$ to the maximal magnitude at the points $\delta = \pm 2\pi n$.

From the curves showing the phase velocity distribution at different levels of amplitudes it is seen that all curves intersect at the points with $\delta = \pm 2\pi/3 \pm 2\pi n$.

Now we will proceed to the subcase when all parameters K_i $(i = 1, 2, 3)$ are nonzero, namely: $K_1 b^{-1} = K_2 b^{-1} = K_3 b^{-1} = 1$. Then the stream function (22) takes the form

$$G(\xi,\delta) = (\frac{1}{4} - \xi)\xi\cos\delta - (\frac{1}{4} - \xi)^2 + \frac{1}{2}\xi^2 = G_0(\xi_0, \delta_0).$$

The stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Fig. 9. Reference to Figure 9 shows that the infinite channel $(-\infty < \delta < \infty)$ bounded by the lines $\xi = 0$ and $\xi = 1$ is divided into two rectangles by the lines $\xi = \frac{1}{4}$ and $\delta = \pm\frac{\pi}{2} \pm 2\pi n$ $(n = 0, 1, 2, ...)$.

In the case under consideration, the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0 \left(\frac{1}{4} - \xi\right) \xi \sin\delta \, e^{-sT_2},$$

$$v_\delta = \dot{\delta} = 2bE_0 \left[\left(\frac{1}{4} - 2\xi\right)\cos\delta - \xi + \frac{1}{2}\right] e^{-sT_2}.$$

Along the rectilinear part of the separatrixes with $G(\xi,\delta) = 1/32$, i.e. along the line $\xi = 1/4$, the phase modulated regime is realized:

$$\xi(T_2) = \xi_0 = \frac{1}{4}$$
$$\cot\left(\frac{1}{2}\delta\right)\Big|_{\delta_0}^{\delta} = -\frac{bE_0}{2s}\left(1 - e^{-sT_2}\right).$$

Along the low boundary $\xi = 0$ the second phase modulated regime is realized:

$$\xi(T_2) = \xi_0 = \frac{1}{4}$$
$$\frac{2}{\sqrt{3}}\,\arctan\frac{\tan\left(\frac{1}{2}\delta\right)}{\sqrt{3}}\Big|_{\delta_0}^{\delta} = \frac{bE_0}{2s}\left(1 - e^{-sT_2}\right).$$

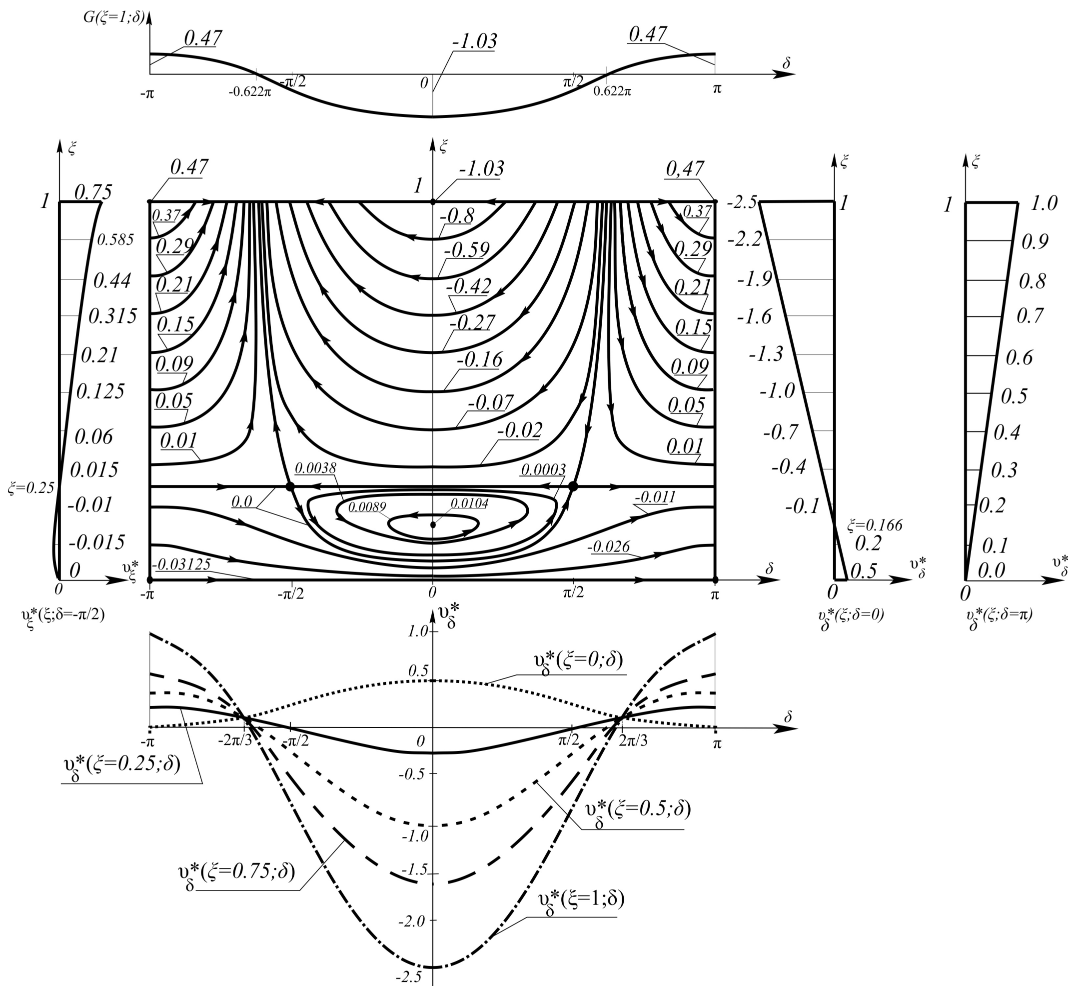

Figure 8: Phase portrait for the case of the additive combinational internal resonance at $K_1b^{-1} = 1$, $K_2 = K_3 = 0$, and $c_1 = c_2 = \frac{1}{4}$, $c_3 = 0$

Within the low rectangle bounded by the lines $\xi = 0$ and $\xi = 1/4$ all stream-lines are nonclosed, and the phase fluid flows with positive phase velocities. Non closed stream-lines correspond to vibratory motions with amplitudes varying periodically and phases varying aperiodically.

The curvilinear parts of the separatrixes are connected with its rectilinear part at the saddle-like points with coordinates $\xi_0 = 1/4$, $\delta_0 = \pm 2\pi n$. Within the upper rectangle the curvilinear separatrixes divide it into zones wherein the phase fluid flows in one direction. As this takes place, in the upper rectangles all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm \pi n$ $(n = 0, 1, 2, ...)$ (Fig. 9).

3.2.2 The Subcase $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, $c_3 = -\frac{1}{4}$

First we will study the subcase when $K_1b^{-1} = 1$, while $K_2 = K_3 = 0$. Then the stream function (22) takes the form

$$G(\xi, \delta) = \left(\xi - \frac{1}{4}\right)\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)}\cos\delta - \frac{1}{2}\left(\frac{2}{5} - \xi\right)^2 = G_0(\xi_0, \delta_0),$$

and the velocities of the phase fluid particles could be calculated as follows

$$v_\xi = \dot{\xi} = 2bE_0\left(\xi - \frac{1}{4}\right)\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)} \times \sin\delta\, e^{-sT_2},$$

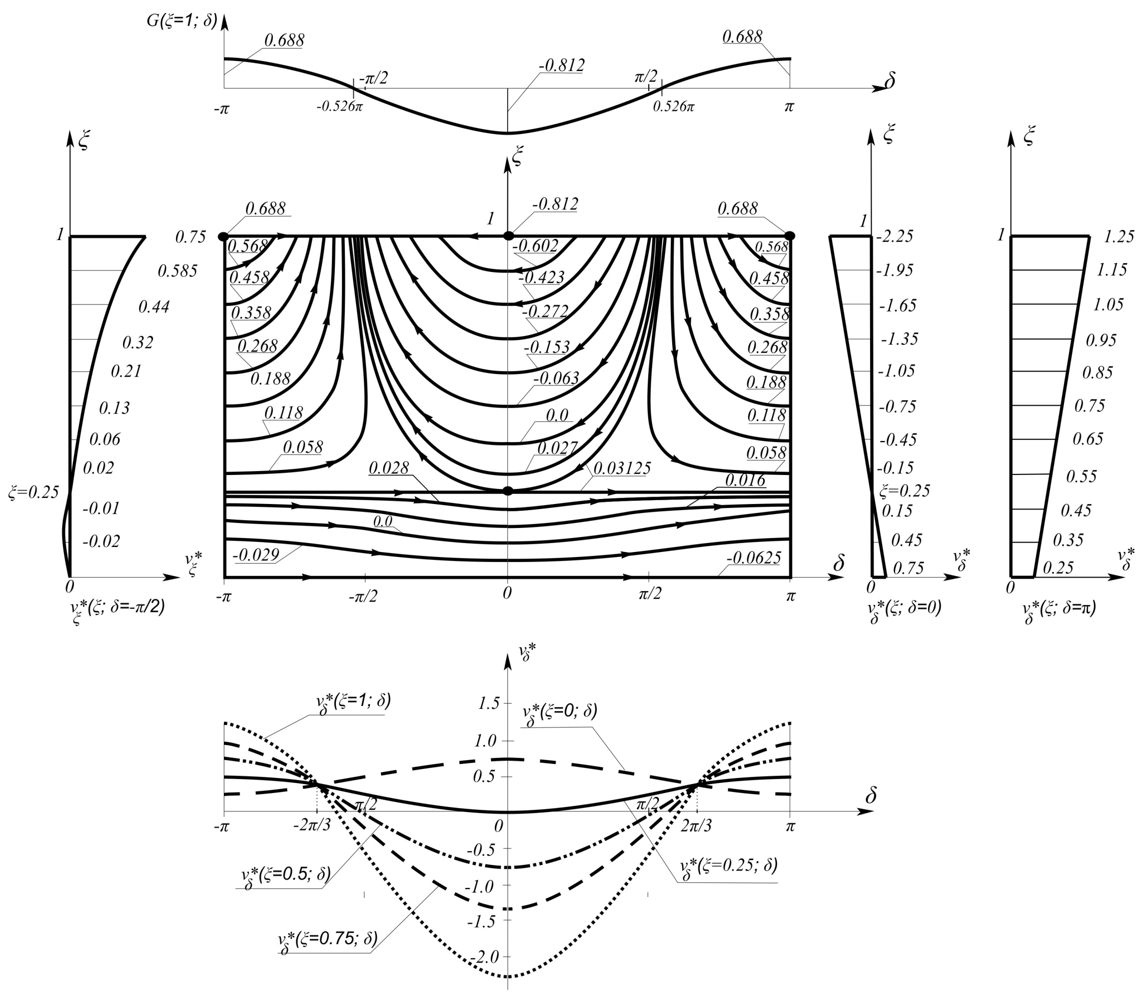

Figure 9: Phase portrait for the case of the additive combinational internal resonance at $K_1b^{-1} = K_2b^{-1} = K_3b^{-1} = 1$, and $c_1 = c_2 = \frac{1}{4}$, $c_3 = 0$

$$v_\delta = \dot{\delta} \ = \ 2bE_0 \left\{ \left[\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)} \right. \right.$$
$$\left. - \ \frac{\left(\xi - \frac{1}{4}\right)(1 - 2\xi)}{2\sqrt{\left(\frac{2}{5} - \xi\right)\left(\frac{3}{5} - \xi\right)}} \right] \cos\delta$$
$$\left. + \ \left(\frac{2}{5} - \xi\right) \right\} e^{-sT_2}.$$

The stream-lines are presented in Figure 10, from which it is evident that horizontal rectilinear separatrixes $\xi = 1/4$, $\xi = 2/5$, and $\xi = 3/5$ remain the same as in the case shown in Figure 7 with the only difference that the magnitudes of the stream-function G now equal, respectively, to -9/800(=-0.01125), 0, and -1/50(=-0.02).

If one of the coefficients K_i $(i = 1, 2, 3)$ is nonzero, then, as it takes place in the case under consideration, vertical rectilinear separatrixes transform to curvilinear ones. Thus, in the upper zone of the infinite channel $(-\infty < \delta < \infty)$ bounded by the lines $\xi = 3/5$ and $\xi = 1$, the separatrixes, along which $G = -0.02$, connect these boundary lines at the points with the coordinates $\xi = 1$, $\delta = \pm 0.3566\pi \pm 2\pi n \quad (n = 0, 1, 2, ...)$ and $\xi = 3/5$, $\delta = \pm\pi/2 \pm 2\pi n \quad (n = 0, 1, 2, ...)$, in so doing they separate the upper zone in subzones within which the phase fluid moves in one direction: clockwise or counter-clockwise. In each such a subzone all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 10).

In the middle and bottom zones there are two types of curvilinear separatrixes along which $G = 0$ and -0.01125, between which in the middle zone there exist nonclosed stream-lines corresponding to peri-

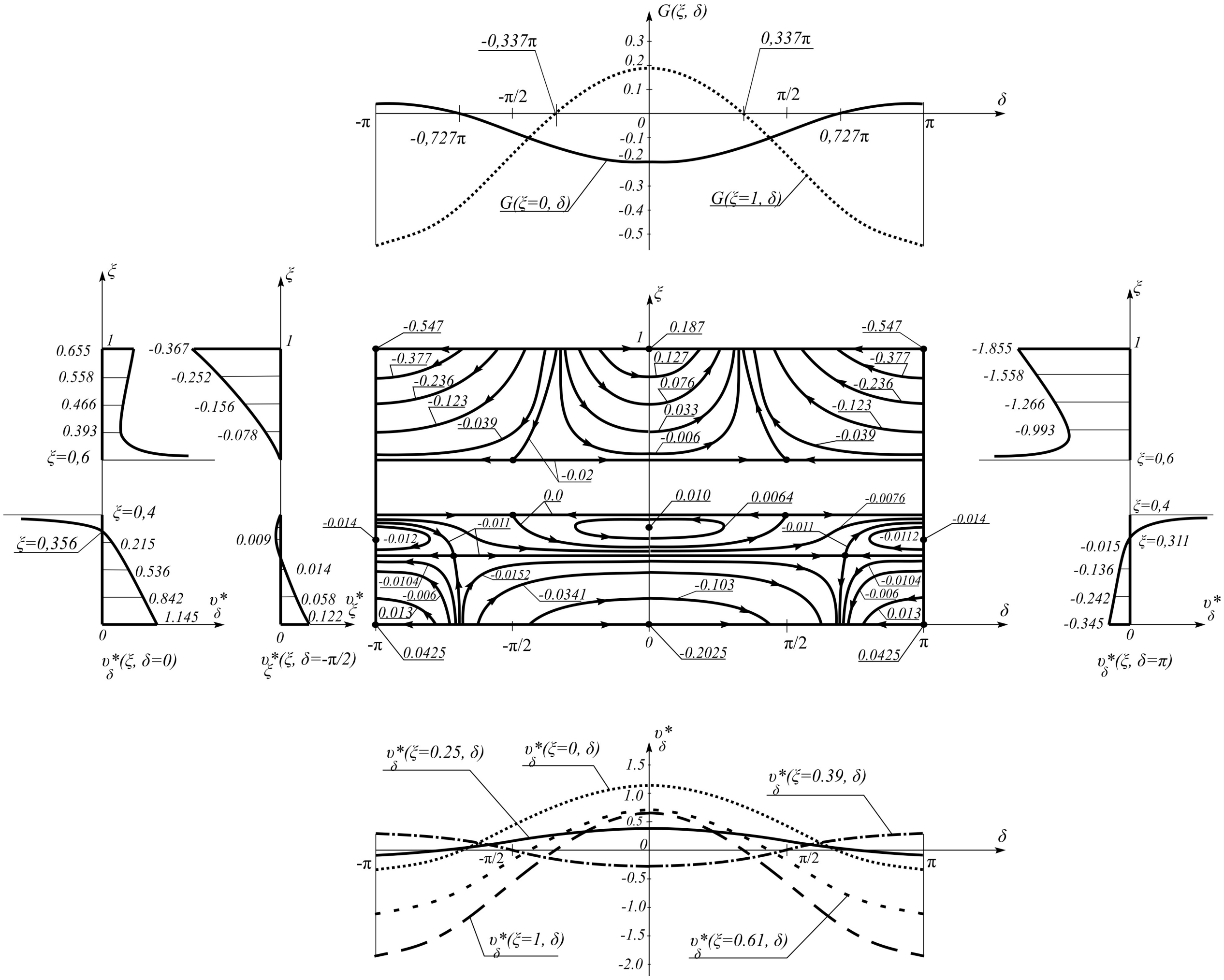

Figure 10: Phase portrait for the case of the additive combinational internal resonance at $K_1 b^{-1} = 1$, $K_2 = K_3 = 0$, and $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, $c_3 = -\frac{1}{4}$

odic changes of amplitudes and aperiodic changes of phase difference.

The stream-lines with $G = 0$ separate in the middle zone the subzones wherein closed stream-lines surround the center-like points with the coordinates $\xi = 0.356$, $\delta = \pm 2\pi n \ \ (n = 0, 1, 2, ...)$, and $G_{\max} = 0.01$. The curvilinear parts of the separatrixes with $G = 0$ are connected with the corresponding rectilinear part $\xi = const = 2/5$ at the points with $\xi = 2/5$, $\delta = \pm\pi/2 \pm 2\pi n \ \ (n = 0, 1, 2, ...)$, in so doing the phase fluid particle moves in the counter-clockwise direction from the point $\xi = 2/5$, $\delta = -\pi/2$ to the point $\xi = 2/5$, $\delta = \pi/2$, crossing the minimal point on this separatrix with coordinates $\xi = 0.285$, $\delta = \pm 2\pi n \ (n = 0, 1, 2, ...)$.

The curvilinear separatrixes with $G = -0.01125$ intersect the rectilinear separatrix $\xi = const = 1/4$ at the saddle-like points with coordinates $\xi = 1/4$, $\delta = \pm 0.727\pi \pm 2\pi n \ \ (n = 0, 1, 2, ...)$, their initial and terminal points locate on the boundary line $\xi = 0$ at $\delta = \pm 0.6897\pi \pm 2\pi n \ \ (n = 0, 1, 2, ...)$, and the uppermost points have coordinates $\xi = 0.363$, $\delta = \pm 0.6897\pi \pm 2\pi n \ (n = 0, 1, 2, ...)$.

Along the line $\xi = 1/4$ the phase modulated regime takes place

$$
\begin{aligned}
&\xi(T_2) = \xi_0 = \frac{1}{4}, \\
&\frac{1}{\sqrt{b^2 - a^2}} \ln \left| \frac{(b-a)\tan\frac{\delta}{2} + \sqrt{b^2 - a^2}}{(b-a)\tan\frac{\delta}{2} - \sqrt{b^2 - a^2}} \right|_{\delta_0}^{\delta} \\
&= \frac{2bE_0}{s}\left(1 - e^{-sT_2}\right),
\end{aligned}
$$

where $a = 3/20$, and $b = \sqrt{21}/20$.

Within the middle zone these separatrixes bound the closed stream-lines surrounding the center-line points with $\xi = 0.3107$, $\delta = \pm\pi \pm 2\pi n \quad (n = 0, 1, 2, ...)$. Within the bottom zone, separatrixes with $G = -0.01125$ divide it into subzones where phase fluid particles move in one direction: clockwise or counter-clockwise. In each such a subzone all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 0$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 0$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 10).

Now we will proceed to the subcase when all parameters K_i $(i = 1, 2, 3)$ are nonzero, namely: $K_1 b^{-1} = K_2 b^{-1} = K_3 b^{-1} = 1$. Then the stream function (22) takes the form

$$G(\xi,\delta) = \left(\xi - \frac{1}{4}\right)\sqrt{\left(\frac{2}{5}-\xi\right)\left(\frac{3}{5}-\xi\right)}\cos\delta - \frac{1}{2}\left(\frac{2}{5}-\xi\right)^2 - \frac{1}{2}\left(\frac{3}{5}-\xi\right)^2 + \frac{1}{2}\left(\xi - \frac{1}{4}\right)^2$$

The stream-lines are presented in Figure 11, from which it is evident that horizontal rectilinear separatrixes $\xi = 1/4$, $\xi = 2/5$, and $\xi = 3/5$ remain the same as in the cases shown in Figures 7 and 10 with the only difference that the magnitudes of the stream-function G now equal, respectively, to -29/400(=-0.0725), -7/800(=-00875), and 33/800(=0.04125).

The velocities of the phase fluid particles could be calculated in this case as follows

$$\begin{aligned} v_\xi = \dot{\xi} &= 2bE_0\left(\xi - \frac{1}{4}\right)\sqrt{\left(\frac{2}{5}-\xi\right)\left(\frac{3}{5}-\xi\right)} \\ &\times \sin\delta\, e^{-sT_2}, \end{aligned}$$

$$\begin{aligned} v_\delta = \dot{\delta} &= 2bE_0\left\{\left[\sqrt{\left(\frac{2}{5}-\xi\right)\left(\frac{3}{5}-\xi\right)} \right.\right. \\ &- \left.\frac{\left(\xi-\frac{1}{4}\right)(1-2\xi)}{2\sqrt{\left(\frac{2}{5}-\xi\right)\left(\frac{3}{5}-\xi\right)}}\right]\cos\delta \\ &+ \left.\left(\frac{3}{4}-\xi\right)\right\}e^{-sT_2}. \end{aligned}$$

In the upper zone of the infinite channel $(-\infty < \delta < \infty)$ bounded by the lines $\xi = 3/5$ and $\xi = 1$, the separatrixes, along which $G = 0.04125$, connect these boundary lines at the points with the coordinates $\xi = 1$, $\delta = \pm 0.4829\pi \pm 2\pi n \quad (n = 0, 1, 2, ...)$ and $\xi = 3/5$, $\delta = \pm\pi/2 \pm 2\pi n \quad (n = 0, 1, 2, ...)$, in so doing they separate the upper zone in subzones within which the phase fluid moves in one direction: clockwise or counter-clockwise. In each such a subzone all stream-lines are nonclosed with initial and terminal points locating on the boundary line $\xi = 1$, along which the stream-function changes periodically attaining its extreme magnitudes at the points with the coordinates $\xi = 1$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ (Fig. 11).

In the middle zone nearby its upper boundary line $\xi = 2/5$ there exist center-like points with coordinates $\xi = 0.3927$, $\delta = \pm 2\pi n \quad (n = 0, 1, 2, ...)$ and $G_{\min} = -0.017$ located within narrow subzones separated by the curvilinear separatrixes with G =-0.00875, which are connected with the corresponding rectilinear separatrix $\xi = 2/5$ at the points with $\xi = 2/5$, $\delta = \pm\pi/2 \pm 2\pi n \ (n = 0, 1, 2, ...)$.

In the bottom zone bounded by the lines $\xi = 0$ and $\xi = 1/4$ one part of the nonclosed streamlines with initial and terminal points located at the lower boundary $\xi = 0$ is separated from the other nonclosed infinitely long stream-lines located nearby the rectilinear separatrix $\xi = 1/4$ is separated by the stream-line with $G = -0.1063$ and with initial and terminal points $\xi = 0$, $\delta = \pm\pi \pm \pi n \ (n = 0, 1, 2, ...)$.

Along the line $\xi = 1/4$ the phase modulated regime takes place

$$\begin{aligned} &\xi(T_2) = \xi_0 = \frac{1}{4}, \\ &\frac{2}{\sqrt{a^2-b^2}}\arctan\frac{(a-b)\tan\frac{\delta}{2}}{\sqrt{a^2-b^2}}\Big|_{\delta_0}^{\delta} \\ &= \frac{2bE_0}{s}\left(1 - e^{-sT_2}\right), \end{aligned}$$

where $a = 1/2$, and $b = \sqrt{21}/20$.

4 Conclusion

The proposed analytical approach for investigating the damped vibrations of a nonlinear plate in a fractional derivative viscoelastic medium subjected to the combinational internal resonances of additive-difference type has been possible owing to the new procedure suggested recently in [6, 7], resulting in decoupling linear parts of equations with further utilization of the method of multiple scales for solving nonlinear governing equations of motion.

The phenomenological analysis carried out for the additive combinational internal resonance using the phase portraits constructed for different magnitudes of the plate parameters reveals the great variety of vibrational motions: stationary vibrations, two-sided energy exchange between two subsystems, and complete one-sided energy transfer. The analysis of the phase portraits for various oscillatory regimes shows that they contain closed and non-closed stream-lines separated by the rectilinear and curvilinear separatrixes, along which analytic solutions have been

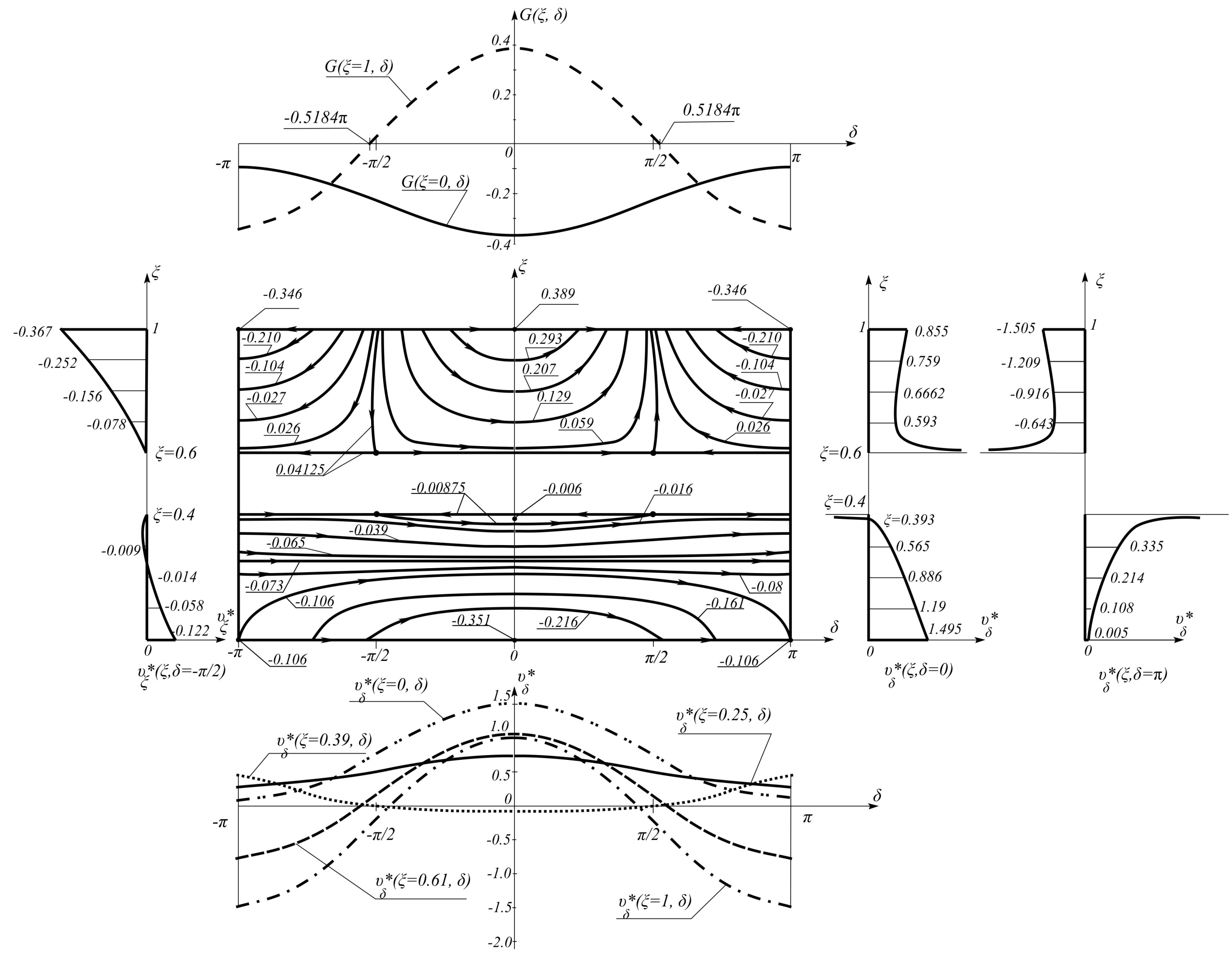

Figure 11: Phase portrait for the case of the additive combinational internal resonance at $K_1 b^{-1} = K_2 b^{-1} = K_3 b^{-1} = 1$ and $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, $c_3 = -\frac{1}{4}$

found, which define the irreversible energy transfer from one subsystem into another and are inherently soliton-like solutions in the theory of vibrations.

The location of horizontal rectilinear separatrixes is defined by the constants of integration c_i ($i = 1, 2, 3$) governed by the initial conditions. These constants not only govern the distribution of the initial energy between the modes coupled by the additive internal resonance, but also influence the type of streamlines locating between horizontal separatrixes, in so doing the magnitudes of c_i do not change the rectilinear character of the horizontal separatrixes. The coefficients K_i ($i = 1, 2, 3$), which depend on the frequencies of interacting modes and on plate's parameters, define the character of separatrixes connecting the boundaries $\xi = 0$ and $\xi = 1$. If all coefficients $K_i = 0$, then all separatrixes are pure vertical, but if at least one of them is nonzero, then vertical separatrixes transform into curvilinear separatrixes, the initial and terminal points of which locate on horizontal separatrixes and/or on boundary lines $\xi = 0$ and $\xi = 1$.

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